Split noncommutativity and compactified brane solutions in matrix models

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Motivation

- NC space(time): $\theta^{\mu\nu}$ breaks Lorentz invariance, UV/IR mixing
- protection through maximal SUSY:

  IKKT model $S = Tr([X^a, X^b][X^a, X^b] + \Psi\Gamma_a[X^a, \Psi])$

  $\rightarrow$ 4-dimensional solutions, e.g. $\mathbb{R}^4_\theta$: too simple for physics
- natural way out: compact extra dimensions

  $\rightarrow$ symmetry breaking, etc.

  usually: $\mathcal{M}^4_\theta \times \mathcal{K}_N,$, e.g. $\mathcal{K}_N = S^2_N$

  problem: need cubic term, breaks (super)symm., scale!

here: split NC = Poisson structure relates $\mathcal{M}$ with $\mathcal{K}$

solutions of *undeformed* IKKT model (Minkowski)
milder Lorentz violation

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Basic idea:

NC spaces with geometry $\mathcal{M}^{2n} = \mathcal{M}^4 \times \mathcal{K}$ s.t.
NC structure mixes spacetime $\mathcal{M}^4$ with the compact space $\mathcal{K}$

Poisson tensor: $\Pi = \theta^{\mu i}(x,y)\frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial y^i} \quad + \ldots$
i.e. $[x^\mu, y^i] \neq 0$

... "split noncommutativity" x^μ on $\mathcal{M}^4$, y^i on $\mathcal{K}$

in particular: $\dim(\mathcal{K}) = 4 \;\Rightarrow\; \mathcal{M}$ may be isotropic! $[x^\mu, x^\nu] = 0$

always assume Π nondeg., $\mathcal{M}^{2n} = \mathcal{M}^4 \times \mathcal{K}$ symplectic space

- may get strict UV cutoff on $\mathcal{M}^4$
  since: $\Delta x^\mu \Delta y^i \geq \theta, \; \Delta y^i \leq R \;\Rightarrow\; \Delta x^\mu \geq \theta R^{-1}$
- Lorentz inv. on $\mathcal{M}^4$ may be respected (almost ...), good for gravity
- find solutions of IKKT model (Minkowski!), scales = dynamical

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basic example: fuzzy cylinder $S^1 \times_\xi \mathbb{R}$

Chaichian Demichev Presnajder 1998

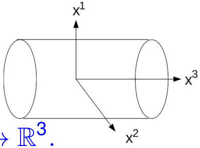
3 hermitian matrices X^1, X^2, X^3, define $U = X^1 + iX^2$,

$$\begin{aligned} UU^\dagger &= U^\dagger U = R^2 \qquad \text{hence } [X^1, X^2] = 0 \\ [U, X^3] &= \xi U, \qquad [U^\dagger, X^3] = -\xi U^\dagger \end{aligned}$$

representation on $\mathcal{H}$:

$$\begin{aligned} U|n\rangle &= R|n+1\rangle, \qquad U^\dagger|n\rangle = R|n-1\rangle \\ X^3|n\rangle &= \xi n|n\rangle, \qquad n \in \mathbb{Z},\ \xi \in \mathbb{R} \end{aligned}$$

interpretation: quantized embedding functions



$$\begin{pmatrix} X^1 + iX^2 \\ X^3 \end{pmatrix} \sim \begin{pmatrix} Re^{iy_3} \\ x^3 \end{pmatrix} : \quad S^1 \times \mathbb{R} \hookrightarrow \mathbb{R}^3.$$

... quantization of $T^\star S^1$, Poisson structure: $\{x^3, y^3\} = \xi$

wavefunctions: basis of functions on $S^1 \times_\xi \mathbb{R}$:

$$\{e^{ipX^3} U^n, \quad p \in [-\frac{\pi}{\xi}, \frac{\pi}{\xi}], \quad n \in \mathbb{Z}\},$$

hence

$$\phi = \sum_{n\in\mathbb{Z}} \int_{-\pi/\xi}^{\pi/\xi} dp\, \tilde\phi_n(p)\, e^{ipX^3} U^n.$$

note: momenta p compactified on a circle, since $e^{ipX^3} \equiv e^{i(p+\frac{2\pi}{\xi})X^3}$,
noncompact space is lattice ! (since $X^3|n\rangle = \xi n|n\rangle, \quad n \in \mathbb{Z}$)

(matrix) Laplacian:

$$\Box := [X^a, [X^b, .]]\delta_{ab} \quad \sim e^\sigma \Box_G, \qquad G^{\mu\nu} \sim \theta^{\mu\mu'}\theta^{\mu\mu'} g_{\mu'\nu'}$$

check:

$$\Box X^3 = 0$$
$$\Box X^i = \xi^2 X^i, \qquad i = 1, 2$$

$\Rightarrow$ is not solution of matrix model ☹ but: it is, if rotating! ☺

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Matrix Models

IKKT (IIB) model — Ishibashi, Kawai, Kitazawa and Tsuchiya 1996

$$S[X] = -Tr\left([X^a, X^b][X^{a'}, X^{b'}]\eta_{aa'}\eta_{bb'} + \bar{\Psi}\gamma_a[X^a, \Psi]\right)$$

$$X^a = X^{a\dagger} \in Mat(N, \mathbb{C}), \qquad a = 0, ..., 9, \quad N \to \infty$$

gauge symmetry $X^a \to UX^aU^{-1}$, $SO(9,1)$, SUSY, etc.

e.o.m.

$$\Box X^c \equiv [X^a, [X^b, X^c]]\eta_{ab} = 0$$

as
1) nonpert. def. of IIB string theory (on $\mathbb{R}^{10}$) (*IKKT*)
2) $\mathcal{N} = 4$ SUSY Yang-Mills gauge thy. on $\mathbb{R}^4_\theta$

NC branes $\mathcal{M} \subset \mathbb{R}^{10}$, (NC) gauge theory & (emergent) gravity (H.S. 2007 ff)

BFSS model — Banks Fischler Shenker Susskind 1996

9 time-dependent matrices $X^a(t)$, e.o.m.

$$\ddot{X}^c + [X^a, [X^b, X^c]]\delta_{ab} = 0, \qquad (a, b = 1, ..., 9),$$

... M-theory in DLCQ, resp. IIA string theory

let (U, X) ... fuzzy cylinder; $U = X^i + iX^2 \sim Re^{i\xi y}$
not solution of IKKT.

but: consider rotating fuzzy cylinder

$$\begin{pmatrix} X^1(t) + iX^2(t) \\ X^3(t) \end{pmatrix} = \begin{pmatrix} Ue^{i\xi t} \\ X^3 \end{pmatrix} \sim \begin{pmatrix} Re^{i\xi(y+t)} \\ x \end{pmatrix}.$$

is solution of the BFSS model:

$$\ddot{X}^{1,2} + \Box X^{1,2} = (-\xi^2 + \xi^2)X^{1,2} = 0, \qquad \Box = [X^a, [X^b, .]]\delta_{ab}$$

”rotating supertube“ Bak, Lee 2001

expect similar solution also for IKKT model; how?

new solutions of the IKKT model

propagating fuzzy clinder: let (U, X) ... fuzzy cylinder

embed non-compact direction along light-like direction:

$$\begin{pmatrix} X^0 \\ X^1 + iX^2 \\ X^3 \end{pmatrix} = \begin{pmatrix} 0 \\ U \\ 0 \end{pmatrix} + v^a X \sim \begin{pmatrix} x \\ Re^{iy} \\ x \end{pmatrix}, \qquad v^a = \begin{pmatrix} 1 \\ 0 + i0 \\ 1 \end{pmatrix}.$$

easy to check:

$$\Box U = (v^a \eta_{ab} v^b)\,[X,[X,U]] + [X^1,[X^1,U]] + [X^2,[X^2,U]] = 0$$

remarks:

- cylinder, propagates in light-like direction v
- degenerate metric
- radius, ξ are arbitrary moduli

higher-dimensional solutions

Assume Y^i describes fuzzy space $\mathcal{K} \subset \mathbb{R}^D$ with

$$\Box_Y Y^i = \xi_d^2 Y^i, \qquad \Box_Y \equiv [Y^i,[Y^j,.]]\delta_{ij}, \quad i,j = 1,...,D.$$

examples: $S^2_N, T^2_N, ...$

construct solution of BFSS model: define $Z^\alpha := Y^{2\alpha-1} + iY^{2\alpha}$, and

$$Z^\alpha(t)' := Z^\alpha e^{i\omega t} =: Y^{2\alpha-1}(t) + iY^{2\alpha}(t), \qquad \omega^2 = \xi_d^2.$$

is solution of BFSS

$$\ddot{Y}^i + \Box_Y Y^i = 0$$

stabilized by angular momentum

for IKKT model: time is noncommutative $\rightarrow$ twist via fuzzy cylinder!

Lemma

Suppose $Y^i, \; i = 1, ..., 6$ *are hermitian matrices which satisfy*

$$\Box_Y Y^j \equiv \sum_i [Y^i, [Y^i, Y^j]] \quad = \xi_Y^2 \, Y^j$$

Let (X, U) *be a fuzzy cylinder with* $(R = 1, \xi)$ *which commutes with the above matrices. Collect the* Y^i *into complex matrices*

$$Z^\alpha = Y^{2\alpha-1} + iY^{2\alpha}, \qquad \alpha = 1, 2, 3.$$

Then the 6 hermitian matrices $Y^{i\prime}$ *defined via*

$$\begin{pmatrix} Z^{1\prime} \\ Z^{2\prime} \\ Z^{3\prime} \end{pmatrix} = \begin{pmatrix} Z^1 \, U^{n_1} \\ Z^2 \, U^{n_2} \\ Z^3 \, U^{n_3} \end{pmatrix}$$

satisfy

$$\begin{aligned} [X, [X, Z^{\alpha\prime}]] &= n_\alpha^2 \xi_X^2 \, Z^{\alpha\prime} \\ \Box_{Y'} Y^{j\prime} &= \xi_Y^2 \, Y^{j\prime} \\ \Box_{Y'} X &= 0 \end{aligned}$$

geometrical significance of the construction, semi-classically:

$Y^i \sim y^i : \ \mathcal{M} \hookrightarrow \mathbb{R}^D$ ... quantized embedding

$$\begin{aligned} (\mathbb{R} \times S^1) \times \mathcal{M} &\to \mathcal{M}' \subset \mathbb{R}^{D+1} \\ (x, e^{i\varphi}, p) &\mapsto (x, e^{i\varphi} \cdot p) \end{aligned}$$

where $e^{i\varphi} \cdot p$ ... S^1 action on $\mathcal{M}$ (should be free) and

$$(X, Y^{i'}) \sim (x, y^{i'}) : \quad \mathcal{M}' \hookrightarrow \mathbb{R}^{D+1}$$

$\rightarrow$ many new solutions of IKKT model of type $\mathcal{M}^4 \times \mathcal{K}$,

$\mathcal{K} = T^2$, $\mathcal{K} = T^4$, $\mathcal{K} = S^2 \times T^2$ and $\mathcal{K} = S^2 \times S^2$

Examples of IKKT solutions:

1) $\mathbb{R}^3 \times_\xi T^3$: take 3 fuzzy cylinders $S^1 \times_\xi \mathbb{R}$ realized by (X^i, U_i), embedded along space-like directions:

$$\begin{pmatrix} X^0 \\ X^i, \ i = 1,2,3 \\ Z^1 \\ Z^2 \\ Z^3 \end{pmatrix} = \begin{pmatrix} 0 \\ X^i, \ i = 1,2,3 \\ U_1 \\ U_2 \\ U_3 \end{pmatrix}$$

add "time-like" fuzzy cylinder $(U_0, X^0) \sim (e^{iy_0}, x_0)$, redefine

$$\begin{pmatrix} X^{0\prime} \\ X^{i\prime}, \ i = 1,2,3 \\ Z^{1\prime} \\ Z^{2\prime} \\ Z^{3\prime} \end{pmatrix} = \begin{pmatrix} X^0 \\ X^i, \ i = 1,2,3 \\ U_1 U_0 \\ U_2 U_0 \\ U_3 U_0 \end{pmatrix}$$

Lemma $\Rightarrow$ is solution of IKKT model $\Box X^{a\prime} = 0$ if $-\xi_0^2 + \xi_i^2 = 0$

however: constraint $X^0 - \sum_i X^i = C$, symplectic leaves $\mathbb{R}^3 \times_\xi T^3$

avoid using $SO(D)$ trafo $\Rightarrow$ can get non-degenerate $\mathbb{R}^4 \times_\xi T^4$

fluctuations on NC brane $\mathcal{M}$ $\Rightarrow$ NC gauge theory on $\mathcal{M}$

- induced metric on $\mathbb{R}^n \times_\xi T^n$:

$$g_{AB} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & R^2\delta_{\mu\nu} \end{pmatrix}.$$

- effective metric on $\mathbb{R}^n \times_\xi T^n$:

$$G^{AB} \sim \theta^{AA'}\theta^{BB'} g_{A'B'} = \xi^2 \begin{pmatrix} R^2\delta_{\mu\nu} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}$$

  because of split NC

note: time-like direction is compactified, unphysical ☹

way out: re-introduce some NC in non-compact direction

propagating $\mathbb{R}^3 \times_\xi S^1$

take fuzzy cylinder (U, X^2), and $\mathbb{R}^2_\theta$ with $[X^\mu, X^\nu] = i\theta^{\mu\nu}$, $\mu = 0,1$
define

$$\begin{pmatrix} X^\mu, \ \mu = 0,1 \\ X^2 \\ Z \end{pmatrix} = \begin{pmatrix} X^\mu, \ \mu = 0,1 \\ X^2 \\ U\, e^{ik_\mu X^\mu} \end{pmatrix}$$

Lemma $\Rightarrow$ solution $\Box X^a = 0$ provided

$$k \cdot k + \xi^2 = 0, \qquad k \cdot k := G^{\mu\nu}_{(2)} k_\mu k_\nu \ \text{(Minkowski)}$$

turns out: eff. metric on non-compact $\mathbb{R}^3$ has Minkowski signature ☺

... analogous for $\mathbb{R}^4 \times_\xi T^2$, $\mathbb{R}^4 \times_\xi T^2 \times T^2$, $\mathbb{R}^4 \times_\xi S^2 \times T^2$ etc.

Kaluza-Klein modes

- fuzzy cylinder (Euclidean):

$$\Box e^{ipX^3} U^n = \left(4R^2 \sin^2(p\xi/2) + n^2\xi^2\right) e^{ipX^3} U^n$$
$$\stackrel{p\xi \ll 1}{\sim} \left(R^2 p^2 + n^2\right) \xi^2 e^{ipX^3} U^n$$

$n \in \mathbb{Z}$ ... Kaluza-Klein modes; strict UV cutoff on $\mathbb{R}$!

$\Rightarrow$ looks like S^1 in UV, $\mathbb{R}$ in IR

- fuzzy cylinder with time-like X^0:

$$\Box e^{ipX^0} U^n = \left(4R^2 \sin^2(p\xi/2) - n^2\xi^2\right) e^{ipX^0} U^n.$$

time-like direction compactified

- propagating cylinder $\mathbb{R}^3 \times_\xi S^1$:

$$\Box(e^{ip_j X^j} U^n) = \left(4R^2 \sin^2\left(\frac{(p_2 - nk_2)\xi - k_\mu \theta^{\mu\nu} p_\nu}{2}\right) + G^{\mu\nu}_{(2)} p_\mu p_\nu + n^2\xi^2\right) e^{ip_j X^j} U^n$$

$\Rightarrow$ effective metric for lowest KK modes on prop. cylinder:

$$\Box e^{ip_j X^j} \approx (p \cdot p)\, e^{ip_j X^j}, \qquad p\xi \ll 1$$

where

$$p \cdot p \sim \ (p_\mu, p_2) \begin{pmatrix} R^{-2}\bar{G}^{\mu\nu}_{(2)} + \tilde{k}^\mu \tilde{k}^\nu & -\xi\tilde{k}^\mu \\ -\xi\tilde{k}^\nu & \xi^2 \end{pmatrix} \begin{pmatrix} p_\mu \\ p_2 \end{pmatrix}.$$

Minkowski signature

consistent with direct computation of $G \sim \theta\theta g$

further remarks, conclusion

- new solutions of IKKT model with $\mathbb{R}^4 \times_\xi \mathcal{K}$
- split NC solutions have zero action (not energy)

  cf. 3+1-dim space-times from Monte Carlo

  Kim Nishimura Tsuchiya arXiv:1108.1540
- moduli are arbitrary

  for cylinder solutions: cross-section is arbitrary, cf. left/right-movers (strings)
- no preferred embedding $\Rightarrow$ suitable for (emergent) gravity on branes
- good building blocks for intersecting branes, $\Rightarrow$ standard model

  cf. A. Chatzistavrakidis, H. Steinacker, G. Zoupanos. arXiv:1107.0265 2011