Scaling dimensions from the mirror TBA

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Outline

"Harmonic oscillator" of AdS/CFT:

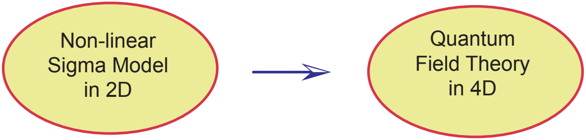
Maldacena '97

$\mathcal{N}=4$ $SU(N_c)$ SYM $\Longleftrightarrow$ IIB strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry

YM coupling g_{YM}	$\Longleftrightarrow$	string coupling $g_s = \frac{g_{YM}^2}{4\pi}$
't Hooft coupling $\lambda = g_{YM}^2 N_c$	$\Longleftrightarrow$	String tension $\frac{R^2}{2\pi\alpha'} \equiv g = \frac{\sqrt{\lambda}}{2\pi}$
SYM operators	$\Longleftrightarrow$	String states
Scaling dimension $\Delta(\lambda)$	$=$	String energy $E(g)$

Exact spectra of $\mathcal{N}=4$ SYM and strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$

From sigma models to four-dimensional QFT



Planar scaling dimensions $\Delta(\lambda)$ in Yang-Mills theory should be computable by string theory! Simultaneously, this would test the conjecture.

- Green-Schwarz superstring — Metsaev, Tseytlin '98

$$S = -\frac{g}{2}\int d\tau d\sigma\, \sqrt{-h}h^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N G_{MN}(X) + \text{fermions}$$

- L.c. string sigma model : $E - J = \int_{-J/2}^{J/2} \mathcal{H}_{l.c.}$ — Arutyunov, Frolov '04

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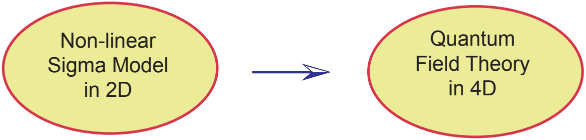

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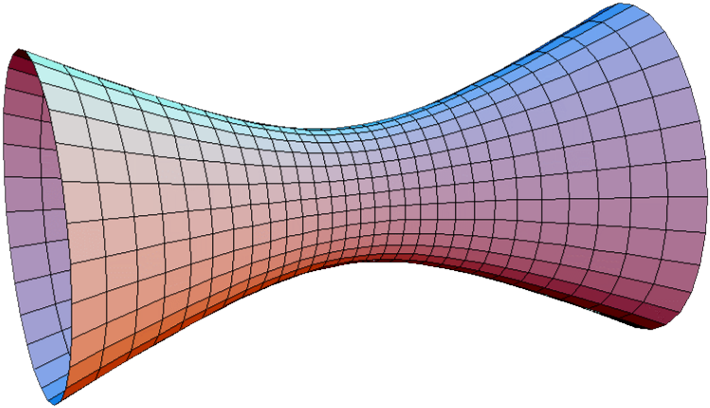
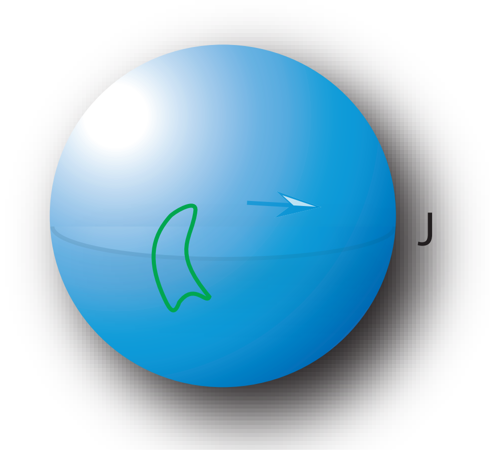
Anti-de Sitter space

Space of constant negative curvature

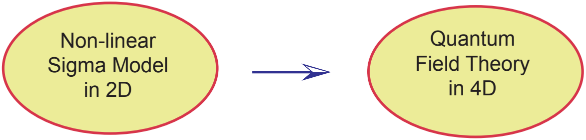


String energy E is a conserved Noether charge corresponding to the $\mathrm{SO}(2)$ subgroup of the conformal group $\mathrm{SO}(4,2)$



J is a conserved Noether charge corresponding to one of the Cartan generators of $\mathrm{SO}(6)$

From sigma models to four-dimensional QFT



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- L.c. string sigma model : $E - J = \int_{-J/2}^{J/2} H_{l.c.}$ — Arutyunov, Frolov '04
- To compute $E(g)$ and therefore $\Delta(g)$, one needs to solve the 2-dim quantum sigma model on a cylinder!
- String integrability is the key to the solution

N=4 super Yang-Mills theory

- Maximally supersymmetric gauge theory in 4dim:

$$A_\mu\,, \quad \Phi^i, \quad i=1,\ldots,6 \quad \text{and} \quad 4 \text{ Weyl fermions}$$

- Introduce $X=\Phi^1+i\Phi^2$, $Y=\Phi^3+i\Phi^4$, $Z=\Phi^5+i\Phi^6$, $D=D_+$
- The $\mathfrak{sl}(2)$-sector consists of linear combinations of operators

$$\mathrm{Tr}\left(\prod_{k=1}^{J} D^{n_k} Z\right)\,, \qquad \sum_{k=1}^{J} n_k = N\,, \quad n_k \geq 0$$

  J is the twist, and N is the spin.

- These operators are dual to N-particle states of l.c. string theory.
- Spin-2 operators

$$\mathrm{Tr}(Z^{J-1}D^2Z)\,, \quad \mathrm{Tr}(Z^{k-2}DZ\,Z^{J-k}DZ)$$

- If $N=2$ and $J=2$ only one operator is unprotected, and it is a susy descendent of the Konishi operator

$$\mathrm{Tr}\,\Phi_i^2$$

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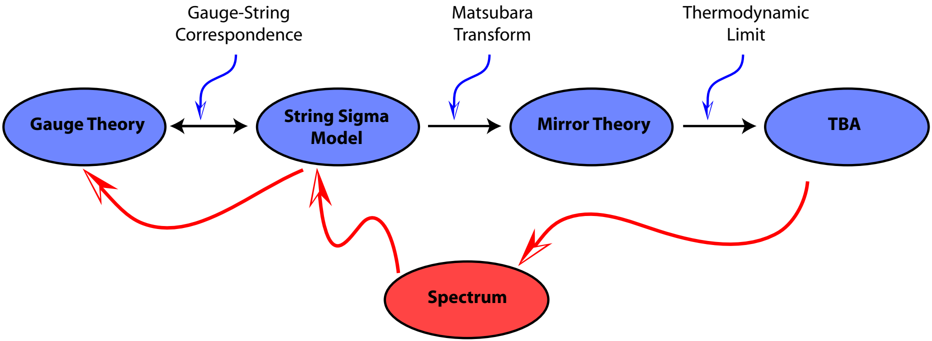
Summary of the TBA approach



String theory and N=4 SYM results

- Semi-classical strings — Berenstein, Maldacena,Nastase '02; Gubser, Klebanov, Polyakov '02; Frolov, Tseytlin '02, '03; Bena, Polchinski, Roiban '03; Kazakov, Marshakov, Minahan, Zarembo '04;
- L.c. strings in $AdS_5 \times S^5$ — Arutyunov, Frolov '04, '05; Frolov, Plefka, Zamaklar '06; Arutyunov, Frolov, Plefka, Zamaklar '06;
- Decompactification: $J \to \infty$ — Ambjorn, Janik, Kristjansen '05; Janik '06; Arutyunov, Frolov '06; Hofman, Maldacena '06;
- Symmetry algebra — Beisert '05, '06; Arutyunov, Frolov, Plefka, Zamaklar '06;
- Dispersion relations — Beisert, Dippel, Staudacher '04; Beisert '05; N.Dorey '06;
- S-matrix — Staudacher '04; Beisert '05; Arutyunov, Frolov, Zamaklar '06;
- Dressing factor and crossing eqs — Arutyunov, Frolov, Staudacher '04; Beisert, Tseytlin '05; Janik '06; Hernandez, Lopez '06; Arutyunov, Frolov '06; Beisert, Hernandez, Lopez '06; Beisert, Eden, Staudacher '06;
- Bethe ansatz — Minahan, Zarembo '02; Beisert, Dippel, Staudacher '04; Arutyunov, Frolov, Staudacher '04; Staudacher '04; Beisert, Staudacher '05; Beisert, Eden, Staudacher '06;

Comparison chart

Arutyunov, Frolov '07

	Strings	Mirrors
Dispersion relation	$\mathcal{E}_Q = \sqrt{Q^2 + 4g^2 \sin^2 \frac{p}{2}}$	$\widetilde{\mathcal{E}}_Q = 2\,\mathrm{arcsinh}\left(\frac{1}{2g}\sqrt{Q^2 + \widetilde{p}^2}\right)$
Momentum	$-\pi \le p < \pi$	$-\infty < \widetilde{p} < \infty$
Type of theory	Lattice model	Continuum model
Giant magnon	Soliton in $\mathrm{R} \times \mathrm{S}^5$	Soliton in AdS_5
Bound states	Symmetric irrep su(2) sector	Antisymmetric irrep sl(2) sector
Physical region	"Fish" (?)	"Leaf" (?)
S – matrix	$S(z_1, z_2)$	$S(z_1 + \frac{\omega_2}{2}, z_2 + \frac{\omega_2}{2})$
Bethe – Yang eqs	BS; $\mathrm{P} = 0$	extra $\sqrt{x^+/x^-}$
Dressing factor	$\sigma(1,2)^* \sigma(1,2) = 1$	$\sigma(1,2)^* \sigma(1,2) = \frac{x_1^+}{x_1^-}\frac{x_2^-}{x_2^+}$

Mirror TBA

- Ground state energy is related to the free energy of the mirror theory at temperature $T = 1/J$ Al. Zamolodchikov '90

$$E(J) = J\,\mathcal{F}(J)$$

- Mirror TBA for the ground state is a set of nonlinear integral equations on Y–functions. Its solution computes the free energy
- TBA eqs follow from the string hypothesis for the mirror model Takahashi '72 Arutyunov, Frolov '09(a)
- A Bethe string leads to a Y–function ($Q = 1, 2, \ldots$)

$$Y^{(-)}_{Q|w}\,,\ Y^{(-)}_{Q|vw}\,,\ Y^{(-)}_{+}\,,\ Y^{(-)}_{-}\,,\ Y_Q\,,\ Y^{(+)}_{-}\,,\ Y^{(+)}_{+}\,,\ Y^{(+)}_{Q|vw}\,,\ Y^{(+)}_{Q|w}$$

- Ground state energy

$$E - J = \underbrace{-\frac{1}{2\pi}\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty} d\tilde{p}\,\log(1+Y_Q)}_{\text{finite-size contribution}}$$

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Mirror TBA

- TBA eqs can be written in various forms
  - Canonical — Arutyunov, Frolov ’09(b); Bombardelli, Fioravanti, Tateo ’09
  - Simplified — Arutyunov, Frolov ’09(b), ’09(d)
  - Hybrid — Arutyunov, Frolov, Suzuki ’09
  - Quasi-local — Balog,Hegedus ’11
- TBA eqs for excited states via the contour deformation trick (inspired by P. Dorey, Tateo ’96) — Gromov, Kazakov, Kozak, Vieira ’09v3; Arutyunov, Frolov, Suzuki ’09
- or via the Y-system and jump discontinuities (following Bazhanov, Lukyanov, Zamolodchikov ’96) — Cavaglia, Fioravanti, Tateo ’10; Balog, Hegedus ’11

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Excited states TBA and CDT

Arutyunov, Frolov, Suzuki '09

Inspired by P. Dorey, Tateo '96

- Q-particles (sum over $\alpha = -, +$):

$$\log Y_Q = -L\,\widetilde{\mathcal{E}}_Q + \log(1+Y_M) \star_{C_M} \left(K^{MQ}_{\mathfrak{sl}(2)} + 2\, s \star K^{M-1,Q}_{vwx}\right)$$
$$+ \log(1+Y^{(\alpha)}_{1|vw}) \star_{C^{(\alpha)}_{1|vw}} s \,\hat{\star}\, K_{yQ} + \log(1+Y^{(\alpha)}_{Q-1|vw}) \star_{C^{(\alpha)}_{Q-1|vw}} s$$
$$- \log \frac{1-Y^{(\alpha)}_-}{1-Y^{(\alpha)}_+} \star_{C^{(\alpha)}_\pm} s \star K^{1Q}_{vwx} + \log\left(1-\frac{1}{Y^{(\alpha)}_-}\right) \star_{C^{(\alpha)}_-} K^{yQ}_- + \log\left(1-\frac{1}{Y^{(\alpha)}_+}\right) \star_{C^{(\alpha)}_+} K^{yQ}_+$$

- y-particles: $\log \frac{Y^{(\alpha)}_+}{Y^{(\alpha)}_-} = \log(1+Y_Q) \star_{C_Q} K_{Qy}\,,$

$$\log Y^{(\alpha)}_+ Y^{(\alpha)}_- = 2\log \frac{1+Y^{(\alpha)}_{1|vw}}{1+Y^{(\alpha)}_{1|w}} \star_{C^{(\alpha)}_{1|(v)w}} s - \log(1+Y_Q) \star_{C_Q} K_Q + 2\log(1+Y_Q) \star_{C_Q} K^{Q1}_{xv} \star s$$

- $M|vw$-strings:

$$\log Y^{(\alpha)}_{M|vw} = \log \frac{(1+Y^{(\alpha)}_{M-1|vw})(1+Y^{(\alpha)}_{M+1|vw})}{1+Y_{M+1}} \star_C s + \delta_{M1} \log \frac{1-Y^{(\alpha)}_-}{1-Y^{(\alpha)}_+} \star_{C^{(\alpha)}_\pm} s$$

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- Ground state energy

$$E = J - \underbrace{\frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_{C_Q} d\widetilde{p}\, \log(1+Y_Q)}_{\text{finite-size corr.}}$$

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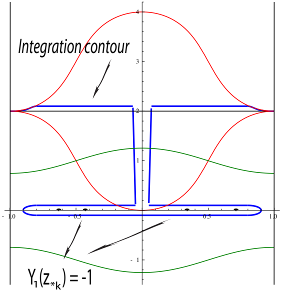

- **TBA equations for excited states differ from each other only by a choice of the integration contour $\Longrightarrow$ contour deformation trick**
- If $Y(z_*) = -1$ (or ∞) then taking the contour back to real mirror line produces driving term $-\log S(z_*, z)$ from $\log(1+Y)\star K$; $K(w,z) = \frac{1}{2\pi i}\frac{d}{dw}\log S(w,z)$

$$\log(1+Y)\star K \quad\Longrightarrow\quad \frac{1}{2\pi i}\oint_{z_*}\log(1+Y(w))\frac{d}{dw}\log S(w,z) =$$

$$= -\frac{1}{2\pi i}\oint_{z_*}\frac{d}{dw}\log(w-z_*)\,\log S(w,z) = \underbrace{-\log S(z_*,z)}_{\text{new driving term}}$$

Exact Bethe equations

The spectrum of excited states

$$E = J + \underbrace{\sum_{k=1}^{N} \mathcal{E}(p_k)}_{\text{Bethe-Yang}} \underbrace{- \frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} d\tilde{p} \log(1 + Y_Q)}_{\text{finite-size contribution}}$$

Momenta p_k (or rapidities u_k) are found from the *exact Bethe equations* (quantization cond.)

Bazhanov, Lukyanov, Zamolodchikov '96; P. Dorey, Tateo '96

$$Y_{1_*}(p_k) = -1$$

The EBE work fine for *real* p_k.
Do they need a modification for *complex* ones?

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Bajnok-Janik asymptotic solution (large J or small g and finite J)

- Generalized Lüscher formulae give the large J asymptotic solution Bajnok, Janik '08

$$Y_Q^o(v) = \Upsilon_Q(v)\, T_{Q,-1}(v|\vec{u})\, T_{Q,1}(v|\vec{u})$$

$$\Upsilon_Q(v) = e^{-J\widetilde{\mathcal{E}}_Q(v)} \prod_{i=1}^{N} S_{\mathfrak{sl}(2)}^{Q1_*}(v,u_i) \quad \leftarrow \quad \text{exp suppressed}$$

- $T_{Q,1}$ is an eigenvalue of a properly normalized $\mathfrak{su}(2|2)_C$ transfer matrix

$$\mathcal{T}_{Q,1}(v|\vec{u}) = \mathrm{str}_A\, S_{A1}^{Q,1}(v,u_1) S_{A2}^{Q,1}(v,u_2)\cdots S_{AN}^{Q,1}(v,u_N)\,,$$

conjectured by Beisert '06
and derived by Arutyunov, de Leeuw, Suzuki, Torrieli '09

- The BY equations (in the $\mathfrak{sl}(2)$ sector) follow from $\widetilde{\mathcal{E}}_{1_*}(u_k) = -i\,p_k$ and

$$T_{1_*,\pm1}(u_k|\vec{u}) = 1 \implies -1 = e^{i\,J\,p_k} \prod_{j=1}^{N} S_{\mathfrak{sl}(2)}^{1_*1_*}(u_k,u_j)$$

- All auxiliary Y^o-functions are fixed by Y_Q^o Arutyunov, Frolov '11
and (almost) agree with Gromov, Kazakov, Vieira '09(a)

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Known excited state TBA equations

Almost all states are from the $sl(2)$ sector

$$\mathrm{Tr}\left(\prod_{k=1}^{J} D^{n_k} Z\right), \qquad \sum_{k=1}^{J} n_k = N, \quad n_k \geq 0$$

J is the twist, and N is the spin or the number of particles.

- Canonical TBA eqs for two-particle Konishi-like states, $\lambda < \lambda_{\mathrm{cr}}$

  Gromov, Kazakov, Kozak, Vieira '09v3;
- Hybrid, simplified and canonical TBA eqs for arbitrary two-particle states

  Arutyunov, Frolov, Suzuki '09;
- Hybrid TBA eqs for twist-2 N-particle lightest state, $\lambda < \lambda_{\mathrm{cr}}$

  Balog, Hegedus '10;
- TBA eqs for a subsector of the $sl(2)$ sector, $\lambda < \lambda_{\mathrm{cr}}$

  Balog, Hegedus '11;
- TBA eqs for the two states not from the $sl(2)$ sector which are degenerate asymptotically, $\lambda < \lambda_{\mathrm{cr}}$

  Sfondrini, van Tongeren. '11;

Two-particle states in perturbation theory

- dual to spin-2 operators

$$\mathrm{Tr}(Z^{J-1}D^2Z)\,,\quad \mathrm{Tr}(Z^{k-2}DZ\,Z^{J-k}DZ)$$

- Log of the BY equation for two-particle states with $p_1+p_2=0$

$$ip(J+1)-\log\frac{1+\frac{1}{x^{+2}}}{1+\frac{1}{x^{-2}}}-2i\underbrace{\theta(p,-p)}_{\text{dressing phase}}=2\pi i\,n\,,\quad n>0$$

- At $g\to 0$ the momentum is

$$p^o_{J,n}=\frac{2\pi n}{J+1}\,,\quad n=1\,,\dots\,,\left[\frac{J+1}{2}\right]\,,$$

- The corresponding rapidity

$$u_{J,n}\to\frac{1}{g}u^o_{J,n}\,,\quad u^o_{J,n}=\cot\frac{\pi n}{J+1}$$

- At large g the integer n coincides with the string level
- For Konishi $J=2$ and $n=1$

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Types of states

At $g \sim 0$ the following classification of two-particle states in the $\mathfrak{sl}(2)$-sector takes place

Arutyunov, Frolov, Suzuki '09

Type of a state	Y-functions	Number of zeros
I	$Y_{1\|vw}$	2
II	$Y_{1\|vw}$, $Y_{2\|vw}$	2+2
III	$Y_{1\|vw}$, $Y_{2\|vw}$, $Y_{3\|vw}$	4+2+2
IV	$Y_{1\|vw}$, $Y_{2\|vw}$, $Y_{3\|vw}$, $Y_{4\|vw}$	4+4+2+2
⋮	⋮	⋮
$k \to \infty$	$Y_{1\|vw}$, $Y_{2\|vw}$, $\ldots$	4+4+ $\ldots$

Type of a state depends on how many Y_{vw}-functions have zeroes in the rescaled analyticity strip $|\mathrm{Im}(u)| < 1$

Evolution of asymptotic Y-functions

Arutyunov, Frolov, Suzuki '09

Initial cond. $\rightarrow$	$Y_{1\|vw}$, $Y_{2\|vw}$	2+2
	$Y_{1\|vw}$, $Y_{2\|vw}$, $Y_{3\|vw}$	4+2+2
$g\ \downarrow$	$Y_{1\|vw}$, $Y_{2\|vw}$, $Y_{3\|vw}$, $Y_{4\|vw}$	4+4+2+2
	$Y_{1\|vw}$, $Y_{2\|vw}$, $Y_{3\|vw}$, $Y_{4\|vw}$, $Y_{5\|vw}$	4+4+4+2+2
$\vdots$	$\vdots$	$\vdots$
	$Y_{1\|vw}$, $Y_{2\|vw}$, ...	4+4+ ...

The change of analytic properties of Y's in the analyticity strip changes the TBA equations. This leads to the issue of critical values of the coupling.

Strong coupling expansion

$$E_{(J,n)}(\lambda) = c_{-1}\sqrt[4]{n^2\lambda} + c_0 + \frac{c_1}{\sqrt[4]{n^2\lambda}} + \frac{c_2}{\sqrt{n^2\lambda}} + \frac{c_3}{(n^2\lambda)^{3/4}} + \frac{c_4}{n^2\lambda} + \frac{c_5}{(n^2\lambda)^{5/4}} + \cdots$$

- Expansion is in $1/\sqrt[4]{n^2\lambda}$, n is the string level of a 2-particle state
- $c_{-1} = 2$ from the flat space string spectrum — Gubser, Klebanov, Polyakov '98
  and BYE — Arutyunov, Frolov, Staudacher '04
- $c_0 = 0$ from BYE and free fermion model — Arutyunov, Frolov '05
- Other coefficients are functions of J and n
- $c_2 = 0$??? due to supersymmetry — Roiban, Tseytlin '09
- $c_{2k} = 0$!?!?!? — Gromov, Kazakov, Vieira '09(b)

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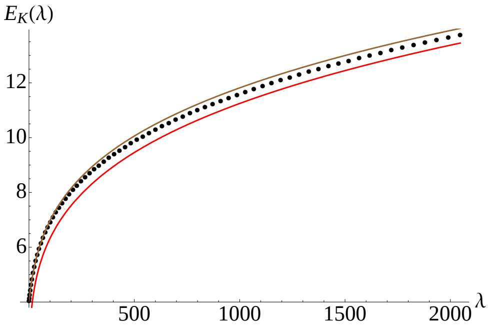
Konishi dimension from TBA

TBA eqs were solved numerically up to $g = 7.2$ ($\lambda \approx 2047$) Frolov '10



dots – numerics,
red – $2\sqrt[4]{\lambda}$,
brown –
$$E_{\rm asym} = 2 + 2\sqrt{1 + 4g^2 \sin^2 \frac{p_{\rm asym}}{2}}$$

- Up to $g = 4.1$ ($\lambda \approx 664$) agreement with Gromov, Kazakov, Vieira '09(b)
- Setting $c_0 = c_2 = c_4 = 0$, one gets $c_1 = 2$ from GKV numerics

$$\overline{E}_K^{GKV}(\lambda) = \sqrt[4]{\lambda}\left(2.0004 + \frac{1.988}{\sqrt{\lambda}} - \frac{2.60}{\lambda} + \frac{6.2}{\lambda^{3/2}}\right)$$

Large λ expansion from numerics

- In general an asymptotic series cannot be found reliably from numerical data

$$2\frac{1-e^{100-\sqrt{\lambda+1}}}{1+e^{100-\sqrt{\lambda+1}}}\sqrt[4]{\lambda+1} \;\rightarrow\; 2\sqrt[4]{\lambda}$$

From numerics one would conclude that it asymptotes to $-2\sqrt[4]{\lambda}$

- We have to assume that exponentially suppressed terms become very small already at the values of λ we are dealing with.
- If λ is not large enough then one needs to make an assumption about the structure of the large λ expansion, for example to decide if the series contains all possible terms or some of them vanish.
- Fitting numerical data one should decide how many terms to keep in an asymptotic series, and what fitting interval to use.
- A function can approach its asymptotic series monotonically or in oscillations, and it does not seem possible to single out one from numerics. In fact, using the standard least-square fitting procedure would always lead to an oscillating behavior of numerical data about a fitting function.

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Large λ expansion from numerics

Fitting the data for $g \in [1.4, 7.2]$, one gets

Frolov '10

- No condition on c_i

$$\overline{E}_K(\lambda) = 1.99\sqrt[4]{\lambda} + 0.21 - \frac{0.06}{\sqrt[4]{\lambda}} + \frac{10.43}{\sqrt{\lambda}} - \frac{31.78}{\lambda^{3/4}} + \frac{42.20}{\lambda} - \frac{17.56}{\lambda^{5/4}}$$

- $c_{-1} = 2$

$$\overline{E}_K(\lambda) = 2\sqrt[4]{\lambda} - 0.027 + \frac{2.59}{\sqrt[4]{\lambda}} - \frac{5.16}{\sqrt{\lambda}} + \frac{18.86}{\lambda^{3/4}} - \frac{44.17}{\lambda} + \frac{42.89}{\lambda^{5/4}}$$

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$$\overline{E}_K(\lambda) = 2\sqrt[4]{\lambda} + \frac{1.99}{\sqrt[4]{\lambda}} + \frac{0.15}{\sqrt{\lambda}} - \frac{4.01}{\lambda^{3/4}} + \frac{4.15}{\lambda} + \frac{2.79}{\lambda^{5/4}}$$

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$$\overline{E}_K(\lambda) = 2.00005\sqrt[4]{\lambda} + \frac{1.99237}{\sqrt[4]{\lambda}} - \frac{2.72847}{\lambda^{3/4}} + \frac{7.45145}{\lambda^{5/4}}$$

$c_1 = 2$ was confirmed ??? by

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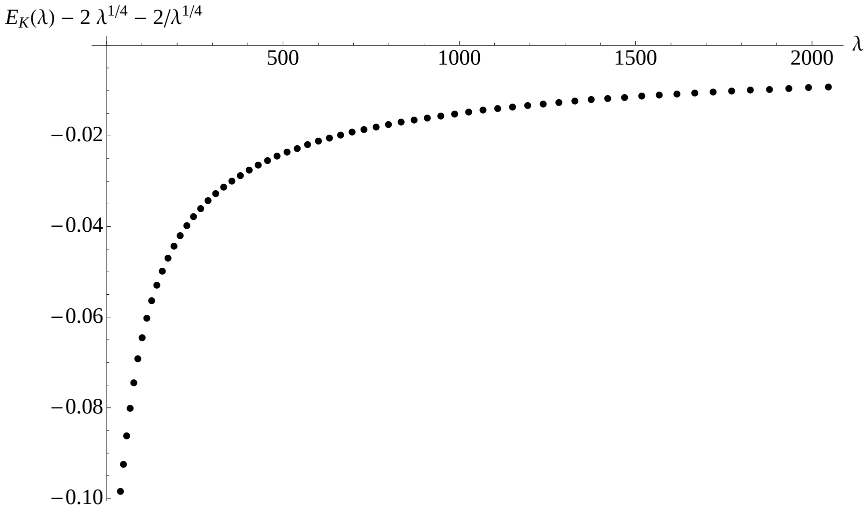
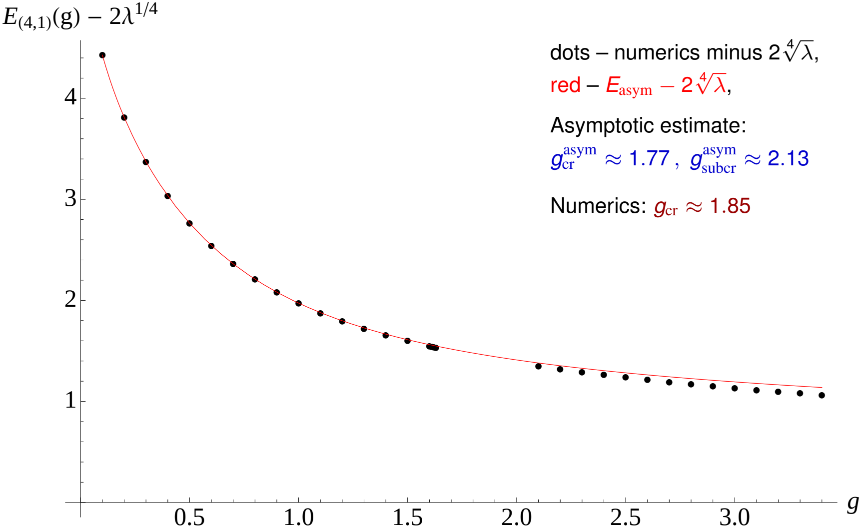


J=4, n=1 state

Frolov '??



dots – numerics minus $2\sqrt[4]{\lambda}$,

red – $E_{\rm asym} - 2\sqrt[4]{\lambda}$,

Asymptotic estimate:

$g_{\rm cr}^{\rm asym} \approx 1.77\,,\; g_{\rm subcr}^{\rm asym} \approx 2.13$

Numerics: $g_{\rm cr} \approx 1.85$

J=4, n=1 state

Frolov '??

- Fitting the data for $g \in [1.4, 3.4]$, one gets
  - $c_0 = 0,\ c_2 = 0,\ c_4 = 0$

$$\overline{E}_{(4,1)}(\lambda) = 2.0022\sqrt[4]{\lambda} + \frac{4.91748}{\sqrt[4]{\lambda}} - \frac{1.24309}{\lambda^{3/4}} + \frac{9.64361}{\lambda^{5/4}}$$

  - $c_{-1} = 2,\ c_0 = 0,\ c_2 = 0,\ c_4 = 0$

$$\overline{E}_{(4,1)}(\lambda) = 2\sqrt[4]{\lambda} + \frac{5.00948}{\sqrt[4]{\lambda}} - \frac{2.47288}{\lambda^{3/4}} + \frac{14.9027}{\lambda^{5/4}}$$

  - Is $c_1 = 5$???

J=4, n=1 state

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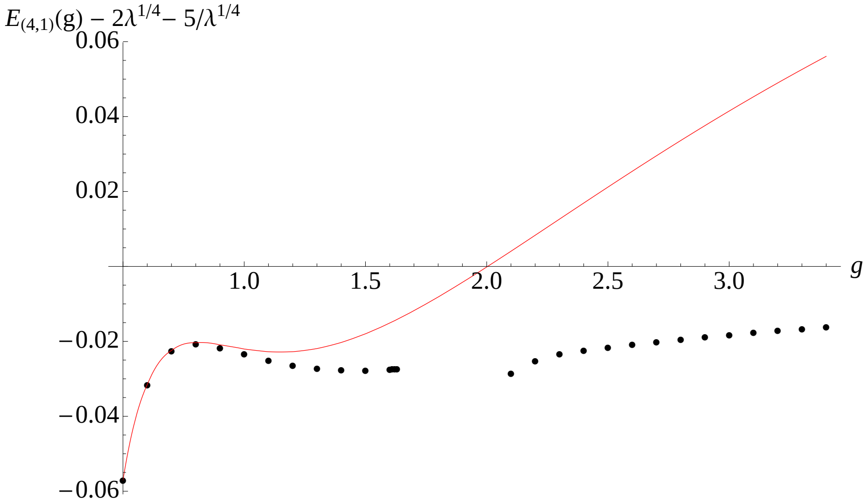

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- $c_1^{\text{asym}}(J,n) = \frac{J^2}{4} + \frac{1}{2}$ from BYE and free fermions — Arutyunov, Frolov '05
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  - $c_2(2,1) \approx 0$, $c_1(2,1) \approx 2$, $\Rightarrow c_1(J,1) = \frac{J^2}{4} + 1$
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The mirror TBA: from string hypothesis to the exact spectrum

1. Ground state TBA eqs follow from the string hypothesis
2. TBA eqs for excited states are determined by the ground state ones via the CDT
3. TBA eqs are written for the whole superconformal multiplet. $PSU(2,2|4)$ invariance is built-in — Arutyunov, Frolov '11
4. TBA eqs for states composed of particles with *real momenta* can be obtained
5. States containing particles with *complex momenta* (e.g. bound states) require special treatment — Arutyunov, Frolov, van Tongeren (to appear)
6. The form of the equations depends on λ. For infinitely many states *there are critical values of* λ, crossing which the TBA eqs must be modified.
7. For a given operator, is the number of critical values *infinite* or *finite* (or even 0)?
8. We found *no evidence* that up to the overall factor $2\sqrt[4]{n^2\lambda}$ the large λ expansion is in powers of $1/\sqrt{\lambda}$. Is the expansion in powers of $1/\sqrt[4]{\lambda}$?
9. The numerics we performed is not sufficient to give definite answers to (m)any questions. Analytical methods are necessary. NLIE ???

Assumptions

1. Quantum integrability of l.c. string theory and its mirror
2. Symmetry algebra of l.c. string theory and its mirror in the decompactification limit — Beisert '05; Arutyunov, Frolov, Plefka, Zamaklar '06
3. BES dressing factor — Beisert, Eden, Staudacher '06
4. Bajnok-Janik asymptotic solution — Bajnok, Janik '08
5. String hypothesis for the mirror model — Arutyunov, Frolov '09
6. Universality of the contour deformation trick
7. Universality of the exact Bethe equations $Y_{1_*}(p_k) = -1$