Spectral action, scale anomaly

and the Higgs-Dilaton potential

Fedele Lizzi

Università di Napoli Federico II

Work in collaboration with A.A. Andrianov (St. Petersburg) and M.A. Kurkov (Napoli)

JHEP 1005:057,2010 (arXiv:1001.2036), arXiv:1103.0478 (proceedings of Corfù 2010)

Corfù 2011

I will report on some work which I have been doing for some time with my collaborators, and which is a continuation of the work I presented last year in this conference. Nevertheless this talk will be selfcontained

I will make some considerations about relevant physical aspects of the general framework of spectral geometry

The framework in which I will be presenting the work is that of the spectral triples, although probably the whole work could be recast in a way which does not make any mention of noncommutative geometry.

The starting point of Connes' approach to is that geometry and its (noncommutative) generalizations are described by the spectral data of three basic ingredients:

- An algebra $\boxed{\mathcal{A}}$ which describes the topology of spacetime.

- An Hilbert space $\boxed{\mathcal{H}}$ on which the algebra act as operators, and which also describes the matter fields of the theory.

- A (generalized) Dirac Operator $\boxed{D}$ which carries all the information of the metric structure of the space, as well as other crucial information about the fermions.

While the formalism is geared towards the construction of genuine noncommutative spaces, such as the noncommutative torus, spectacular results are obtained considering almost commutative geometries

Those are simply the product of ordinary (commutative) spacetimes, times an internal finite dimensional noncommutative structure, represented by a matrix algebra

Choosing the internal space (the choice is almost unique) and writing an action (the spectral action), based on the spectrum of the Dirac operator, one reproduces the standard model coupled to gravity

And along the way the mass of the Higgs is predicted at a mass of $\boxed{\sim 170\text{GeV}}$, a value experimentally disfavoured, but certainly not far from the actual value. A surprising result for a totally geometrically theory

This indicates that we may be on the right track

For a realistic theory another crucial ingredient is the chirality $\gamma = \gamma^\dagger$, with $\gamma^2 = 1$. A generalization of γ_5, which cause the splitting V

$$\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R$$

Also important is another operator, J, representing charge conjugation, which however plays no crucial role in this seminar

The central idea behind spectral geometry is that these ingredients are sufficient to describe not only a geometry, but also the behaviour of the fields defined on them, and their couplings to the geometry of spacetime (gravity). Treating on an equal footing the *external geometry (spacetime), with the inner one, gauge degrees of freedom*

The main success of this view is the spectral action, where the algebra is the product of the algebra of functions on spacetime, the Hilbert space is that of fermion matter fields, and the Dirac operator contains all information on the metric of spacetime, as well as the mass, couplings and mixings of fermions.

The spectral action contains two part, one is the bosonic action, to be read in a Wilsonian renormalization group sense:

$$\boxed{S_B = \text{Tr}\,\chi\left(\frac{D_A}{\Lambda}\right)}$$

where $\boxed{D_A = D + A}$ is a fluctuation of the Dirac operator, $\boxed{\chi}$ is the characteristic function of the interval $\boxed{[0,1]}$, or some smoothened version of it, and $\boxed{\Lambda}$ is a cutoff

Then there is a "standard" fermionic action $\boxed{\langle\Psi|\, D_A \,|\Psi\rangle}$

The bosonic action is finite by construction, the fermionic part needs to be regularized

Consider the fermionic action alone, a theory in which fermions move in a fixed background

The classical action is invariant for Weyl rescaling

$$g^{\mu\nu} \longrightarrow \mathrm{e}^{2\phi} g^{\mu\nu}$$

$$\psi \longrightarrow \mathrm{e}^{-\frac{3}{2}\phi} \psi$$

$$D \longrightarrow \mathrm{e}^{-\frac{1}{2}\phi} D \mathrm{e}^{-\frac{1}{2}\phi}$$

This is a symmetry of the classical action, not of the quantum partition function

$$Z(D) = \int [\mathrm{d}\psi][\mathrm{d}\bar{\psi}] e^{-S_\psi}$$

and therefore there is an anomaly because a classical symmetry is not preserved at the quantum level by a regularized measure.

We can therefore either "correct" the action to have an invariant theory, or consider a theory in which the symmetry is explicitly broken by a physical scale

In fact we need a scale to regularize the theory. The expression of the partition function can be formally written as a determinant:

$$Z(D,\mu) = \int [\mathrm{d}\psi][\mathrm{d}\bar{\psi}]e^{-S_\psi} = \det\left(\frac{D}{\mu}\right)$$

The determinant is still infinite and we need to introduce a cutoff

The regularization can be done in several ways. In the spirit of noncommutative geometry the most natural one is a truncation of the spectrum of the Dirac operator. This was considered long ago by Andrianov, Bonora, Fujikawa, Novozhilov, Vassilevich

The cutoff is enforced considering only the first N eigenvalues of D

Consider the projector $P_N = \sum_{n=0}^{N} |\lambda_n\rangle \langle\lambda_n|$ with λ_n and $|\lambda_n\rangle$ the eigenvalues and eigenvectors of D

N is a function of the cutoff defined as $N = \max n \text{ such that } \lambda_n \leq \Lambda$

We effectively use the N^{th} eigenvalue as cutoff

The choice of a sharp cutoff could be changed in favour of a cutoff function, similar to the choice of χ

Define the regularized partition function

$$Z(D,\mu) = \prod_{n=1}^{N} \frac{\lambda_n}{\mu} = \det\left(\mathbb{1} - P_N + P_N \frac{D}{\mu} P_N\right)$$

$$= \det\left(\mathbb{1} - P_N + P_N \frac{D}{\Lambda} P_N\right) \det\left(\mathbb{1} - P_N + \frac{\Lambda}{\mu} P_N\right)$$

$$= Z_\Lambda(D,\Lambda) \det\left(\mathbb{1} - P_N + \frac{\Lambda}{\mu} P_N\right)$$

The cutoff Λ can be given the physical meaning of the energy in which the effective theory has a phase transition, or at any rate an energy in which the symmetries of the theory are fundamentally different (unification scale)

The quantity μ in principle different and is a normalization scale, the one which changes with the renormalization flow

Under the change $\boxed{\mu \longrightarrow \gamma\mu}$ the partition function changes

$$\boxed{Z(D,\mu) \to Z(D,\mu)e^{\frac{1}{\gamma}\,\mathrm{tr}\,P_N}}$$

On the other side

$$\boxed{\mathrm{tr}\,P_N = N = \mathrm{tr}\,\chi\left(\frac{D}{\Lambda}\right) = S_B(\Lambda, D)}$$

for the choice of $\boxed{\chi}$ the characteristic function on the interval, a consequence of our sharp cutoff on the eigenvalues.

We found the spectral action.

We could have started without it and the renormalization flow would have provided it for free.

Let us now consider the Dirac operator for the standard model, in its barest essentiality (for our purposes). In the left-right splitting of $\mathcal{H}$, the operator D it is a 2×2 matrix

$$D = \begin{pmatrix} i\gamma^\mu D_\mu + \mathbb{A} & \gamma_5 S \\ \gamma_5 S^\dagger & i\gamma^\mu D_\mu + \mathbb{A} \end{pmatrix}$$

where

$D_\mu = \partial_\mu + \omega_\mu$, ω_μ the spin connection.

$\mathbb{A}$ contains all gauge fields

S contains the Higgs field, Yukawa couplings, mixings. . .

Technically the bosonic spectral action is a sum of residues and can be expanded in a power series in terms of Λ^{-1} as

$$S_B = \sum_n f_n \, a_n(D^2/\Lambda^2)$$

where the f_n are the momenta of χ

$$\begin{aligned} f_0 &= \int_0^\infty \mathrm{d}x \, x\chi(x) \\ f_2 &= \int_0^\infty \mathrm{d}x \, \chi(x) \\ f_{2n+4} &= (-1)^n \partial_x^n \chi(x) \Big|_{x=0} \quad n \geq 0 \end{aligned}$$

the a_n are the Seeley-de Witt coefficients which vanish for n odd. For D^2 of the form

$$D^2 = -(g^{\mu\nu}\partial_\mu\partial_\nu \mathbb{1} + \alpha^\mu \partial_\mu + \beta)$$

defining (in term of a generalized spin connection containing also the gauge fields)

$$\begin{aligned}
\omega_\mu &= \frac{1}{2} g_{\mu\nu} \left(\alpha^\nu + g^{\sigma\rho} \Gamma^\nu_{\sigma\rho} \mathbb{1} \right) \\
\Omega_{\mu\nu} &= \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu] \\
E &= \beta - g^{\mu\nu} \left(\partial_\mu \omega_\nu + \omega_\mu \omega_\nu - \Gamma^\rho_{\mu\nu} \omega_\rho \right)
\end{aligned}$$

then

$$\begin{aligned}
a_0 &= \frac{\Lambda^4}{16\pi^2} \int \mathrm{d}x^4 \sqrt{g}\, \mathrm{tr}\, \mathbb{1}_F \\
a_2 &= \frac{\Lambda^2}{16\pi^2} \int \mathrm{d}x^4 \sqrt{g}\, \mathrm{tr} \left(-\frac{R}{6} + E \right) \\
a_4 &= \frac{1}{16\pi^2} \frac{1}{360} \int \mathrm{d}x^4 \sqrt{g}\, \mathrm{tr} \, (-12 \nabla^\mu \nabla_\mu R + 5R^2 - 2R_{\mu\nu} R^{\mu\nu} \\
&\quad +2R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} - 60RE + 180E^2 + 60 \nabla^\mu \nabla_\mu E + 30 \Omega_{\mu\nu} \Omega^{\mu\nu})
\end{aligned}$$

tr is the trace over the inner indices of the finite algebra $\mathcal{A}_F$ and in Ω and E are contained the gauge degrees of freedom including the gauge stress energy tensors and the Higgs, which is given by the inner fluctuations of D

We can split the partition function in the product of a term invariant for Weyl transformations, and another not invariant, which will depend on the field ϕ, the dilaton

$$Z = Z_{inv} Z_{not}$$

The terms in Z_{not} exist due to the Weyl anomaly and we can calculate them. Using $D_\phi = \mathrm{e}^{-\frac{1}{2}\phi} D \mathrm{e}^{-\frac{1}{2}\phi}$ consider the identity

$$Z(D) = \left(\int [d\phi] \frac{1}{Z(D_\phi)} \right)^{-1} \int [d\phi] \frac{Z(D)}{Z(D_\phi)}$$

Since the first term is invariant by construction, $Z_{inv} = \left(\int [d\phi] \frac{1}{Z(D_\phi)} \right)^{-1}$, the second is the not invariant one

$$Z_{not}(D) = \int [d\phi] e^{-S_{not}} = \int [d\phi] \frac{Z(D)}{Z(D_\phi)}$$

hence

$$S_{not} = \ln \frac{Z(D_\phi)}{Z(D)}$$

The calculation of S_{not} can be done easily for ϕ constant and the result is

$$S_{not} = \int_0^\phi \mathrm{d}t' \left(1 - \Lambda^2 \log \frac{\Lambda^2}{\mu^2}\partial_{\Lambda^2}\right) \mathrm{Tr}\, \Theta \left(1 - \frac{(e^{-\frac{t'}{2}} D e^{-\frac{t'}{2}})^2}{\Lambda^2}\right)$$

$$= \int_0^\phi \mathrm{d}t' \left(1 - \Lambda^2 \log \frac{\Lambda^2}{\mu^2}\partial_{\Lambda^2}\right) S_B(\Lambda,\ (e^{-\frac{t'}{2}} D e^{-\frac{t'}{2}})^2)$$

This is a slight modification of the spectral action

Let me stress the fact that we used very few ingredients and the analysis is quite independent on the details. We have a Higgs field and a dilaton. We can therefore ask ourselves if we can say something about the effective potential involving these two fields, and its possible role in the early universe

Therefore we make the brutal approximations of neglecting all other fields and the derivative of the Higgs, and retain in the heath kernel expansion only the terms involving the Higgs field H and the dilaton ϕ

The behaviour of D under Weyl rescaling gives the transformation of H under such transformation. Only the H^4 term in the effective potential is invariant, and it can be multiplied by a constant quantity (ϕ_0). This gives, in this approximation, the invariant part of the effective potential

The other terms of the effective potential can be calculated using the heath kernel, they have been calculated fore the general case and it is sufficient to set to zero everything except $\boxed{H}$ and $\boxed{\phi}$ and their powers (but not their derivatives)

The effective potential, sum of the invariant and not invariant part has the form has the form

$$\boxed{V = V_0 + a(e^{2\phi} - 1) + bH^2(e^{2\phi} - 1) - cH^4(\phi + \phi_0) + EH^2}$$

The coefficients are in principle calculable at one loop, and are functions of the the parameters Λ and μ and there is another (integration) constant ϕ_0, in principle also calculable.

with a shift $\phi \rightarrow \phi - \phi_0$ and a redefinition if the constants the potential can be written as

$$V = V_0 + Ae^{4\phi} + BH^2e^{2\phi} - CH^4 + EH^2$$

Again the constants depend on Λ and μ, some relations among them are fixed

The full calculation of all these constants is possible, and is partially under way. Nevertheless we can immediately learn something just by imposing

- The existence of a local minimum

- The existence of an unbroken phase from which the potential may roll down to the broken phase

A choice of signs is possible: $\boxed{A > 0, B > 0, C > 0, E < 0}$

The choice of signs, and the relation among the constant, imply:

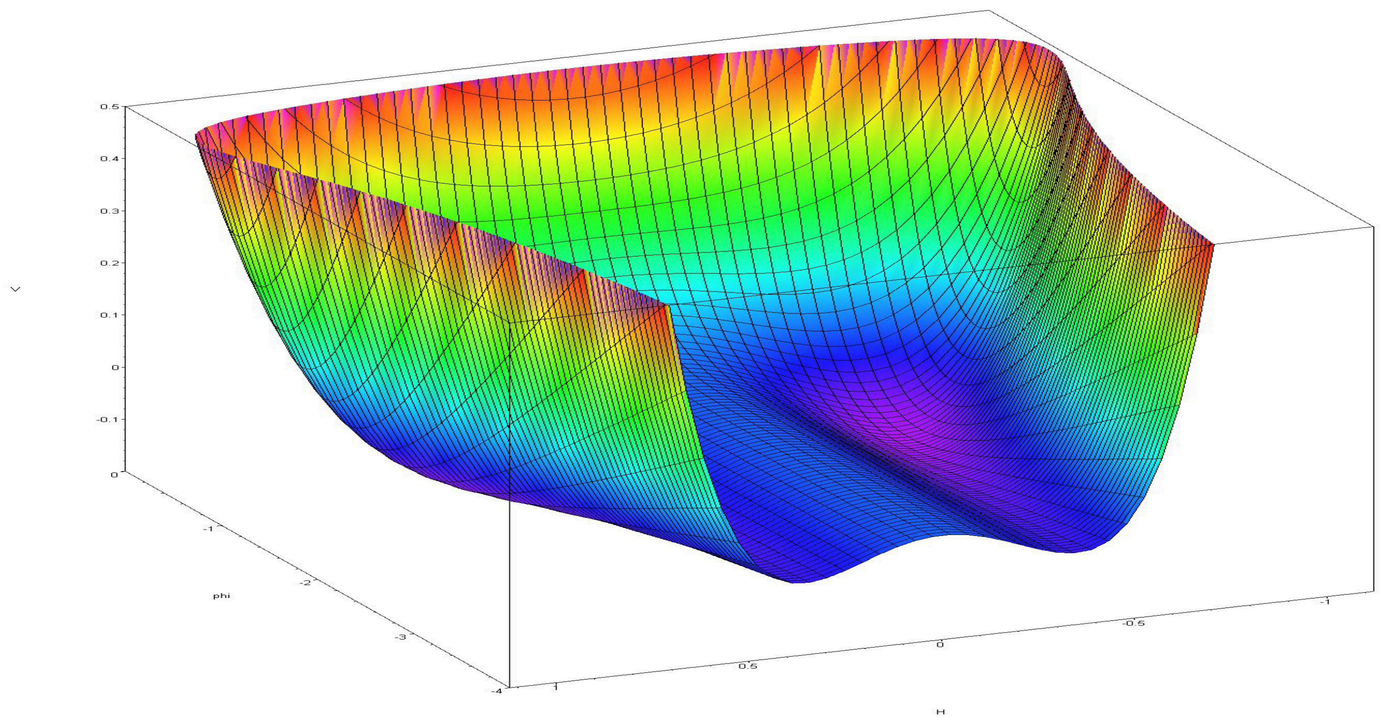
$$A > 0 \Rightarrow 1 - 2\ln\frac{\Lambda^2}{\mu^2} > 0 \qquad B > 0 \Rightarrow 1 - \ln\frac{\Lambda^2}{\mu^2} < 0 \qquad B + E > 0$$

which in turn implies that $\phi_0 < 0$ and

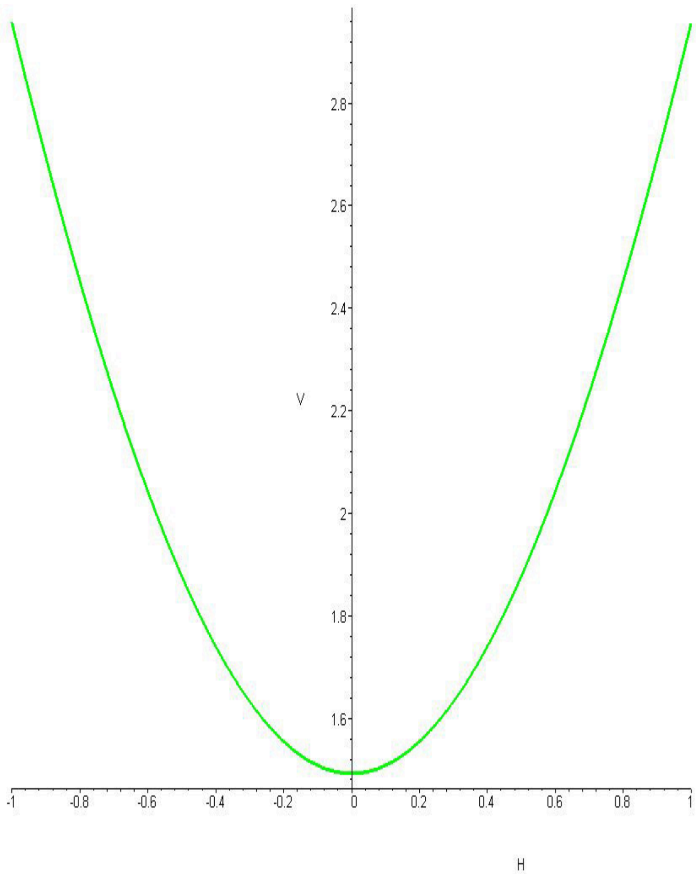
$$e^{\frac{1}{2}}\mu = 1.64\mu > \Lambda > e^{\frac{1}{4}}\mu = 1.28\mu$$

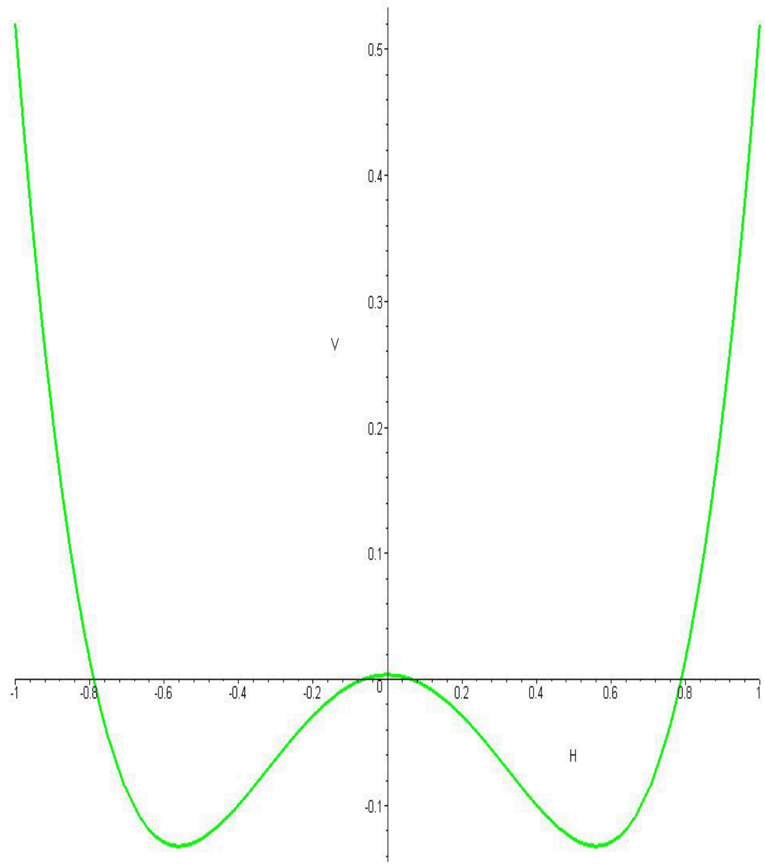
Hence Λ and μ must be of the same order, but not equal, at least within the scopes of our (1-loop) approximation

We can now plot the effective Higgs-Dilaton potential for a reasonable choice of parameters:



We see that for different values of $\boxed{\phi}$, the potential $\boxed{V(H)}$ has a transition from a symmetric to a broken phase.





Conclusions

- The bosonic spectral action emerges from the fermionic one and Weyl anomaly

- The renormalizion flow (scaling of $\boxed{\mu}$) plays a central role

- The effective Higgs-dilaton potential also emerges with desirable features: broken and symmetrical phases, roll down