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Gauge theory on twisted κ-Minkowski space-time

Marija Dimitrijević

University of Belgrade, Faculty of Physics,
Belgrade, Serbia

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M. Dimitrijević, L. Jonke and L. Möller, *JHEP* **0509** (2005) 068.

Overview

Introduction

κ-Minkowski space-time is defined by

$$[\hat{x}^0, \hat{x}^j] = ia\hat{x}^j, \quad [\hat{x}^i, \hat{x}^j] = 0,$$

with $a = 1/\kappa$ and $i, j = 1, 2, 3$.

-a dimensionful deformation of the global Poincaré group, the κ-Poincaré group [Lukierski, Nowicki, Ruegg, '92].

-an arena for formulating new physical concepts: Double Special Relativity [Amelino-Camelia '02], The principle of relative locality [Amelino-Camelia, Freidel, Kowalski-Glikman, Smolin, '11]; potentially interesting phenomenology.

$\star$-product approach [Dimitrijević, Meyer, Möller Wess '04; Dimitrijević, Jonke, Möller '05] has problems with: non-unique derivatives, diferential calculus, non-cyclic integral. $\Rightarrow$ Difficult to do field theory...

Suggestion: apply the twist formalism!

Reminder: twist formalism

Consider first a deformation (twist) of a classical symmetry algebra g (Lorentz, SUSY, gauge,...). Then deform the space-time itself.

A twist $\mathcal{F}$ (introduced by Drinfel'd in 1983-1985) is:
-an element of $Ug \otimes Ug$
-invertible
-fulfills the cocycle condition (ensures the associativity of the $\star$-product)

$$\mathcal{F} \otimes 1(\Delta \otimes id)\mathcal{F} = 1 \otimes \mathcal{F}(id \otimes \Delta)\mathcal{F}.$$

-additionally: $\mathcal{F} = 1 \otimes 1 + \mathcal{O}(h)$; h-deformation parameter.

Notation: $\mathcal{F} = \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha$ and $\mathcal{F}^{-1} = \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_\alpha$.

- $\mathcal{F}$ applied to $\mathcal{A}_x$ (algebra of smooth functions on $\mathcal{M}$): $\mathcal{A}_x^\star$

  pointwise multiplication: $\mu(f \otimes g) = f \cdot g$

  $\Downarrow$

  $\star$-multiplication: $\mu_\star(f \otimes g) \equiv \mu \circ \mathcal{F}^{-1}(f \otimes g) = f \star g$

- $\mathcal{F}$ applied to Ω (exterior algebra of forms): $\Omega^\star$

  wedge product: $\omega_1 \wedge \omega_2 = \omega_1 \otimes \omega_2 - \omega_2 \otimes \omega_1$

  $\Downarrow$

  $\star$-wedge product: $\omega_1 \wedge_\star \omega_2 = \wedge \circ \mathcal{F}^{-1}(\omega_1 \otimes \omega_2)$

- Differential calculus is classical: $\mathrm{d} : \mathcal{A}_x^\star \to \Omega^\star$.

$$\mathrm{d}^2 = 0, \quad \mathrm{d}(f \star g) = \mathrm{d}f \star g + f \star \mathrm{d}g,$$
$$\mathrm{d}f = (\partial_\mu f)\mathrm{d}x^\mu = (\partial_\mu^\star f) \star \mathrm{d}x^\mu.$$

- Integral of a maximal form is graded cyclic:

$$\int \omega_1 \wedge_\star \omega_2 = (-1)^{d_1 d_2} \int \omega_2 \wedge_\star \omega_1.$$

Kappa-Minkowski via twist

Consider twisting the global Poincaré symmetry $iso(1,3)$ with

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{ab}X_a \otimes X_b} = e^{-\frac{ia}{2}(\partial_0 \otimes x^j\partial_j - x^j\partial_j \otimes \partial_0)},$$

with $X_1 = \partial_0$, $X_2 = x^j\partial_j$, $[X_1, X_2] = 0$ and $\theta^{ab} = a\epsilon^{ab}$.

Consequences:

-the vector field X_2 not in universal enveloping algebra of Poincaré algebra, we enlarge it to get twisted $igl(1,3)$ [Borowiec, Pachol, '09].

-$\star$-product of functions

$$\begin{aligned} f \star g &= \mu\{\mathcal{F}^{-1} f \otimes g\} \\ &= f \cdot g + \frac{ia}{2} x^j \big((\partial_0 f)\partial_j g - (\partial_j f)\partial_0 g\big) + \mathcal{O}(a^2) \\ &= f \cdot g + + \frac{i}{2} C^{\rho\sigma}_{\lambda} x^{\lambda} (\partial_\rho f) \cdot (\partial_\sigma g) + \mathcal{O}(a^2), \end{aligned}$$

with $C^{\rho\sigma}_{\lambda} = a(\delta^{\rho}_{0}\delta^{\sigma}_{\lambda} - \delta^{\sigma}_{0}\delta^{\rho}_{\lambda})$.

Especially: $[x^0 \stackrel{\star}{,} x^j] = iax^j$ and $[x^i \stackrel{\star}{,} x^j] = 0$.

-differential calculus

$$\begin{aligned}
&\mathrm{d}f = (\partial_\mu^\star) \star \mathrm{d}x^\mu, \quad \partial_0^\star = \partial_0, \quad \partial_j^\star = e^{-\frac{i}{2}a\partial_0}\partial_j, \\
&f \star \mathrm{d}x^0 = \mathrm{d}x^0 \star f, \quad f \star \mathrm{d}x^j = \mathrm{d}x^j \star e^{ia\partial_0} f, \\
&\mathrm{d}x^\mu \wedge_\star \mathrm{d}x^\nu = \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \quad \mathrm{d}^4 x = \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^3.
\end{aligned}$$

-integral:

$$\int \omega_1 \wedge_\star \omega_2 = (-1)^{d_1 d_2} \int \omega_2 \wedge_\star \omega_1,$$

with $d_1 + d_2 = 4$.

$U(1)$ gauge theory coupled with matter: 1st approach

The NC matter field ψ transforms under infinitesimal NC gauge transformation as

$$\delta^\star \psi = i\Lambda \star \psi,$$

with the NC gauge parameter Λ. The NC covariant derivative $D\psi$ is defined by:

$$D\psi \ = \ d\psi - iA \star \psi \ = \ D^\star_\mu \psi \star \mathrm{d}x^\mu$$
$$D^\star_0 = \partial^\star_0 \psi - iA_0 \star \psi, \quad D^\star_j = \partial^\star_j \psi - iA_j \star e^{-ia\partial_0} \psi$$

where the NC connection is $A = A_\mu \star \mathrm{d}x^\mu$. From

$$\delta^\star D\psi = i\Lambda \star D\psi,$$

it follows

$$\delta^\star A \ = \ \mathrm{d}\Lambda + i[\Lambda \stackrel{\star}{,} A], \quad \delta^\star A_0 = \partial_0 \Lambda + i[\Lambda \stackrel{\star}{,} A_0]$$
$$\delta^\star A_j \ = \ \partial^\star_j \Lambda + i\Lambda \star A_j - iA_j \star e^{-ia\partial_0} \Lambda.$$

The NC field-strength tensor is a two-form given by

$$F = \frac{1}{2} F_{\mu\nu} \star \mathrm{d}x^{\mu} \wedge_{\star} \mathrm{d}x^{\nu} = \mathrm{d}A - iA \wedge_{\star} A$$

or in components

$$\begin{aligned} F_{0j} &= \partial_0^{\star} A_j - \partial_j^{\star} A_0 - iA_0 \star A_j + iA_j \star e^{-ia\partial_0} A_0 \\ F_{ij} &= \partial_i^{\star} A_j - \partial_j^{\star} A_i - iA_i \star e^{-ia\partial_0} A_j + iA_j \star e^{-ia\partial_0} A_i \end{aligned}$$

One can check that field-strength tensor transforms covariantly:

$$\delta^{\star} F = i[\Lambda \stackrel{\star}{,} F].$$

Next step: construction of the action.
The NC action for matter fields is a straightforward generalization of the commutative action.

The NC gauge field action should look like

$$S \propto \int F \wedge_\star (*F)$$

where $*F$ is the noncommutative Hodge dual. The obvious choice $*F = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \star \mathrm{d}x^\mu \wedge_\star \mathrm{d}x^\nu$ does not lead to a gauge invariant action!

A way out: assume that $*F$ has the form

$$*F := \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}X^{\alpha\beta} \star \mathrm{d}x^\mu \wedge_\star \mathrm{d}x^\nu,$$

where $X^{\alpha\beta}$ components are determined demanding

$$\delta^\star(*F) = i[\Lambda \stackrel{\star}{,} *F]$$

Up to first order we obtain

$$X^{0j} = F^{0j} - aA_0 \star F^{0j}, \quad X^{jk} = F^{jk} + aA_0 \star F^{jk}.$$

Action

The NC action for gauge fields is

$$\begin{aligned} S_g &= \int F \wedge_\star (*F) \\ &= -\frac{1}{4}\int \left\{2F_{0j} \star e^{-ia\partial_0} X^{0j} + F_{ij} \star e^{-2ia\partial_0} X^{ij}\right\} \star \mathrm{d}^4 x. \end{aligned}$$

For fermions we obtain

$$S_m \propto \int \left((\overline{D\psi})_B \star \psi_A - \bar{\psi}_B \star (D\psi)_A\right) \wedge_\star (V \wedge_\star V \wedge_\star V\gamma_5)_{BA}.$$

In flat space-time $V = V_\mu \star \mathrm{d}x^\mu = \delta^a_\mu \gamma_a \star \mathrm{d}x^\mu = \gamma_\mu \mathrm{d}x^\mu$ and after tracing over spinor indices we find

$$S_m = \frac{1}{2}\int \left(\bar{\psi} \star (i\gamma^\mu D^\star_\mu - m)\psi - (i\overline{D^\star_\mu \psi}\gamma^\mu + m\bar{\psi}) \star \psi\right) \star \mathrm{d}^4 x.$$

Seiberg-Witten map

Idea: the NC gauge transformations are induced by commutative ones, $\delta^\star \to \delta^\star_\alpha$. Then:

$$\Lambda = \Lambda_\alpha(A^c), \quad A = A(A^c), \quad \psi = \psi(\psi^c, A^c), \ldots \tag{1}$$

where α is the commutative gauge parameter and A^c, ψ^c are commutative fields.

The consistency relation for gauge transformations

$$(\delta^\star_\alpha \delta^\star_\beta - \delta^\star_\beta \delta^\star_\alpha)\psi(x) = \delta^\star_{-i[\alpha,\beta]}\psi$$

yields the solution for $\Lambda_\alpha(A^c)$. The transformation laws $\delta^\star_\alpha \psi = i\Lambda_\alpha \star \psi(x), \quad \delta^\star_\alpha A = \mathrm{d}\Lambda_\alpha + i[\Lambda_\alpha \stackrel{\star}{,} A]$, can be solved order by order in deformation parameter a. The solutions for the fields have free parameters (SW freedom), e.g.

$$\psi = \psi^c - \frac{1}{2} C^{\rho\sigma}_\lambda x^\lambda A^c_\rho (\partial_\sigma \psi^c) + i d_1 C^{\rho\sigma}_\lambda x^\lambda F^c_{\rho\sigma} \psi^c + d_2 a (D_0 \psi)^c.$$

Expanded action

Finally, the NC action expanded up to first order in a (expanding $\star$-product and using the SW map) reads:

$$
\begin{aligned}
S_g^{(1)} &= -\frac{1}{4}\int \mathrm{d}^4x \Big\{ F^c_{\mu\nu}F^{c\mu\nu} - \frac{1}{2}C^{\rho\sigma}_{\lambda}x^{\lambda}F^{c\mu\nu}F^c_{\mu\nu}F^c_{\rho\sigma} + \\
&\quad +2C^{\rho\sigma}_{\lambda}x^{\lambda}F^{c\mu\nu}F^c_{\mu\rho}F^c_{\nu\sigma}\Big\}, \\
S_m^{(1)} &= \frac{1}{2}\int \mathrm{d}^4x \Big\{ \bar\psi^c\Big(i\gamma^\mu (D_\mu\psi)^c - m\psi^c + \frac{a}{2}\gamma^j (D_0D_j\psi)^c + \\
&\quad +\frac{i}{2}C^{\rho\sigma}_{\lambda}x^{\lambda}\gamma^\mu F^c_{\rho\mu}(D_\sigma\psi)^c\Big) - \\
&\quad -\Big(i\overline{D_\mu\psi}^c\gamma^\mu + m\bar\psi^c - \frac{a}{2}\overline{D_0D_j\psi}^c\gamma^j \\
&\quad +\frac{i}{2}C^{\rho\sigma}_{\lambda}x^{\lambda}\overline{D_\sigma\psi}^c\gamma^\mu F^c_{\rho\mu}\Big)\psi^c\Big\}.
\end{aligned}
$$

No free parameters! Calculate EOM, conserved $U(1)$ current, dispersion relations,...

2nd approach: natural basis

$$x^\mu = (t = x^0, x, y, z), \quad \mathrm{d}x^\mu = (\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z), \quad \partial_\mu = (\partial_t, \partial_x, \partial_y, \partial_z)$$

$\downarrow$ [Schenkel, Uhlemann '10]

$$x^a = (t, r, \theta, \phi), \quad \theta^a = (\mathrm{d}t, \frac{\mathrm{d}r}{r}, \mathrm{d}\theta, \mathrm{d}\phi), \quad e_a = (\partial_t, r\partial_r, \partial_\theta, \partial_\phi)$$

$$\mathcal{F} \rightsquigarrow \mathcal{F} = e^{-\frac{ia}{2}(\partial_0 \otimes r\partial_r - r\partial_r \otimes \partial_0)}, \quad f \star g = \dots, \mathrm{d}f = \dots$$

But: $f \star \theta^a = \theta^a \star f = f \cdot \theta^a$! Also: $\theta^a \wedge_\star \theta^b = \theta^a \wedge \theta^b$!

$\eta_{\mu\nu} \rightsquigarrow g_{ab} = diag(1, -r^2, -r^2, -r^2 \sin^2\theta) \rightsquigarrow$ Hodge dual in CURVED space-time!

$$*F^c = \frac{1}{2}\epsilon_{abcd}\sqrt{-g}g^{am}g^{bn}F_{mn}\theta^c \wedge \theta^d$$

$$\downarrow$$

$$*F = \frac{1}{2}\epsilon_{abcd}G^{ambn} \star F_{mn} \star \theta^c \wedge_\star \theta^d, \quad \delta^\star_\alpha G^{ambn} = i[\Lambda_\alpha \stackrel{\star}{,} G^{ambn}].$$

Doable, but not finished. One has to check if the results are basis independent. To be expected, but...

Conclusions

- Advantages of twist formalism:
  -mathematically well defined
  -differential calculus
  -cyclic integral
  -no SW ambiguities
- Disadvantages:
  -Hodge dual is difficult to generalize
  -global Poinaceré symmetry $iso(1,3)$ is replaced by global inhomogenious general linear symmetry $igl(1,3)$
  -problem of conserved charges
- Possibilities:
  -natural basis
  -new definition of $\star$-Hodge dual
  -twisted gauge symmetry

Outlook

- Better understanding of the model
  -x-dependent term in the action: geometrical interpretation?
  -renormalization?
  -phenomenological consequences?
- Generalization
  -$su(n)$ gauge theory; SW freedom
  -twisted gauge symmetry helps?