Quantization of 2-plectic Manifolds

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Based on:

- CS and Richard Szabo, work in progress

2-plectic Manifolds

Multisymplectic manifolds are a natural generalization of symplectic manifolds.

Symplectic manifolds

Manifold M with closed 2-form ω such that $\iota_v \omega = 0 \Leftrightarrow v = 0$.

- Poisson structure $\rightarrow$ Phase spaces in Hamiltonian dynamics.
- Starting point for quantization.

p-plectic manifolds

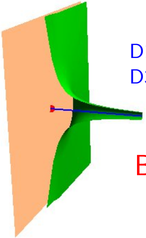
Manifold M with closed $p+1$-form ω such that $\iota_v \omega = 0 \Leftrightarrow v = 0$.

- 1-plectic: symplectic, 2-plectic: 3-form ω
- (Often) Nambu-Poisson structure $\rightarrow$ multiphase spaces in Nambu mechanics.
- Starting point for higher quantization (?)

- Why should we be interested in such manifolds?
- Why should we quantize them?

Monopoles and Self-Dual Strings

Lifting monopoles to M-theory yields self-dual strings.



	0	1	2	3	4	5	6
D1	×						×
D3	×	×	×	×			

BPS configuration!

Nahm equation:

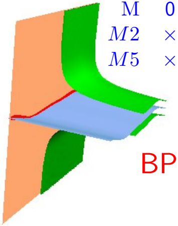
$$\frac{\mathrm{d}}{\mathrm{d}x^6}X^i + \varepsilon^{ijk}[X^j, X^k] = 0$$

Sol. $X^i = \frac{1}{x^6}G^i$,

$$G^i = \varepsilon^{ijk}[G^j, G^k]$$

$\Rightarrow$ Fuzzy Funnel

Nahm, Diaconescu, Tsimpis



M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

BPS configuration!

Basu-Harvey equation:

$$\frac{\mathrm{d}}{\mathrm{d}x^6}X^\mu + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0$$

Sol. $X^\mu = \frac{1}{\sqrt{x^6}}G^\mu$,

$$G^\mu = \varepsilon^{\mu\nu\rho\sigma}[G^\nu, G^\rho, G^\sigma]$$

$\Rightarrow$ Fuzzy Funnel

Basu, Harvey

Further Motivation

There are more appearances of 2-plectic manifolds in string-/M-theory.

- M5-brane perspective: Turning on 3-form background,

$$C = \theta \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \theta' \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \ ,$$

  one gets interesting noncommutative deformations:
  - Noncommutative loop space
    Kawamoto, Sasakura and Bergshoeff et al. (2000)
  - $[x^0, x^1, x^2] = \theta$ and $[x^3, x^4, x^5] = \theta'$ Chu, Smith (2009)
- Non-associative structures from strings in H-field backgrounds
  Blumenhagen, Deser, Lüst, Plauschinn, Rennecke (2010/11)
- Baez et al.: Phase space of bosonic string is 2-plectic

Outline

- Berezin quantization of $\mathbb{C}P^1$
- Quantization of duals of Lie algebras
- From Lie algebras to Lie algebroids
- Hawkins approach to quantization
- Loop spaces and gerbes
- Example: Quantization of $\mathbb{R}^3$

Hilbert space

$\mathscr{H}$: space of global polarized sections of prequantum line bundle.
Line bundle $L_k = \mathcal{O}(k)$, Kähler polarization: $\mathscr{H}_k = H^0(\mathbb{C}P^1, L_k)$.

$$\mathscr{H}_k \cong \mathrm{span}(z_{\alpha_1} \dots z_{\alpha_k}) \cong \mathrm{span}(\hat{a}^\dagger_{\alpha_1} \dots \hat{a}^\dagger_{\alpha_k}|0\rangle)$$

Coherent states

For any $z \in \mathcal{M}$: coherent st. $|z\rangle \in \mathscr{H}$. Here: $|z\rangle = \frac{1}{k!}(\bar{z}_\alpha \hat{a}^\dagger_\alpha)^k|0\rangle$.

Quantization

Quantization is the inverse map on the image $\Sigma = \sigma(\mathcal{C}^\infty(\mathcal{M}))$ of

$$f(z) = \sigma(\hat{f}) = \mathrm{tr}\left(\frac{|z\rangle\langle z|}{\langle z|z\rangle}\hat{f}\right)\ , \quad \text{Bridge: } \mathcal{P} := \frac{|z\rangle\langle z|}{\langle z|z\rangle}$$

Quantization of the Dual of a Lie Algebra

The quantization of the dual of a Lie algebra is straightforward and well known.

Kirillov-Kostant-Souriau Poisson structure on $M = \mathfrak{g}^*$

Linear functions on M: elements of $\mathfrak{g}$. Define

$$\{g_1, g_2\}(x) = \langle x, [g_1, g_2] \rangle \ , \quad g_1, g_2 \in \mathfrak{g} \ .$$

Extend to polynomial functions using the Leibniz identity.

Quantization: $C_0(\mathfrak{g}^*) \rightarrow$ convolution algebra $C^*(G)$

$$\begin{array}{ccccl} C_0(\mathfrak{g}^*) & \xrightarrow{\text{Fourier−transform}} & C_0(\mathfrak{g}) & & \\ & & \downarrow & & \text{identify via exp} \\ C^*(\mathfrak{g}^*) & \xleftarrow{\text{transform back}} & C^*(G) & \leftarrow & \text{convolution product} \end{array}$$

For nilpotent Lie algebras: equiv. to Kontsevich/univ. env. algebra

Example: κ-Minkowski Space

κ-Minkowski space can be obtained from a quantized dual of a Lie algebra.

κ-Minkowski space: $[g^0, g^i] = \mathrm{i}\hbar g^i$, $[g^i, g^j] = 0$, $i > 0$

Group: $G_\hbar \cong \mathbb{R} \rtimes_\alpha \mathbb{R}$ generated by $W(k_0, \vec{k}) = V_{\vec{k}} U_{k_0}$ with $U_{k_0} := \exp\big(\mathrm{i}\, k_0\, g^0\big)$, $V_{\vec{k}} = \exp\big(-\mathrm{i} \sum_i k_i\, g^i\big)$ and

$$W(k_0, \vec{k}\,)\, W(k_0', \vec{k}'\,) = W(k_0 + k_0', \vec{k} + \mathrm{e}^{-\hbar\, k_0}\, \vec{k}'\,)$$

Convolution algebra generated by

$$W(\tilde{f}) := \int_{\mathbb{R}^d} \mathrm{d}k_0\ \mathrm{d}\vec{k}\ \mathrm{e}^{\hbar\, k_0}\ \tilde{f}(k_0, \vec{k}\,)\, W(k_0, \vec{k}\,)$$

Convolution product:

$$(\tilde{f} \circledast_\hbar \tilde{g}\,)(k_0, \vec{k}\,) = \int_{\mathbb{R}^d} \mathrm{d}k_0'\ \mathrm{d}\vec{k}'\ \mathrm{e}^{\hbar\, k_0'}\ \tilde{f}(k_0', \vec{k}'\,)\, \tilde{g}\big(k_0 - k_0'\,,\ \mathrm{e}^{\hbar\, k_0'}\, (\vec{k} - \vec{k}')\big)$$

Lie Groupoids and Lie Algebroids

Lie groupoids and Lie algebroids are generalizations of Lie groups and Lie algebras.

Group:
Category with one object $\mathbb{1}$, every morphism (group elt.) invertible.

Groupoid:
Category with every morphism invertible.

Lie group(oid): objects/morphisms manifolds, maps differentiable.

Equivalently: Symmetric digraph (vertices B and arrows $\mathcal{G}$) with:

- source- and target maps $s, t : \mathcal{G} \to B$: $s(a) \xrightarrow{a} t(a)$
- embedding $i : B \hookrightarrow \mathcal{G}$ with $t \circ i = s \circ i = \mathrm{id}_B$
- partial associative product $m : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ with $m(a, iv) = a$

Examples:

- Group G: one object pt, arrows: group elements, $i : pt \mapsto \mathbb{1}_G$
- Pair groupoid $M \times M$: $x \xrightarrow{(x,y)} y \xrightarrow{(y,z)} z$, $i(x) = (x, x)$.

Lie Groupoids and Lie Algebroids

Lie groupoids and Lie algebroids are generalizations of Lie algebras and Lie groups.

Lie algebra: tangent space of Lie group at identity.

Lie algebroids: $\mathsf{Lie}(\mathcal{G}) = \cup_{x \in B} T_{i(x)}(t^{-1}(x)) \subset T\mathcal{G}$:

$$\begin{array}{ccc} \mathsf{Lie}(\mathcal{G}) & \xrightarrow{\#} & TB \\ & \searrow & \downarrow \\ & & B \end{array}$$

with Lie bracket structure on sections of $\mathsf{Lie}(\mathcal{G})$, $\#$ compatible.

Examples:

- Group G: $\mathsf{Lie}(\mathcal{G}) = \mathfrak{g} \rightarrow pt$, $\#$: trivial.
- Pair groupoid: $\mathsf{Lie}(M \times M) = \cup_{x \in M} x \times T_x M = TM$.

Note: not every Lie algebroid can be integrated to a Lie groupoid!

Quantizing Duals of Lie Algebroids

The elements of Lie algebra quantization can be lifted to Lie algebroids and Lie groupoids.

Goal:

Extend quant. of duals of Lie algebras to duals of Lie algebroids.

Before: $C_0(\mathfrak{g}^*) \;\rightarrow\; C^*(G)$. Now:

$$C_0(\mathsf{Lie}^*(\mathcal{G})) \quad\rightarrow\quad C^*(\mathcal{G})$$

This is interesting, as T^*M of Poisson manifold M is Lie algebroid.

Eli Hawkins (2006), Weinstein, Renault, ...

Yesterday's talk: Podles sphere

Bonechi, Ciccoli, Staffolani, Tarlini

Hawkins' Groupoid Approach to Quantization

In Hawkins' approach, the elements of geometric quantization are lifted to groupoids.

Groupoid $\mathcal{G}$, composable arrows: $\mathcal{G}_2$. Maps: pr_1, pr_2, m: $\mathcal{G}_2 \to \mathcal{G}$

Key idea in Hawkins: Reduce polarization issues as follows
T^*M, M Poisson manifold, integrates to Lie groupoid $\Leftrightarrow$
$\exists\ s, t : (\Sigma, \omega) \rightrightarrows M$ with t Poisson, $\partial^*\omega = 0$, $\partial^* = \mathrm{pr}_1^* + \mathrm{pr}_2^* - m^*$

Crainic, Fernandes (2002)

Hawkins' quantization algorithm:

1. Integrating groupoid $s, t : \Sigma \rightrightarrows M$, ω, $\partial^*\omega = 0$, t Poisson
2. Construct a prequantization of Σ with data (L, ∇)
3. Endow Σ with a groupoid polarization
4. Construct a twist element
5. Obtain twisted polarized convolution algebra of Σ.

Note: Many questions concerning existence and uniqueness remain.

Approach avoids Hilbert spaces, this might help in 2-plectic case.

Example: Groupoid Quantization of $\mathbb{R}^2$

The Moyal plane is conveniently reproduced in groupoid language.

Starting point: $M = \mathbb{R}^2$, Poisson structure θ^{ij}, $i, j = 1, 2$.

1. Lie groupoid: $\Sigma = M \times M^*$, coords. (x^i, y_i), $\omega = \mathrm{d}x^i \wedge \mathrm{d}y_i$

$$x^i + \tfrac{1}{2}\theta^{ij} y_j \xrightarrow{(x^i, y_i)} x^i - \tfrac{1}{2}\theta^{ij} y_j$$

   Note: t is indeed a Poisson map: $\{t^* f, t^* g\}_\omega = t^*\{f, g\}_\pi$

$$x^i + \theta^{ij}(y_j + y'_j) \longrightarrow x^i + \theta^{ij}(y_j - y'_j) \longrightarrow x^i - \theta^{ij}(y_j + y'_j)$$

   From this: pr_1, pr_2 and m. $\partial^*\omega = pr_1^*\omega - m^*\omega + pr_2^*\omega = 0$

2. Prequantization: L trivial line bundle over Σ, $F = -\mathrm{i}2\pi\omega$
3. Polarization: Induced by symplectic prepotential $\vartheta = -x^i \mathrm{d}y_i$
4. Twist element: $\partial^*\vartheta = \sigma_0^{-1}\mathrm{d}\sigma_0 = \mathrm{d}(-\tfrac{1}{2}\theta^{ij} y_i y'_j)$
5. Twisted polarized convolution algebra: Moyal product on M

Hawkins

Other Examples

Hawkins' method reproduces all canonical examples.

Analogously:

- Quantization of κ-Minkowski space with $\Sigma = T^*G$, $\sigma_0 = 0$
- Berezin Quantization of Kähler manifolds M: $\Sigma = M \times M$.

Symplectic manifold (M, ω) with $\omega \in H^2(M, \mathbb{Z})$:
$\Rightarrow$ Prequantum line bundle with connection ∇, $F_\nabla = 2\pi \mathrm{i} \omega$.

2-plectic manifold (M, ϖ) with $\varpi \in H^3(M, \mathbb{Z})$:
$\Rightarrow$ Prequant. abelian gerbe with connect. struct. incl. $H = 2\pi \mathrm{i} \varpi$.

Idea: Categorify Hawkins' approach (2-groupoids, etc.)
(work in progress, cf. Freed, Baez, Rogers ...)

Line Bundles over Loop Spaces from Gerbes

A prequantum gerbe over a manifold yields a prequantum line bundle over its loop space.

Alternative approach:

Map the 2-plectic form to a symplectic form over loop space.

Consider the following double fibration:

$$\begin{array}{ccc} & \mathcal{L}M \times S^1 & \\ {}^{ev}\swarrow & & \searrow^{pr} \\ M & & \mathcal{L}M \end{array}$$

Transgression

$$\mathcal{T} : H^{k+1}(M) \to H^k(\mathcal{L}M)\ , \quad \mathcal{T} = pr! \circ ev^*$$

$$(\mathcal{T}\omega)_x(v_1(\tau), \dots, v_k(\tau)) := \int_{S^1} \mathrm{d}\tau\, \omega(v_1(\tau), \dots, v_k(\tau), \dot{x}(\tau))$$

Previously successfully applied: Lift ADHMN constr. to M-theory

CS, Papageorgakis&CS, Palmer&CS

Explicitly, this works as follows:

We start from $\mathbb{R}^3$ with 2-plectic form $\varpi = \varepsilon_{ijk}\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$.

Transgression yields a symplectic form on loop space $\mathcal{L}\mathbb{R}^3$:

$$\omega = \oint \mathrm{d}\tau \oint \mathrm{d}\sigma\ \varepsilon_{ijk}\dot{x}^k(\tau)\delta(\tau-\sigma)\ \delta x^i(\tau) \wedge \delta x^j(\sigma)$$

Kernel of ω:

$$\iota_X(\mathcal{T}\varpi) = 0 \quad \text{for} \quad X = \oint \mathrm{d}\rho\ \dot{x}^i(\rho)\,\frac{\delta}{\delta x^i(\rho)}$$

This vector field generates reparameterizations of the loops in $\mathcal{L}\mathbb{R}^3$.

We can therefore invert ω and obtain the Poisson bracket

$$\{f,g\} := \oint \mathrm{d}\tau\ \oint \mathrm{d}\rho\,\delta(\tau-\rho)\,\theta^{ijk}\,\frac{\dot{x}_k(\rho)}{|\dot{x}(\rho)|^2}\left(\frac{\delta}{\delta x^i(\tau)}f\right)\left(\frac{\delta}{\delta x^j(\rho)}g\right)$$

Start: $M = \mathbb{R}^3$, Poisson structure $\theta^{i\sigma,j\rho} := \delta(\sigma - \rho)\theta^{ijk}\dot{x}_k(\sigma)$

1. Int. groupoid.: $\Sigma = \mathcal{L}(T^*M) \cong T^*\mathcal{L}M$, coords. $(x^i(\sigma), y_i(\sigma))$

Groupoid structure on $T^*\mathcal{L}M$:

$$x^i + \tfrac{1}{2}\theta^{ijk}y_j\dot{x}_k \xrightarrow{(x^i(\sigma),y_i(\sigma))} x^i - \tfrac{1}{2}\theta^{ijk}y_j\dot{x}_k$$

2-nerve $\Sigma_2 = \mathcal{L}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3)$, coords. $(x^i(\sigma), y_i(\sigma), y_i'(\sigma))$:

$$x^i + \theta^{ijk}(y_j + y_j')\dot{x}_k \to x^i + \theta^{ijk}(y_j - y_j')\dot{x}_k \to x^i - \theta^{ijk}(y_j + y_j')\dot{x}_k$$

Symplectic structure:

$$\omega = \oint \mathrm{d}\rho \oint \mathrm{d}\sigma\ \delta(\rho - \sigma)\delta x^i(\rho) \wedge \delta y_i(\sigma)$$

t is Poisson map and $\partial^*\omega = 0$ only to lowest order in θ!

Groupoid Quantization of $\mathbb{R}^3$

The groupoid quantization can be extended to $\mathcal{L}\mathbb{R}^3$, reproducing M-theory results.

2. Prequantization: L trivial line bundle over Σ
   Connection ∇ with $F = 2\pi\mathrm{i}\omega = 2\pi\mathrm{i}\mathcal{T}\varpi$
3. Polarization: Induced by symplectic prepotential:
$$\vartheta = \oint \mathrm{d}\rho\, x^i(\rho)\delta y_i(\rho)$$
4. Twist element: $\partial^*\vartheta = \sigma_0^{-1}\mathrm{d}\sigma_0$ with
$$\sigma_0(y_i(\rho), y_j'(\sigma)) = -\tfrac{1}{2}\theta^{ijk} y_i(\rho) y_j'(\sigma)\dot{x}_k(\sigma)\delta(\rho-\sigma)$$
5. Twisted polarized convolution algebra: difficult, but we have
$$[x^i(\tau), x^j(\sigma)] = \theta^{ijk}\dot{x}_k(\tau)\delta(\tau-\sigma) + \mathcal{O}(\theta)$$

Agrees with one-form quantization of Baez et al.
Agrees with Kawamoto, Sasakura and Bergshoeff et al. (2000)
Remains: Compute $\mathcal{O}(\theta)$-corrections and compare to KS

Summary:

- ✓ Groupoids offer a nice approach to quantization
- ✓ Can be extended to loop spaces/2-plectic manifolds

Future directions:

- ▷ Complete the loop space picture, other spaces (S^3, T^3).
- ▷ Extend quantization of $\mathbb{R}^3$ to 2-groupoid.
- ▷ Unify picture: Higher Poisson structures? Courant algebroids?
- ▷ Rewrite M2-brane models using the new function algebras.

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