Double Field Theory and Duality

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Strings on a Torus

- States: momentum p, winding w
- String: Infinite set of fields $\psi(p, w)$
- Fourier transform to doubled space: $\psi(x, \tilde{x})$
- "Double Field Theory" from closed string field theory. Some non-locality in doubled space
- Subsector? e.g. $g_{ij}(x, \tilde{x}),\ b_{ij}(x, \tilde{x}),\ \phi(x, \tilde{x})$

Double Field Theory

- Double field theory on doubled torus
- General solution of string theory: involves doubled fields $\psi(x, \tilde{x})$
- DFT needed for non-geometric backgrounds
- *Real* dependence on *full* doubled geometry, dual dimensions not auxiliary or gauge artifact. Double geom. *physical* and *dynamical*

- Novel symmetry, reduces to diffeos + B-field trans. in *any* half-dimensional subtorus
- Backgrounds depending on $\{x^a\}$ seen by particles, on $\{\tilde{x}_a\}$ seen by winding modes.
- Captures exotic and complicated structure of interacting string
- Non-polynomial, algebraic structure homotopy Lie algebra, cocycles.
- T-duality symmetry manifest
- Generalised T-duality: no isometries needed

Earlier work on double fields:
Siegel, Tseytlin

Strings on Torus

D=n+d

Target space $\mathbb{R}^{n-1,1} \times T^d$
Coordinates $x^i = (x^\mu, x^a)$
Momenta $p_i = (p_\mu, p_a)$
Winding $w^i = (w^\mu, w^a)$
Dual coordinates (conjugate to winding) $\tilde{x}_i = (\tilde{x}_\mu, \tilde{x}_a)$
Constant metric and B-field $E_{ij} = G_{ij} + B_{ij}$

<u>Compact dimensions</u>

p_a, w^a discrete, in Narain lattice, $x^a, \tilde{x}_a$ periodic

<u>Non-compact dimensions</u> x^μ, p_μ continuous

Usually take $w^\mu = 0$ so $\dfrac{\partial}{\partial \tilde{x}_\mu} = 0$, fields $\psi(x^\mu, x^a, \tilde{x}_a)$

T-Duality

- Interchanges momentum and winding
- Equivalence of string theories on dual backgrounds with very different geometries
- String field theory symmetry, provided fields depend on both $x, \tilde{x}$ Kugo, Zwiebach
- For fields $\psi(x^\mu)$ not $\psi(x^\mu, x^a, \tilde{x}_a)$ Buscher
- Generalise to fields $\psi(x^\mu, x^a, \tilde{x}_a)$

Generalised T-duality Dabholkar & CMH

String Field Theory on Minkowski Space

Closed SFT: Zwiebach

String field $\quad \Phi[X(\sigma), c(\sigma)]$

$$X^i(\sigma) \to x^i, \text{ oscillators}$$

Expand to get infinite set of fields

$$g_{ij}(x), b_{ij}(x), \phi(x), \ldots, C_{ijk\ldots l}(x), \ldots$$

Integrating out massive fields gives field theory for

$$g_{ij}(x), b_{ij}(x), \phi(x)$$

String Field Theory on a torus

String field $\qquad \Phi[X(\sigma), c(\sigma)]$

$$X^i(\sigma) \to x^i, \tilde{x}_i, \text{ oscillators}$$

Expand to get infinite set of double fields

$$g_{ij}(x,\tilde{x}), b_{ij}(x,\tilde{x}), \phi(x,\tilde{x}), \ldots, C_{ijk\ldots l}(x,\tilde{x}), \ldots$$

Seek **double field theory** for

$$g_{ij}(x,\tilde{x}), b_{ij}(x,\tilde{x}), \phi(x,\tilde{x})$$

Free Field Equations (B=0)

$$L_0 + \bar{L}_0 = 2$$

$$p^2 + w^2 = N + \bar{N} - 2$$

$$L_0 - \bar{L}_0 = 0$$

$$p_i w^i = N - \bar{N}$$

Free Field Equations (B=0)

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$$p^2 + w^2 = N + \bar{N} - 2$$

Treat as field equation, kinetic operator in doubled space

$$G^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + G_{ij} \frac{\partial^2}{\partial \tilde{x}_i \partial \tilde{x}_j}$$

$$L_0 - \bar{L}_0 = 0$$

$$p_i w^i = N - \bar{N}$$

Treat as constraint on double fields

$$\Delta \equiv \frac{\partial^2}{\partial x^i \partial \tilde{x}_i} \qquad (\Delta - \mu)\psi = 0$$

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$$G^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + G_{ij} \frac{\partial^2}{\partial \tilde{x}_i \partial \tilde{x}_j}$$

Laplacian for metric

$$ds^2 = G_{ij} dx^i dx^j + G^{ij} d\tilde{x}_i d\tilde{x}_j$$

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Laplacian for metric

$$ds^2 = dx^i d\tilde{x}_i$$

$$g_{ij}(x,\tilde{x}), b_{ij}(x,\tilde{x}), \phi(x,\tilde{x})$$

$$N = \bar{N} = 1$$

$$p^2 + w^2 = 0$$

$$p \cdot w = 0$$

“Double Massless”

Constrained fields $\psi(x^\mu, x^a, \tilde{x}_a)$

$$(\Delta - \mu)\psi = 0$$

Momentum space $\psi(p_\mu, p_a, w^a)$ $\Delta = p_a w^a$

Momentum space: Dimension n+2d
Cone: $p_a w^a = 0$ or hyperboloid: $p_a w^a = \mu$
dimension n+2d-1

DFT: fields on cone or hyperboloid, with discrete p,w
Problem: naive product of fields on cone do not lie on cone. Vertices need projectors

Restricted fields: Fields that depend on d of 2d torus momenta, e.g. $\psi(p_\mu, p_a)$ or $\psi(p_\mu, w^a)$
Simple subsector, no projectors needed, no cocycles.

Torus Backgrounds

$$G_{ij} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & G_{ab} \end{pmatrix} , \quad B_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ab} \end{pmatrix} \qquad E_{ij} \equiv G_{ij} + B_{ij}$$

Fluctuations $\quad e_{ij} = h_{ij} + b_{ij}$

Take $\quad B_{ij} = 0 \qquad \tilde{\partial}_i \equiv G_{ik} \dfrac{\partial}{\partial \tilde{x}_k}$

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Usual action $\quad \displaystyle\int dx \sqrt{-g} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right]$

Quadratic part $\quad \displaystyle\int dx \; L[\, h, b, d; \partial \,]$

$e^{-2d} = e^{-2\phi}\sqrt{-g} \qquad$ (d invariant under usual T-duality)

Double Field Theory Action

$$S^{(2)} = \int [dx d\tilde{x}] \Big[\; L[\,h, b, d; \partial\,] + L[\,-h, -b, d; \tilde{\partial}\,]$$

$$+ (\partial_k h^{ik})(\tilde{\partial}^j b_{ij}) + (\tilde{\partial}^k h_{ik})(\partial_j b^{ij}) - 4\, d\, \partial^i \tilde{\partial}^j b_{ij} \; \Big]$$

Action + dual action + strange mixing terms

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Action + dual action + strange mixing terms

$$\delta h_{ij} = \quad \partial_i \epsilon_j \; + \partial_j \epsilon_i \; + \; \tilde{\partial}_i \tilde{\epsilon}_j + \tilde{\partial}_j \tilde{\epsilon}_i \,,$$

$$\delta b_{ij} = -(\tilde{\partial}_i \epsilon_j - \tilde{\partial}_j \epsilon_i) - (\partial_i \tilde{\epsilon}_j - \partial_j \tilde{\epsilon}_i) \,,$$

$$\delta d \; = -\, \partial \cdot \epsilon + \, \tilde{\partial} \cdot \tilde{\epsilon} \,.$$

Invariance needs constraint

Diffeos and B-field transformations mixed.

Invariant cubic action found for full DFT of (h,b,d)

T-Duality Transformations of Background

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d,d;\mathbb{Z})$$ T-duality

$$E' = (aE + b)(cE + d)^{-1}$$

$$X \equiv \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}$$ transforms as a vector

$$X' = \begin{pmatrix} \tilde{x}' \\ x' \end{pmatrix} = gX = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{x} \\ x \end{pmatrix}$$

T-Duality is a Symmetry of the Action

Fields $e_{ij}(x, \tilde{x}), \; d(x, \tilde{x})$

Background E_{ij}

$$E' = (aE + b)(cE + d)^{-1}$$

$$X' = \begin{pmatrix} \tilde{x}' \\ x' \end{pmatrix} = gX = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{x} \\ x \end{pmatrix}$$

Action invariant if:

$$e_{ij}(X) = M_i{}^k \, \bar{M}_j{}^l \, e'_{kl}(X') \qquad M \equiv d^t - E\,c^t$$

$$d(X) = d'(X') \qquad \bar{M} \equiv d^t + E^t c^t$$

With general momentum and winding dependence!

Projectors and Cocycles

Naive product of constrained fields doesn't satisfy constraint

$$L_0^- \Psi_1 = 0, L_0^- \Psi_2 = 0 \quad \text{but} \quad L_0^- (\Psi_1 \Psi_2) \neq 0$$

$$\Delta A = 0, \Delta B = 0 \quad \text{but} \quad \Delta (AB) \neq 0$$

String product has explicit projection

Leads to a symmetry that is not a Lie algebra, but is a homotopy lie algebra.

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Double field theory requires projections.

SFT has non-local cocycles in vertices, DFT should too

Cocycles and projectors not needed in cubic action

General double fields $\psi(x, \tilde{x})$

Fields on Spacetime M $\psi(x)$

Restricted Fields on N, T-dual to M $\psi(x')$

M,N null wrt O(D,D) metric $ds^2 = dx^i d\tilde{x}_i$

Subsector with fields and parameters all restricted to M or N

- Constraint satisfied on all fields and products of fields
- No projectors or cocycles
- T-duality covariant: independent of choice of N
- Can find full non-linear form of gauge transformations
- Full gauge algebra, full non-linear action

Restricted DFT

Double fields restricted to null D-dimensional subspace N
T-duality "rotates" N to N'

Background independent fields: g,b,d:

$$\mathcal{E} \equiv E + \left(1 - \frac{1}{2}\, e\right)^{-1} e \qquad\qquad \mathcal{E}_{ij} = g_{ij} + b_{ij}$$

O(D,D) Covariant Notation

$$X^M \equiv \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix} \qquad\qquad \partial_M \equiv \begin{pmatrix} \partial^i \\ \partial_i \end{pmatrix}$$

$$\eta_{MN} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad\qquad M = 1, ..., 2D$$

Generalised T-duality transformations:

$$X'^M \equiv \begin{pmatrix} \tilde{x}'_i \\ x'^i \end{pmatrix} = hX^M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}$$

h in O(d,d;Z) acts on toroidal coordinates only

$$\mathcal{E}'(X') = (a\mathcal{E}(X) + b)(c\mathcal{E}(X) + d)^{-1}$$

$$d'(X') = d(X)$$

Buscher if fields independent of toroidal coordinates
Generalisation to case without isometries

O(D,D)

<u>Non-compact dimensions</u> x^μ, p_μ continuous

Strings: take $w^\mu = 0$ so $\frac{\partial}{\partial \tilde{x}_\mu} = 0$, fields $\psi(x^\mu, x^a, \tilde{x}_a)$

For DFT, <u>if</u> we allow dependence on $\tilde{x}_\mu$
DFT invariant under

$$O(n,n) \times O(d,d;\mathbb{Z})$$

Subgroup of O(D,D) preserving periodicities

O(D,D) is symmetry if all directions non-compact:
theory has formal O(D,D) covariance

- Can rewrite in terms of "generalised metric" - cubic (!) action
- Gauge transformations new "generalised Lie derivatives"
- Gauge algebra a covariant form of Courant bracket
- Action a generalised curvature

Generalised Metric Formulation

$$\mathcal{H}_{MN} \quad = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix} \, .$$

2 Metrics on double space $\quad \mathcal{H}_{MN}, \; \eta_{MN}$

Generalised Metric Formulation

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2 Metrics on double space $\quad \mathcal{H}_{MN}, \; \eta_{MN}$

$$\mathcal{H}^{MN} \equiv \eta^{MP}\mathcal{H}_{PQ}\eta^{QN}$$

Constrained metric $\quad \mathcal{H}^{MP}\mathcal{H}_{PN} \;=\; \delta^{M}{}_{N}$

Generalised Metric Formulation

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Constrained metric $\quad \mathcal{H}^{MP}\mathcal{H}_{PN} = \delta^{M}{}_{N}$

Covariant Transformation

$$h^{P}{}_{M}h^{Q}{}_{N}\mathcal{H}'_{PQ}(X') = \mathcal{H}_{MN}(X)$$

$$X' = hX \qquad\qquad h \in O(D,D)$$

O(D,D) covariant action

$$S = \int dx d\tilde{x}\, e^{-2d}\; L$$

$$L = \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\,\partial_L\mathcal{H}_{MK}$$

$$- 2\,\partial_M d\,\partial_N\mathcal{H}^{MN} + 4\mathcal{H}^{MN}\,\partial_M d\,\partial_N d$$

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L cubic! Indices raised and lowered with η

O(D,D) covariant action

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$$- 2\,\partial_M d\,\partial_N\mathcal{H}^{MN} + 4\mathcal{H}^{MN}\,\partial_M d\,\partial_N d$$

Gauge Transformation

$$\delta_\xi\mathcal{H}^{MN} = \xi^P\partial_P\mathcal{H}^{MN}$$
$$+ (\partial^M\xi_P - \partial_P\xi^M)\,\mathcal{H}^{PN} + (\partial^N\xi_P - \partial_P\xi^N)\,\mathcal{H}^{MP}$$

O(D,D) covariant action

$$S = \int dx d\tilde{x}\, e^{-2d}\; L$$

$$L = \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\,\partial_L\mathcal{H}_{MK}$$
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Gauge Transformation

$$\delta_\xi\mathcal{H}^{MN} = \xi^P\partial_P\mathcal{H}^{MN}$$
$$+ (\partial^M\xi_P - \partial_P\xi^M)\,\mathcal{H}^{PN} + (\partial^N\xi_P - \partial_P\xi^N)\,\mathcal{H}^{MP}$$

Rewrite as “Generalised Lie Derivative”

$$\delta_\xi\mathcal{H}^{MN} = \widehat{\mathcal{L}}_\xi\mathcal{H}^{MN}$$

2-derivative action

$$S \ = \ S^{(0)}(\partial,\partial) + S^{(1)}(\partial,\tilde{\partial}) + S^{(2)}(\tilde{\partial},\tilde{\partial})$$

Write $S^{(0)}$ in terms of usual fields

Gives usual action (+ surface term)

$$\int dx\sqrt{-g}e^{-2\phi}\left[R + 4(\partial\phi)^2 - \frac{1}{12}H^2\right]$$

$$S^{(0)} = S(\mathcal{E}, d, \partial)$$

2-derivative action

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Gives usual action (+ surface term)

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$$S^{(0)} = S(\mathcal{E}, d, \partial)$$

$$S^{(2)} = S(\mathcal{E}^{-1}, d, \tilde{\partial})$$ T-dual!

$S^{(1)}$ strange mixed terms

- Restricted DFT:
  fields independent of half the coordinates
- If independent of $\tilde{x}$, equivalent to usual action
- Duality covariant: duality changes which half of coordinates theory is independent of
- Equivalent to Siegel's formulation Hohm & Kwak
- Good for non-geometric backgrounds

Double Field Theory

- Constructed (unrestricted) DFT cubic action for fields g,b,d.
- Quartic & higher should have many features of string field theory: cocycles, projectors, symmetry based on a homotopy Lie algebra.
- Is there an (unrestricted) gauge-invariant action for g,b,d alone, or are massive fields needed?
- T-duality symmetry
- Stringy issues in simpler setting than SFT

- Restricted DFT: have non-linear background independent theory, duality covariant
- Geometry? Meaning of curvature?
- Use for non-geometric backgrounds
- General spaces, not tori?
- Doubled geometry *physical* and *dynamical*

- Heterotic DFT: Hohm & Kwak
- Factorization in perturbative gravity; Hohm
- Geometry with projectors; YM DFT: Jeon, Lee, Park
- Early work on strings and doubled space: Duff, Tseytlin...
- Sigma model for doubled geometry: Hull; Hull & Reid-Edwards,...
- Beta functions for doubled sigma model: Copland; Berman, Copland & Thompson
- Branes: Hull; Lawrence, Schulz, Wecht; Bergshoeff & Riccioni
- Doubled geometry of KK monopole; Jensen
- O(10,10) from E_{11} ; West
- Non-geometry: Andriot, Larfors, Lust, Patalong
- Type II DFT: Thompson; Hull; Hohm, Kwak, Zwiebach