Twist deformations of module homomorphisms and connections

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(joint work with Paolo Aschieri)

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Workshop on Noncommutative Field Theory and Gravity

Corfu Summer Institute

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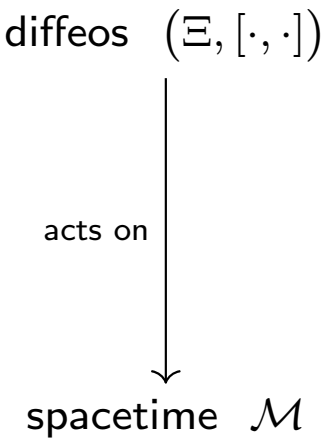
Motivation from noncommutative gravity

Noncommutative gravity à la Wess et al.

◇ Basic idea: [Aschieri,Blohmann,Dimitrijević,Meyer,Schupp,Wess]



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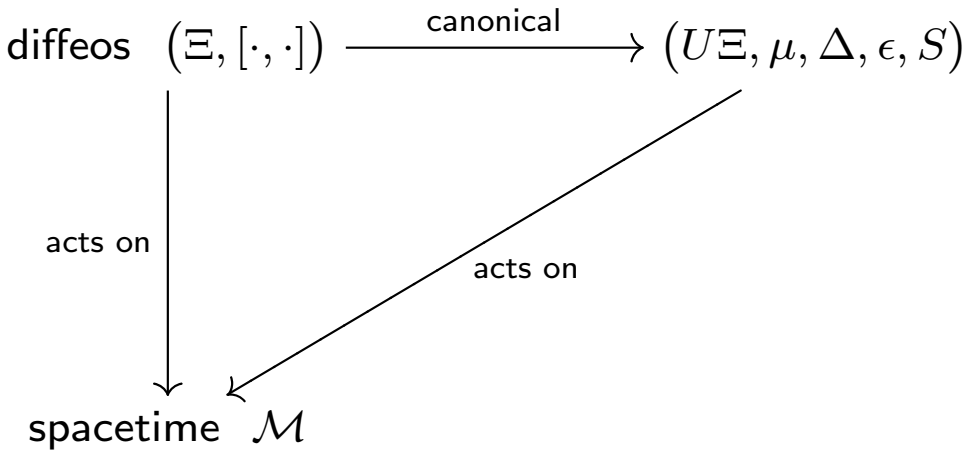

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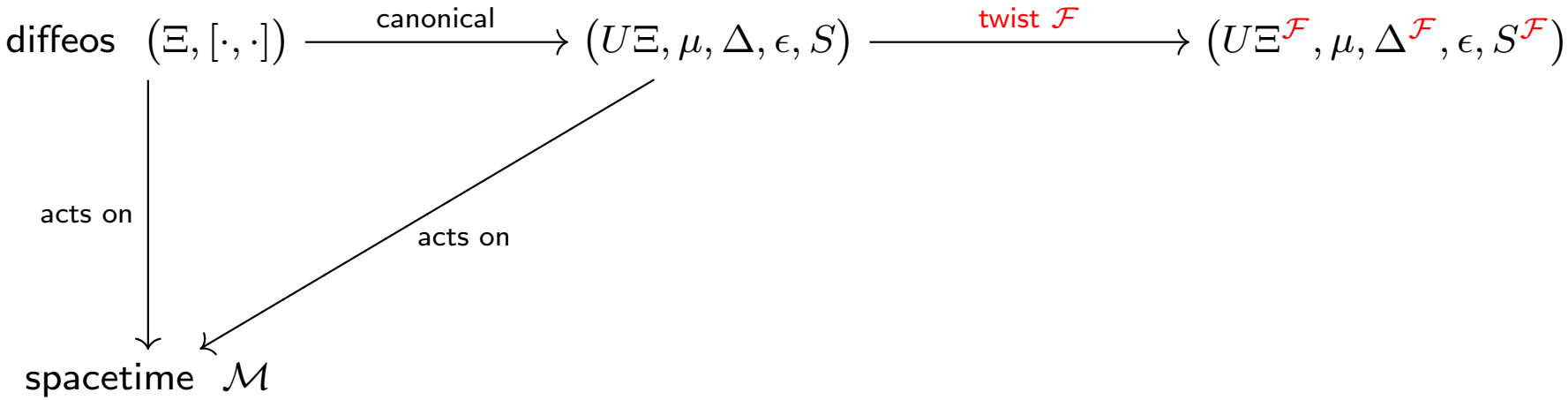

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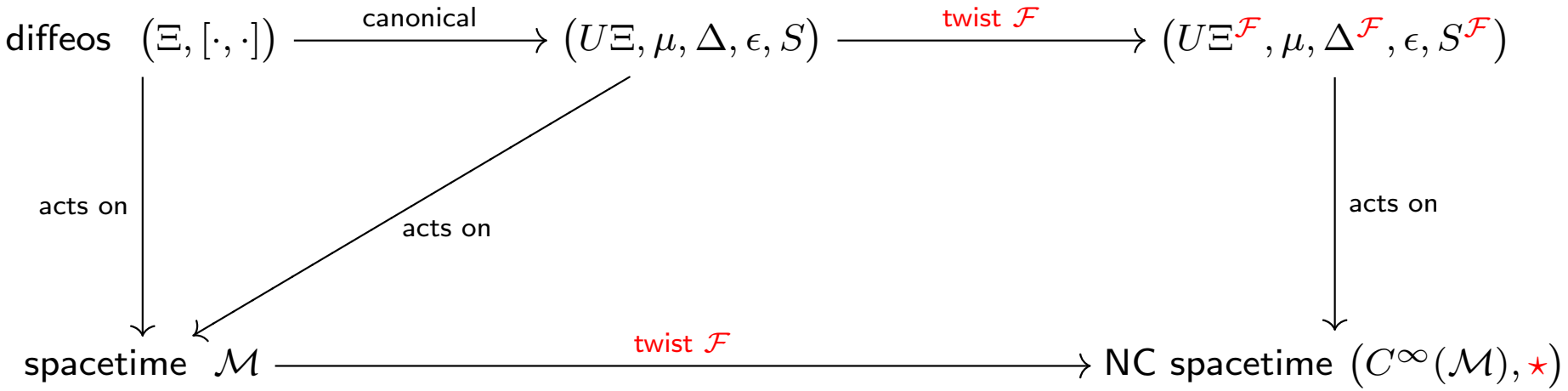

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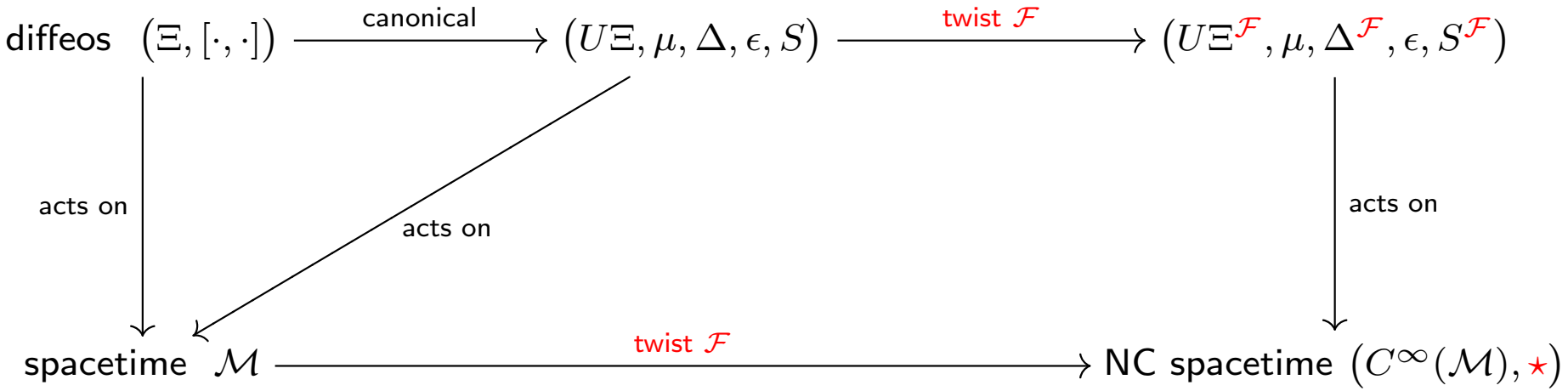

◇ NC geometry via imposing covariance under $U\Xi^{\mathcal{F}}$

– $h \star k = \bar{f}^\alpha(h)\, \bar{f}_\alpha(k) \quad " = h\, e^{\frac{i\lambda}{2} \overleftarrow{\partial_\mu} \Theta^{\mu\nu} \overrightarrow{\partial_\nu}}\, k\, "$

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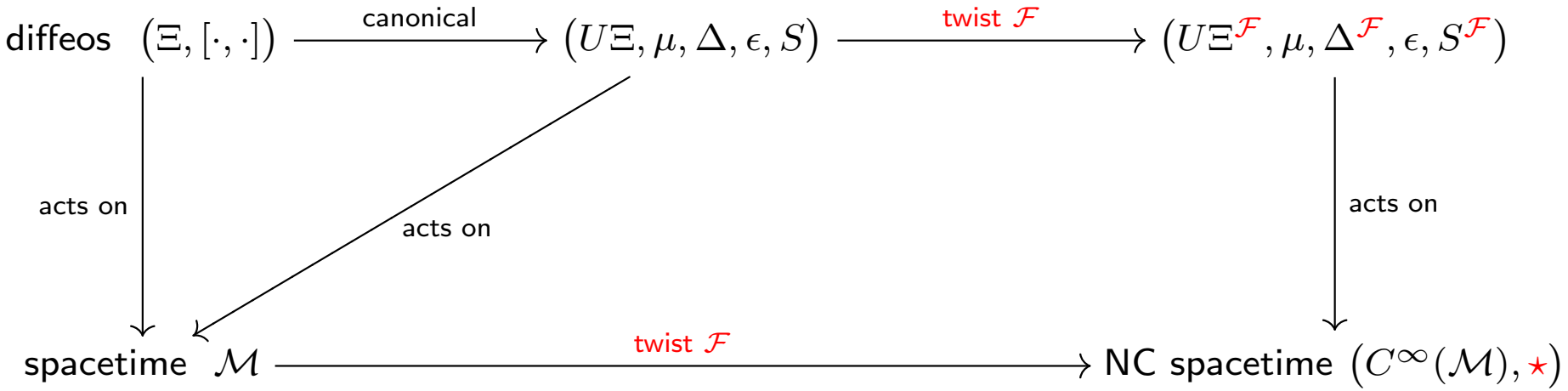

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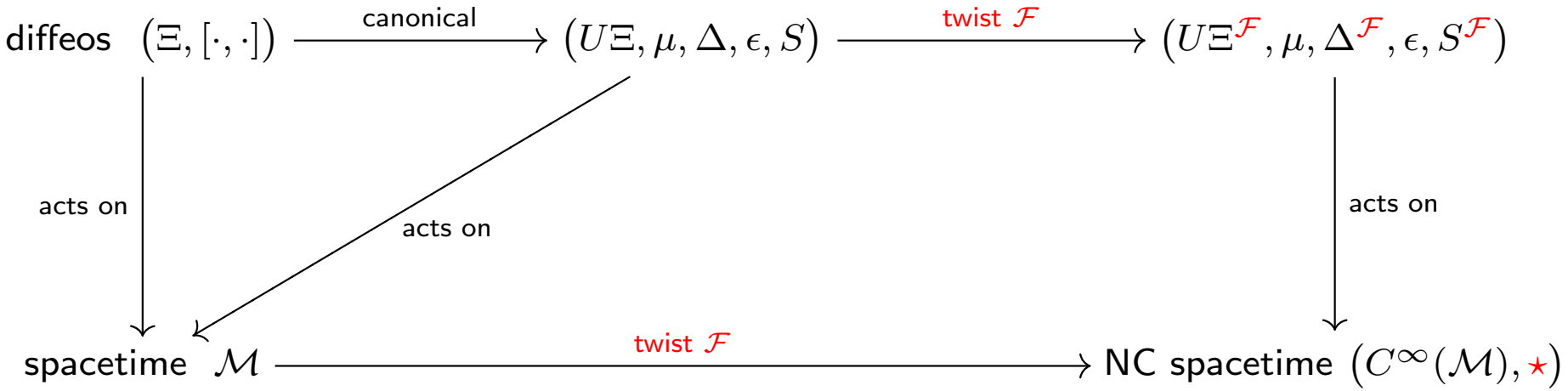

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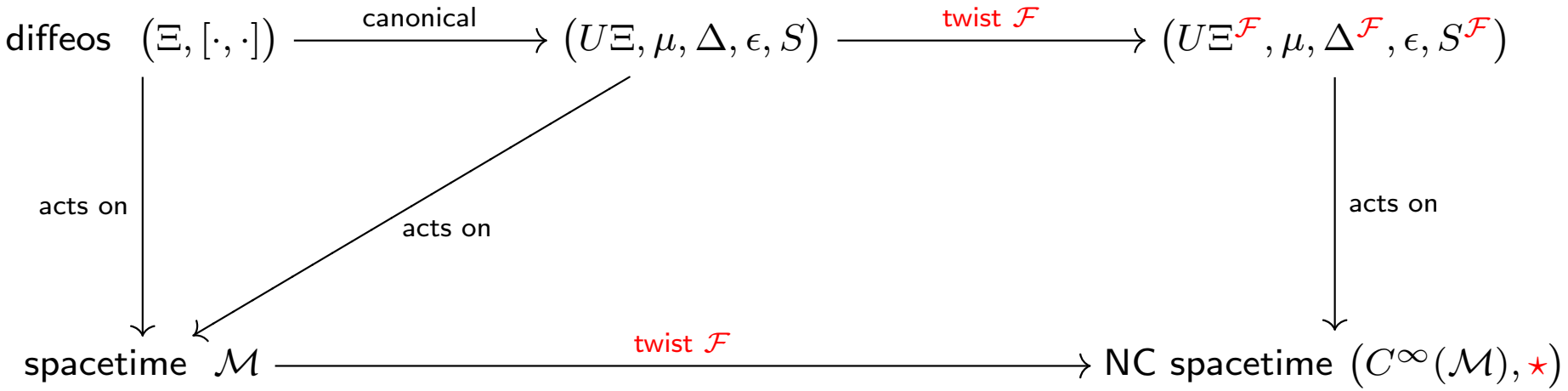

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◇ NC Einstein equation

$$R^\star{}_{ab} - \frac{1}{2} g_{ab} \star \mathfrak{R}^\star = 8\pi G_N\, T^\star{}_{ab}$$

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- deformed diffeomorphisms

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- (quasi)triangular Hopf algebra H

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  - extension (lift) to tensor product modules $V \otimes_A W$
  - NC gravity solutions from a global point-of-view (not using $X^{toomany}_{strangeletters}$)

Twist deformation of homomorphisms and connections

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Theorem (textbook)

- *Given a twist, there is a new Hopf algebra $H^{\mathcal{F}}$ with coproduct $\Delta^{\mathcal{F}}(\cdot) = \mathcal{F}\,\Delta(\cdot)\,\mathcal{F}^{-1}$ and antipode $S^{\mathcal{F}}(\cdot) = \chi\,S(\cdot)\,\chi^{-1}$, where $\chi = f^\alpha S(f_\alpha)$.*

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- *Given also an H-module A-bimodule V, there is an $H^{\mathcal{F}}$-module $A_\star$-bimodule $V_\star$ with $a \star v = (\bar{f}^\alpha \triangleright a) \cdot (\bar{f}_\alpha \triangleright v)$ and $v \star a = (\bar{f}^\alpha \triangleright v) \cdot (\bar{f}_\alpha \triangleright a)$.*

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NB: $D_{\mathcal{F}}$ is a **quantization isomorphism**, mapping one-to-one classical endomorphisms $P(v \cdot a) = P(v) \cdot a$ to deformed ones $P_\star(v \star a) = P_\star(v) \star a$.

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Ex: Consider the dual module $V':=\mathrm{Hom}_A(V,A)$. Then $D_{\mathcal{F}}$ ensures that $(V_\star)'\simeq(V')_\star$.

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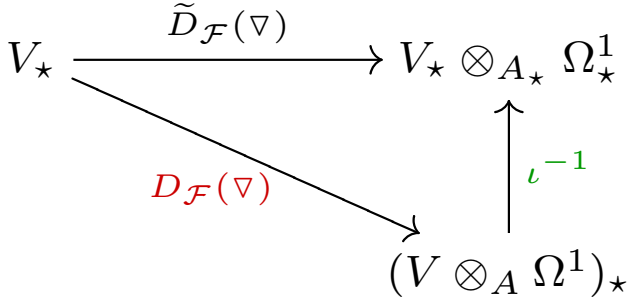

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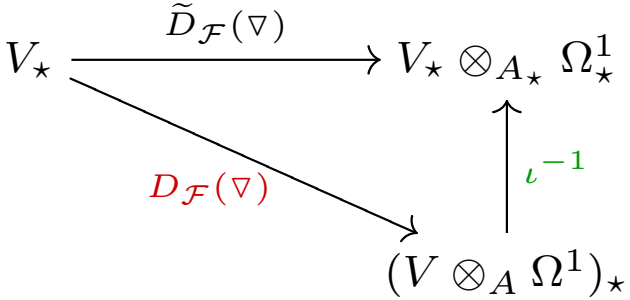

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Theorem

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Product module homomorphisms and connections

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◇ For *central* connections $\mathrm{id} \otimes_R \nabla_W = \mathrm{id} \otimes \nabla_W$, but for noncentral connections the $\otimes_R$ is important!

Noncommutative gravity solutions revisited

Existing results

- Schupp and Solodukhin [arXiv:0906.2724]:
  - NC black hole solution with $[x^i, x^j] = i\,\lambda\,\epsilon^{ijk}x^k$
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Global approach to NC gravity solutions

We use the methods developed above, in particular the isomorphisms

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  - quantization of the classical Levi-Civita connection ∇, i.e. $\nabla_\star := \widetilde{D}_{\mathcal{F}}(\nabla)$, yields deformed LC connection, $(\nabla_\star \oplus_R \nabla_\star)(g_\star) = 0$ and $T_\star = 0$.

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  - deformed Ricci tensor $\mathrm{Ric}_\star := \mathrm{Tr}_\star(R_\star) \in \Omega^1_\star \otimes_{A_\star} \Omega^1_\star$ is related to $\mathrm{Ric} \in \Omega^1 \otimes_A \Omega^1$ via $\mathrm{Ric}_\star = \iota^{-1}(\mathrm{Ric})$

  $\Rightarrow\ \mathrm{Ric}_\star = \iota^{-1}(\mathrm{Ric}) = \iota^{-1}(\Lambda\, g) = \Lambda\, g_\star$

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- Open issues:
  - existence and uniqueness of NC Levi-Civita connection
  - introducing $*$-structures and reality conditions