Classical and quantum Lagrangian field theories with boundary

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Outline

Introduction

- Generalize Segal–Atiyah's axioms to perturbative QFTs
  boundaries $\rightsquigarrow$ Hilbert spaces
  manifolds (with boundaries) $\rightsquigarrow$ states/operators
- Do it for general Lagrangian theories (including gauge theories)
- First understand classical picture

Lagrangian Mechanics

- In Lagrangian mechanics $S = \int_{t_0}^{t_1} L\, dt$ as a functional on the path space $N^{[t_0,t_1]}$.
- Usual example: $L = \frac{1}{2}m||v||^2 - V(q)$.
- Newton's equation are recovered as Euler–Lagrange equations (EL), i.e., critical points: $\delta S = 0$.
- A solution is uniquely specified by its initial conditions. Set $C := TN$, the space of **Cauchy data**.
- For this, one sets conditions at t_0 and t_1 (usually by fixing the path endpoints). Otherwise

$$\delta S = \mathrm{EL} + \alpha|_{t_0}^{t_1},$$

$$\alpha = \sum_i \frac{\partial L}{\partial v^i} dq^i \in \Omega^1(C).$$

Here EL denotes the term containing the EL equations. By *EL* we will denote the space of solutions to EL.

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Symplectic formulation

$\omega := d\alpha$ is symplectic iff L is regular. In this case:

- ω is the pullback on $C = TN$ of the canonical symplectic form on T^*N by the Legendre mapping.
- Time evolution is given by a Hamiltonian flow ϕ. In particular,

$$L := \text{graph}\, \phi_{t_0}^{t_1} \in \overline{TN} \times TN$$

is Lagrangian (**canonical relation**).

Remark

L may also be defined directly as $L = \pi(EL)$ with

$$\begin{array}{rccc} \pi: & N^{[t_0,t_1]} & \to & TN \times TN \\ & \{x(t)\} & \mapsto & ((x(t_0), \dot{x}(t_0)), (x(t_1), \dot{x}(t_1))) \end{array}$$

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Example1: Geodesics

We discuss geodesics on $\mathbb{E}^2$ (Minkowski would be more realistic).

$$L = ||v||,$$

S is defined on $N_0^{[t_0,t_1]} := \{\text{immersed paths}\}$.

- $EL =$ straight lines
- Cauchy data: $C = \mathbb{R}^2 \times \mathbb{R}^2_* = \mathbb{R}^2 \times S^1 \times \mathbb{R}_{>0} \ni (\mathbf{q}, \mathbf{v}, \rho)$.
- $\alpha = \mathbf{v} \cdot \mathrm{d}\mathbf{q}$
- ω degenerate
- $L := \pi(EL) = \{(\mathbf{q}_1, \mathbf{v}, \rho_1), (\mathbf{q}_2, \mathbf{v}, \rho_2)) : \mathbf{q}_1 - \mathbf{q}_2 \,||\, \mathbf{v}\}$ Not a graph!

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Geodesics (continued)

However:

- $\omega|_L = 0$, so L is isotropic (actually Lagrangian).
- $\ker \omega(q, v) = \text{span}\left(\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{q}}, \frac{\partial}{\partial \rho}\right) =$ directions parallel to $\mathbf{v}$ and rescalings of velocity, so

$$\varpi \colon C \to \underline{C} := C/\ker \omega = TS^1$$

  with canonical symplectic form (identify T and T^* using the metric).
- $\underline{L} := \varpi(L) =$ graph Id, so a graph and Lagrangian.
- Actually, no time evolution after reduction (an example of topological theory).

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Example 2: Free 2d particle

- $S_M = \int_M \partial_\mu \phi \, \partial^\mu \phi$ on $\mathbb{R}^M$.
- $EL_M = \{\phi \in \mathbb{R}^M : \Delta\phi = 0\}$.
- Cauchy data (for M a cylinder $S^1 \times I$) $C_{S^1} = (\mathbb{R}^{S^1})^2$: field on S^1 together with its normal derivative.
- If ∂M consistst of n circles $\partial_1 M, \ldots, \partial_n M$:

$$\begin{array}{rccc} \pi : & \mathbb{R}^M & \to & C^n_{S^1} \\ & \phi & \mapsto & ((\phi_{\partial_1 M}, \mathbf{n} \cdot \nabla \phi_{\partial_1 M}), \ldots) \end{array}$$

- $L_M := \pi(EL_M)$ is a graph for M a cylinder, otherwise not a graph.
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General case (after V. Fock)

- Let $S_M = \int_M L$ be a class of local actions determined by a Lagrangian L. Here M is a d-manifold.
- S_M is defined on a space of fields F_M (e.g., maps from M to another manifold, connections on M, ...)
- $EL_M :=$ solutions to $\delta S_M = 0$ modulo boundary terms.
- Cauchy data: Let Σ be a $(d-1)$-dimensional manifold. $C_\Sigma :=$ fields on Σ that determine a unique solution to EL_M for $M = \Sigma \times [0, \epsilon]$, ϵ small.

By restricting the fields on the boundary, we have

$$\pi \colon F_M \to C_{\partial M}$$

The variation is now

$$\delta S_M = \mathrm{EL}_M + \pi^* \alpha_{\partial M} \tag{1}$$

where $\alpha_{\partial M}$ is determined by the boundary contributions.
Actually, working on $\Sigma \times [0, \epsilon]$, we have for every $(d-1)$-manifold Σ

$$\alpha_\Sigma \in \Omega^1(C_\Sigma)$$

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Boundary structure

- $\omega_\Sigma := d\alpha_\Sigma$ is a (pre)symplectic structure on C_Σ (symplectic iff L is regular).
- $L_M := \pi(EL_M)$ is isotropic in $C_{\partial M} \Longleftarrow$ (1) (in general not a graph) i.e., $\omega_{\partial M}|_{L_{\partial M}} = 0$(in all relevant examples $L_{\partial M}$ is Lagrangian)

Remark (Composition)

If $M = M_1 \cup_\Sigma M_2$, where Σ is (part of) the boundary of M_1 and of M_2,

$$L_M = L_{M_1} \circ L_{M_2} \subset C_{(\partial M_1 \setminus \Sigma) \coprod (\partial M_2 \setminus \Sigma)},$$

where $\circ$ denotes the composition of relations.

Definition

We call $L_{\partial M}$ the **evolution relation**. (More precisely, we split $\partial M = \partial_{\text{in}} M \coprod \partial_{\text{out}} M$ and regard L_M as a relation in $\overline{C_{(\partial_{\text{in}} M)^{\text{opp}}}} \times C_{\partial_{\text{out}} M}$.)

For a regular theory on a cylinder $M = \Sigma \times I$, L_M is a graph and the composition of cylinders yields the usual composition of maps.

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Boundary structure (continued)

Remark (EL)

By definition the fiber of EL_M over L_M is just one point if M is a short cylinder, but in general it may be much bigger.
So it makes sense to remember it and think of $EL_M \to C_{\partial_M}$ as a correspondence, the **evolution correspondence**.

Remark (Reduction)

If ω_Σ is degenerate, we may consider symplectic reduction

$$\varpi\colon C_\Sigma \to \underline{C_\Sigma}$$

and also consider reduced evolution relations

$$\underline{L_{\partial M}} := \varpi(L_{\partial M}) \subset \underline{C_{\partial M}}.$$

They are automatically isotropic.
In all known examples, they are Lagrangian. Maybe a theorem.

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Axiomatics

We may then think of a classical Lagrangian field theory in d dimensions as the following data:

- A space of field F_M for every d-manifold M
- A presymplectic space C_Σ for every $(d-1)$-manifold Σ
- An isotropic correspondence $\pi\colon EL_M \to C_{\partial M}$ for every M such that $\pi(EL_M)$ is Lagrangian after reduction.
- $(F_\bullet, C_\bullet)$ should be thought as a functor.

Remark

In the reduced picture (in case of trivial fibers), the target "category" is that of (singular) symplectic manifolds and canonical relations. Notice that the reduced evolution relation for a (short) cylinder is a graph, actually a flow. In particular,

$$\text{“}\lim_{\epsilon\to 0}\text{”}\, \underline{L_{\Sigma\times[0,\epsilon]}} = \text{graph}(\text{Id}_{\underline{C_\Sigma}}).$$

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- If ω_Σ is degenerate, we say that S defines a gauge theory.
- Notice that L_M is not a graph, even if M is a cylinder. In particular,

$$R_\Sigma := \text{“}\lim_{\epsilon\to 0}\text{''}\, L_{\Sigma\times[0,\epsilon]} \subset \overline{C_\Sigma} \times C_\Sigma$$

is not a graph.
It is an equivalence relation (gauge transformation) in C_Σ and

$$\underline{C_\Sigma} = C_\Sigma / R_\Sigma.$$

A topological field theory is a Lagrangian field theory that is invariant under diffeomorphisms.
So, in particular, it is a gauge theory and moreover

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for every interval I (no evolution).
One usually also requires all $\underline{C_\Sigma}$s to be finite dimensional (sometimes even compact).

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Quantization of regular Lagrangian field theories

- In a regular theory, C_Σ is symplectic;
  geometric quantization: Hilbert space H_Σ
- To the canonical relation $L_M \subset C_{\partial M}$ associate a state $\psi_M \in H_{\partial M}$.
  Asymptotically,
  $$\psi_M = \int e^{\frac{i}{\hbar} S_M} \in H_{\partial M}$$
  We integrate over bulk fields perturbing boundary fields (belonging to a section of the chosen polarization of $C_{\partial M}$).
- If $\partial M = \partial_{\text{in}} M \coprod \partial_{\text{out}} M$, then $\psi_M \in H^*_{\partial_{\text{in}} M} \otimes H_{\partial_{\text{out}} M}$.
  Hence, operator $H_{\partial_{\text{in}} M} \to H_{\partial_{\text{out}} M}$.
  Composition of relations goes to composition of operators.
- Cfr. Segal's axiomatization of CFT and Atiyah's axiomatization of TFT.

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Coisotropic submanifolds

If the Lagrangian is not regular, $(C_\Sigma, \omega_\Sigma)$ is not symplectic. It is better to think of $C_\Sigma \subset F^\partial_\Sigma$ with

- F^∂_Σ a symplectic space of fields (for every Σ)
- C_Σ is a coisotropic submanifold of F^∂_Σ i.e., $(T_{C_\Sigma} F^\partial_\Sigma)^\perp \subset TC_\Sigma$

We will call F^∂_Σ the **space of boundary fields**.

Remark

By a Theorem of Gotay every presymplectic manifold may be embedded in a symplectic manifold as a coisotropic submanifold. So we are going to assume that every C_Σ is a presymplectic manifold (i.e., smooth manifold and $\ker \omega_\Sigma$ a smooth subbundle of TC_Σ). This symplectic extension is locally unique.

We assume that π extends to $F_M \to F^\partial_{\partial M}$ as a surjective submersion.
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Coisotropic submanifolds

If the Lagrangian is not regular, $(C_\Sigma, \omega_\Sigma)$ is not symplectic. It is better to think of $C_\Sigma \subset F^\partial_\Sigma$ with

- F^∂_Σ a symplectic space of fields (for every Σ)
- C_Σ is a coisotropic submanifold of F^∂_Σ i.e., $(T_{C_\Sigma} F^\partial_\Sigma)^\perp \subset TC_\Sigma$

We will call F^∂_Σ the **space of boundary fields**.

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The BFV construction

Let C be a coisotropic submanifold of a symplectic manifold (F, ω). Denote by $C^\infty(C)^{\text{invt}}$ the Poisson algebra of functions invariant under the distribution $\ker \omega|_C$.
(If the reduction $\underline{C}$ is smooth, then $C^\infty(C)^{\text{invt}} = C^\infty(\underline{C})$.)
The Koszul–Tate resolution can be given the following form:

Theorem (Batalin–Fradkin–Vilkovisky, Stasheff, Schätz)

One can embed F in a graded symplectic manifold $\mathcal{F}$ and find a function S of degree 1 satisfying $\{S, S\} = 0$ s.t. $C^\infty(C)^{\text{invt}}$ is isomorphic, as a Poisson algebra, to the degree-zero cohomology of $C^\infty(\mathcal{F})$ with differential $Q = \{S, \ \}$.

This requires some assumptions, e.g., the finite dimensionality of C.
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Example

Suppose C is codimension one: $C =$ zeros of a function ϕ.
Let $X := \{\phi, \ \}$. Then

$$C^\infty(C)^{\text{invt}} = (C^\infty(F)/ < \phi >)^X.$$

- First add a new odd coordinate b (degree -1) and set $Qb = \phi$. Hence the degree zero cohomology is $C^\infty(C)$.
- Then add another odd coordinate c (degree $+1$) and set $Qf = c\,X(f)$, $f \in C^\infty(F)$. Now the degree zero cohomology is what we want.
- Extend the symplectic form by the term $\mathrm{d}b\,\mathrm{d}c$ and define $S := c\phi$.

The general case is treated similarly as a starting point. The symplectic form and S are then constructed iteratively in powers of the bs using cohomological perturbation theory [BFV, Stasheff]. It is eventually possible to globalize the construction [Schätz].
In field theory, one may arrange things so that S keeps being a local functional (often at the expense of introducing new coordinates of higher degree).

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Quantization

Working in geometric quantization:

- First assume that $\mathcal{F}$ can be quantized to a graded Hilbert space $\mathcal{H}$.
- Then assume that S can be quantized to an operator Ω (of degree 1) satisfying
$$\Omega^2 = 0$$
Notice that the classical condition $\{S, S\} = 0$ implies the quantum condition only up $\hbar^2$.
- Take the degree zero cohomology of the complex $(\mathcal{H}, \Omega)$ as the Hilbert space quantizing $\underline{C}$.

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Back to our boundary case

Using the BFV construction, we replace the boundary presymplectic manifold C_Σ by the data

$$(\mathcal{F}^\partial_\Sigma, \omega^\partial_\Sigma = \mathrm{d}\alpha^\partial_\Sigma, S^\partial_\Sigma, Q^\partial_\Sigma)$$

of an exact BFV manifold, where

1. ω^∂_Σ symplectic form of degree zero
2. S^∂_Σ of degree 1 satisfying $\{S^\partial_\Sigma, S^\partial_\Sigma\} = 0$
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$$\iota_{Q^\partial_\Sigma}\omega^\partial_\Sigma = \mathrm{d}S^\partial_\Sigma$$

One can also show that $\mathcal{C}^\partial_\Sigma := \{\text{zeros of } Q^\partial_\Sigma\}$ is coisotropic in $\mathcal{F}^\partial_\Sigma$ and $C^\infty(\mathcal{C}^\partial_\Sigma) = H^\bullet_{Q^\partial_\Sigma}$.

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The BV construction

In the bulk we have the problem that EL_M might also be singular. Moreover, if there are symmetries, we wish to consider the quotient $\underline{EL_M}$ as a starting point for perturbation theory in the functional integral.

- If M has no boundary, the Batalin–Vilkovisky (BV) construction yields a BV manifold

$$(\mathcal{F}_M, \omega_M, S_M, Q_M)$$

satisfying the same equations as in BFV but

1. ω_M has degree -1 and S_M has degree zero.
2. F_M is a submanifold of $\mathcal{F}_M$ and S_M is an extension of the classical action.
3. $\underline{EL_M} = \{\text{zeros of } Q_M\}_{\text{degree zero}}$ and $C^\infty(\underline{EL_M})^{\text{invt}} =$ degree zero cohomology of $(C^\infty(\mathcal{F}_M), Q_M)$.

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The case with boundary

The equation

$$\iota_{Q_M}\omega_M = \mathrm{d}S_M$$

no longer holds if M has boundary. We have to deal with the boundary terms as in the first part of this talk.

Putting BV+BFV+Fock together, we get the following axiomatics [C, Mnëv, Reshetikhin]:

- To each $(d-1)$-manifold Σ we associate a BFV-manifold $\mathcal{F}^\partial_\Sigma, \omega^\partial_\Sigma = \mathrm{d}\alpha^\partial_\Sigma, S^\partial_\Sigma, Q^\partial_\Sigma)$.
- To each d-manifold M we associate the data $(\mathcal{F}_M, \omega_M, S_M, Q_M)$ together with a surjective submersion $\pi\colon \mathcal{F}_M \to \mathcal{F}^\partial_{\partial M}$ satisfying:
  1. $Q^\partial_{\partial M} = \mathrm{d}\pi Q_M$;
  2. $\iota_{Q_M}\omega_M = \mathrm{d}S_M + \pi^*\alpha^\partial_{\partial M}$.
- Plus functoriality and some regularity assumptions.

Several examples:

YM, *BF*, CS, PSM (actually, all AKSZ theories)

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Example: Electromagnetism

- Maxwell's equations: $\mathrm{d}^*\mathrm{d}A = 0$, A connection 1-form.
- First-order formalism: $S_M^{\text{cl}} = \int_M B\,\mathrm{d}A + \frac{1}{2}B * B$
  B a $(d-2)$-form. Then $EL = \{*B = \mathrm{d}A,\ \mathrm{d}B = 0\}$.
- BV: $S_M = \int_M B\,\mathrm{d}A + \frac{1}{2}B * B + A^+\,\mathrm{d}c$
  A^+: $(d-1)$-form, ghost number -1; c: 0-form, ghost number 1.
  $\omega_M = \int_M \delta A\,\delta A^+ + \delta B\,\delta B^+ + \delta c\,\delta c^+$,
  B^+ and c^+ do not show up in the action.
  $QA = \mathrm{d}c,\ QA^+ = \mathrm{d}B,\ QB^+ = *B + \mathrm{d}A,\ Qc^+ = \mathrm{d}A^+$.
- Boundary fields: A, B, A^+, c,
  $S_\Sigma^\partial = \int_\Sigma c\,\mathrm{d}B$,
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Properties

The fundamental equation

$$\iota_{Q_M}\omega_M = \mathrm{d}S_M + \pi^*\alpha^\partial_{\partial M} \tag{2}$$

has several consequences:

1. $L_{Q_M}\omega_M = \pi^*\omega^\partial_{\partial M}$ (Q_M not symplectic).
2. $Q_M(S_M) = 2S^\partial_{\partial M} - \pi^*(\iota_{Q^\partial_{\partial M}}\alpha^\partial_{\partial M})$ (modified CME).
3. $\mathcal{EL}_M := \{\text{zeros of } Q_M\}$ coisotropic,

$$\mathcal{L}_M := \pi(\mathcal{EL}_M) \overset{\text{isotropic/Lagrangian}}{\subset} \mathcal{C}^\partial_{\partial M} \overset{\text{coisotropic}}{\subset} \mathcal{F}^\partial_{\partial M}.$$

4. For every $\ell \in \mathcal{L}_M$, let
$\mathcal{E}_\ell := \pi^{-1}$(orbit through ℓ of coisotropic foliation).
Then $\mathcal{E}_\ell$ presymplectic and we have a fibration $\underline{\mathcal{EL}_M} \to \underline{\mathcal{L}_M}$ with finite dimensional odd symplectic fiber $\underline{\mathcal{E}_\ell}$ over $\underline{\ell}$.
BV canonical correspondence

Example EM:
$\underline{\mathcal{E}_\ell} = H^1(M, \partial M) \oplus H^{n-1}(M)[-1] \oplus H^0(M, \partial M)[1] \oplus H^n(M)[-2]$

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   Then $\mathcal{E}_\ell$ presymplectic and we have a fibration $\underline{\mathcal{EL}_M} \to \underline{\mathcal{L}_M}$ with finite dimensional odd symplectic fiber $\underline{\mathcal{E}_\ell}$ over $\underline{\ell}$.

BV canonical correspondence

Example EM:
$\underline{\mathcal{E}_\ell} = H^1(M, \partial M) \oplus H^{n-1}(M)[-1] \oplus H^0(M, \partial M)[1] \oplus H^n(M)[-2]$

Boundaries of boundaries

Sometimes it is possible to push this construction to even lower dimension.

For example in EM:

- Boundary fields: A, B, A^+, c, $S^\partial_\Sigma = \int_\Sigma c\, \mathrm{d}B$, $\alpha^\partial_\Sigma = \int_\Sigma B\,\delta A + A^+\,\delta c$, $Q^\partial A^+ = \mathrm{d}B$, $Q^\partial A = \mathrm{d}c$.
- Boundary of boundary: $\gamma = (d-2)$-manifold
  BB fields: B, c, $\alpha^{\partial\partial}_\gamma = \int_\gamma B\,\delta c$, of degree $+1$
  $S^{\partial\partial}_\gamma = 0$, $Q^{\partial\partial}_\gamma = 0$.
- Again we have $\iota_{Q^\partial_\Sigma}\omega^\partial_\Sigma = \mathrm{d}S^\partial_\Sigma + \pi^*\alpha^{\partial\partial}_{\partial\Sigma}$
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$$\underline{\mathcal{EL}_\Sigma} = \Omega^1(\Sigma)/_{\text{exact}} \oplus \Omega^{d-2}_{\text{closed}}(\Sigma,\partial\Sigma) \oplus H^0(\Sigma,\partial\Sigma)[1] \oplus H^{d-1}(\Sigma)[-1].$$

For $d = 2$ this space is finite dimensional.

In CS, *BF*, and all AKSZ theories, one can go down up to zero dimensions!

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Quantization

1. Fix a polarization on $\mathcal{F}^{\partial}_{\partial M}$ such the quantization $\Omega_{\partial M}$ of $S^{\partial}_{\partial M}$ squares to zero.
2. For simplicity, assume we have a transversal $\mathcal{L}'$ to the polarization. So $\mathcal{H}_{\partial M} =$ functions on $\mathcal{L}'$.
3. Define

$$\psi_M = \int e^{\frac{i}{\hbar} S_M} \in \mathcal{H}_{\partial M}$$

where the integral is over a Lagrangian submanifold of the fiber over a boundary field in $\mathcal{L}'$.

4. By standard techniques in BV, one may prove that

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Perturbative quantization

Usually, the only way of computing the functional integral is to perturb around a Gaussian theory.
Let S^0 be the Gaussian theory and denote by $\mathcal{Z}_M^0$ the space of functions on the fiber of $\underline{\mathcal{E}\mathcal{L}}_M^0$ (“vacua"). Then

1. We get

$$\psi_M = \int e^{\frac{i}{\hbar} S_M} \in \mathcal{H}_{\partial M} \otimes \mathcal{Z}_M^0$$

2. Because of the odd symplectic structure on these fibers, $\mathcal{Z}_M^0$ has a BV structure. The modified CME is quantized as

$$\Delta_{\mathcal{Z}_M^0} \psi_M + \Omega_{\partial M} \psi_M = 0$$

Setting $\psi_M = e^{\frac{i}{\hbar} S_{\text{eff}}}$, we get the modified QME

$$\{S_{\text{eff}}, S_{\text{eff}}\} - i\hbar \Delta_{\mathcal{Z}_M^0} S_{\text{eff}} + (i\hbar)^2 e^{-\frac{i}{\hbar} S_{\text{eff}}} \Omega_M e^{\frac{i}{\hbar} S_{\text{eff}}} = 0.$$

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Axiomatics

- To each $(d-1)$-manifold Σ we associate a complex $(\mathcal{H}_\Sigma, \Omega_\Sigma)$ of Hilbert spaces.
- To each d-manifold we as associate a f.d. BV manifold $\underline{\mathcal{EL}}_M$ ("moduli space of vacua"), the BV algebra $\mathcal{Z}_M$ of functions on $\underline{\mathcal{EL}}_M$ (endowed with a BV operator Δ), and an element ψ_M of $\mathcal{H}_{\partial M} \otimes \mathcal{Z}_M$ satisfying the modified QME.
- Plus functorial properties.

Eventually, we may integrate over a Lagrangian submanifold of $\underline{\mathcal{EL}}_M$ and go to the Ω_Σ-cohomology getting just a state in the physical Hilbert space.

Remark

The full power of this approach is that we may cut the original manifold M into simple, or tiny, pieces; do the perturbative quantization there; and eventually glue and reduce.
This could provide some new insight for physical theories.
In TFTs it yields a perturbative version of Atiyah's axioms. We expect to be able to compute, e.g., perturbative CS invariants.

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Example: *BF* theory

$$S = \int_M \left\langle B, \mathrm{d}A + \frac{1}{2}[A, A] \right\rangle, A \in \Omega(M, \mathfrak{g}), B \in \Omega(M, \mathfrak{g}^*)$$

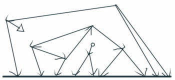
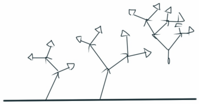


Figure: $\frac{\delta}{\delta B}$-foliation



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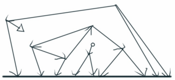

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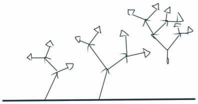

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