Bundles over quantum projective spaces

Tomasz Brzeziński

Swansea University

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References:

- TB & B Zieliński “Quantum principal bundles over quantum real projective spaces”, arxiv:1105.5897
- TB & SA Fairfax “Quantum teardrops”, arxiv:1107.1417

Quantum principal bundles

Definition

Let H be a Hopf algebra with bijective antipode and let A be a right H-comodule algebra with coaction $\varrho : A \to A \otimes H$. Let $B = A^{coH} := \{b \in A \mid \varrho(b) = b \otimes 1\}$. A is a *principal H-comodule algebra* if:

(a) the coaction ϱ is free, that is the canonical map

$$\mathrm{can} : A \otimes_B A \to A \otimes H, \qquad a \otimes a' \mapsto a\varrho(a'),$$

is bijective (the Hopf-Galois condition);

(b) there exists a strong connection in A, that is

$$B \otimes A \to A, \qquad b \otimes a \mapsto ba,$$

splits as a left B-module and right H-comodule map (the equivariant projectivity).

Strong connections

A right H-comodule algebra A with coaction $\varrho : A \to A \otimes H$ is principal if and only if it admits a *strong connection*, that is if there exists a map $\omega : H \longrightarrow A \otimes A$, such that

$$\omega(1) = 1 \otimes 1,$$

$$\mu \circ \omega = \eta \circ \varepsilon,$$

$$(\omega \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varrho) \circ \omega,$$

$$(S \otimes \omega) \circ \Delta = (\sigma \otimes \mathrm{id}) \circ (\varrho \otimes \mathrm{id}) \circ \omega.$$

Here $\mu : A \otimes A \to A$ denotes the multiplication map, $\eta : \mathbb{C} \to A$ is the unit map, $\Delta : H \to H \otimes H$ is the comultiplication, $\varepsilon : H \to \mathbb{C}$ counit and $S : H \to H$ the (bijective) antipode of the Hopf algebra H, and $\sigma : A \otimes H \to H \otimes A$ is the flip.

Trivial bundles

Cleft extension: an H-comodule algebra admitting a right H-colinear map $j : H \to A$ such that $j(1) = 1$, and for which there exists the *convolution inverse* $j^{-1} : H \to A$,

$$j(h_{(1)})j^{-1}(h_{(2)}) = j^{-1}(h_{(1)})j(h_{(2)}) = \varepsilon(h)1,$$

is principal. A strong connection can be defined as

$$\omega = (j^{-1} \otimes j) \circ \Delta.$$

Equivalently: principal H-comodule algebras isomorphic to $A^{coH} \otimes H$ as left A^{coH}-modules and right H-comodules. A right H-colinear algebra map $j : H \to A$ is convolution invertible with $j^{-1} = j \circ S$, so A is a cleft principal comodule algebra termed a *trivial principal comodule algebra*.

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Prolongations

Definition

Let $\bar{A}$ be an $\bar{H}$-comodule algebra. Denote by $B = \bar{A}^{\mathrm{co}\bar{H}}$ a subalgebra of $\bar{H}$-coaction invariant elements. Let $\pi : H \to \bar{H}$ be a Hopf algebra map and consider the cotensor product

$$A = \bar{A}\Box_{\bar{H}}H \quad := \quad \{\sum_i a^i \otimes h^i \in \bar{A} \otimes H \mid \sum_i a^i{}_{(0)} \otimes a^i{}_{(1)} \otimes h^i = \sum_i a^i \otimes \pi(h^i{}_{(1)}) \otimes h^i{}_{(2)}\}.$$

View A as a right H-comodule subalgebra of the tensor algebra $\bar{A} \otimes H$ with the coaction $\mathrm{id} \otimes \Delta_H$. A is called a *prolongation* of $\bar{A}$.

Question: When is A a principal H-comodule algebra?

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[H-J Schneider '90]
If $\bar{A}$ is a principal $\bar{H}$-comodule algebra, then $\bar{A}\Box_{\bar{H}}H$ is a principal H-comodule algebra, with a strong connection

$$\omega : H \longrightarrow (\bar{A}\Box_{\bar{H}}H) \otimes (\bar{A}\Box_{\bar{H}}H),$$

$$\omega = (\sigma \otimes \mathrm{id} \otimes \mathrm{id}) \circ (S \otimes (\bar{\omega} \circ \pi) \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_H) \circ \Delta_H.$$

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Question: When is A cleft?

Cleft prolongations

Theorem

The prolongation $\bar{A}\Box_{\bar{H}}H$ is a cleft extension of B if and only if there exists a right $\bar{H}$-colinear, unital, convolution invertible map $f : H \to \bar{A}$.

The hierarchy of spheres

Let q be a real number, $0 < q < 1$. The coordinate algebra $\mathcal{O}(S_q^{2n+1})$ of the odd-dimensional quantum sphere is the unital complex $*$-algebra with generators $z_0, z_1, \ldots, z_n$ subject to the following relations:

$$z_i z_j = q z_j z_i \quad \text{for } i < j, \qquad z_i z_j^* = q z_j^* z_i \quad \text{for } i \neq j,$$

$$z_i z_i^* = z_i^* z_i + (q^{-2} - 1) \sum_{m=i+1}^{n} z_m z_m^*, \qquad \sum_{m=0}^{n} z_m z_m^* = 1.$$

The coordinate algebra $\mathcal{O}(S_q^{2n})$ of the even-dimensional quantum sphere is the unital complex $*$-algebra with generators $z_0, z_1, \ldots, z_n$ and above relations supplemented with $z_n^* = z_n$.

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The hierarchy of spheres

- Quantum spheres are right comodule algebras over the Hopf algebra $\mathcal{O}(\mathbb{Z}_2)$ generated by a self-adjoint grouplike element u satisfying $u^2 = 1$. The coaction is

$$z_i \longmapsto z_i \otimes u.$$

- The *quantum real projective space* $\mathcal{O}(\mathbb{RP}^m_q)$ is defined as the $\mathcal{O}(\mathbb{Z}_2)$-coinvariant subalgebra of $\mathcal{O}(S^m_q)$.
- $\mathcal{O}(S^{2n}_q)$, $\mathcal{O}(S^{2n+1}_q)$ admit a strong connection

$$\omega(u) = \sum_{i=0}^{n} z_i \otimes z_i^*.$$

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The hierarchy of spheres

Quantum spheres form a hierarchy of right $\mathcal{O}(\mathbb{Z}_2)$-comodule $*$-algebras

$$\cdots \mathcal{O}(S_q^5) \xrightarrow{f_4} \mathcal{O}(S_q^4) \xrightarrow{f_3} \mathcal{O}(S_q^3) \xrightarrow{f_2} \mathcal{O}(S_q^2) \xrightarrow{f_1} \mathcal{O}(S_q^1),$$

where each of the f_m is a surjective $*$-algebra and right $\mathcal{O}(\mathbb{Z}_2)$-colinear map defined on generators as follows. In the odd case

$$f_{2n-1} : \mathcal{O}(S_q^{2n}) \longrightarrow \mathcal{O}(S_q^{2n-1}), \qquad z_i \longmapsto \begin{cases} z_i & \text{if } i \neq n \\ 0 & \text{if } i = n \end{cases}$$

In the even case

$$f_{2n} : \mathcal{O}(S_q^{2n+1}) \longrightarrow \mathcal{O}(S_q^{2n}), \qquad z_i \longmapsto z_i.$$

Quantum unitary groups

- $\mathcal{O}(S^1_q) = \mathcal{O}(U(1))$ is a commutative Hopf algebra generated by a unitary grouplike element v, and there is a Hopf algebra map

$$\pi_2 : \mathcal{O}(U(1)) \longrightarrow \mathcal{O}(\mathbb{Z}_2), \qquad v \longmapsto u.$$

- $\mathcal{O}(S^3_q)$ is the coordinate algebra of the quantum group $SU_q(2)$. The composites $f_1 \circ f_2 : \mathcal{O}(SU_q(2)) \to \mathcal{O}(U(1))$ and

$$\pi = \pi_2 \circ f_1 \circ f_2 : \mathcal{O}(SU_q(2)) \longrightarrow \mathcal{O}(\mathbb{Z}_2),$$

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Prolongations to $SU_q(2)$-bundles

Since the algebra maps f_m are also right $\mathcal{O}(\mathbb{Z}_2)$-colinear, the prolongation along π gives rise to the following hierarchy of quantum principal $SU_q(2)$-bundles over $\mathbb{RP}^m_q$:

$$\mathcal{O}(S_q^{m+1})\Box_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(SU_q(2)) \xrightarrow{f_m\otimes\mathrm{id}} \mathcal{O}(S_q^{m})\Box_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(SU_q(2)) \,.$$

- $f_2 : \mathcal{O}(SU_q(2)) \to \mathcal{O}(S_q^2)$ is a right $\mathcal{O}(\mathbb{Z}_2)$-colinear algebra map, hence $\mathcal{O}(S_q^2)\Box_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(SU_q(2))$ is cleft (trivial).
- $\mathcal{O}(S_q^3)\Box_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(SU_q(2))$ is trivial via the identity map $\mathcal{O}(SU_q(2)) \to \mathcal{O}(S_q^3)$.
- for $m > 3$ probably non-trivial (classically, by the Borsuk-Ulam theorem: there are no continuous odd (i.e. $\mathbb{Z}_2$-equivariant) functions $S^{n+1} \to S^n$).

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Prolongations to $U(1)$-bundles

Use the Hopf algebra map $\pi_2 : \mathcal{O}(U(1)) \to \mathcal{O}(\mathbb{Z}_2)$ to construct prolongations of the $\mathcal{O}(S_q^m)$ to principal $\mathcal{O}(U(1))$-comodule algebras. Then there is a sequence of surjective algebra maps between principal $\mathcal{O}(U(1))$-comodule algebras

$$\mathcal{O}(S_q^{m+1})\Box_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(U(1)) \xrightarrow{f_m \otimes \mathrm{id}} \mathcal{O}(S_q^m)\Box_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(U(1)) \,.$$

Theorem

For all natural numbers $m > 1$, the principal $\mathcal{O}(U(1))$-comodule algebras $\mathcal{O}(S_q^m)\Box_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(U(1))$ are non-trivial.

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The structure of $\mathcal{O}(S_q^m)\square_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(U(1))$

Theorem

1. $\mathcal{O}(S_q^{2n+1})\square_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(U(1))$ *is an algebra isomorphic to* $\mathcal{O}(S_q^{2n+1}) \otimes \mathcal{O}(U(1))$.
2. $\mathcal{O}(S_q^{2n})\square_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(U(1))$ *is isomorphic to a polynomial* $*$*-algebra* $\mathcal{A}^{2n}$ *generated by* $\zeta_0, \zeta_1, \ldots, \zeta_n$ *and a central unitary* ξ *subject to the following relations*

$$\zeta_i\zeta_j = q\zeta_j\zeta_i \quad \text{for } i < j, \qquad \zeta_i\zeta_j^* = q\zeta_j^*\zeta_i \quad \text{for } i \neq j,$$

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Quantum weighted projective spaces [TB & SAF]

- For any $n+1$ pairwise coprime numbers $l_0, \ldots, l_n$, $\mathcal{O}(U(1))$ coacts on $\mathcal{O}(S_q^{2n+1})$, as

$$\varrho_{l_0,\ldots,l_n} : z_i \mapsto z_i \otimes v^{l_i}, \qquad i = 0, 1, \ldots, n.$$

- The *quantum weighted projective space* is the subalgebra of $\mathcal{O}(S_q^{2n+1})$ containing all elements invariant under the coaction $\varrho_{l_0,\ldots,l_n}$, i.e.

$$\mathcal{O}(\mathbb{WP}_q(l_0, l_1, \ldots, l_n)) = \{x \in \mathcal{O}(S_q^{2n+1}) \mid \varrho_{l_0,\ldots,l_n}(x) = x \otimes 1\}.$$

In the case $l_0 = l_1 = \cdots = 1$ one obtains the algebra of functions on the quantum complex projective space $\mathbb{CP}_q^n$ defined in **[YaS Soibel'man & LL Vaksman '90]**.

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Quantum teardrops

The quantum teardrop $\mathcal{O}(\mathbb{WP}_q(k,l))$ is the $*$-algebra generated by a and b subject to the following relations

$$a^* = a, \quad ab = q^{-2l}ba, \quad bb^* = q^{2kl}a^k \prod_{m=0}^{l-1}(1 - q^{2m}a),$$

$$b^*b = a^k \prod_{m=1}^{l}(1 - q^{-2m}a).$$

Representations

Theorem

Up to a unitary equivalence, the following is the list of all bounded irreducible $*$-representations of $\mathcal{O}(\mathbb{WP}_q(k,l))$.

(i) One dimensional representation

$$\pi_0 : a \mapsto 0, \qquad b \mapsto 0.$$

(ii) Infinite dimensional (faithful) representations $\pi_s : \mathcal{O}(\mathbb{WP}_q(k,l)) \to \mathrm{End}(V_s)$, $s = 1, 2, \ldots, l$. $V_s \simeq l^2(\mathbb{N})$ *has orthonormal basis* e_p^s, $p \in \mathbb{N}$, $\pi_s(b)e_0^s = 0$ *and*

$$\pi_s(a)e_p^s = q^{2(lp+s)}e_p^s,$$

$$\pi_s(b)e_p^s = q^{k(lp+s)}\prod_{r=1}^{l}\left(1 - q^{2(lp+s-r)}\right)^{1/2}e_{p-1}^s.$$

A "no-go theorem"

Theorem

The algebra $\mathcal{O}(S_q^3)$ is a principal $\mathcal{O}(U(1))$-comodule algebra over $\mathcal{O}(\mathbb{WP}_q(k,l))$ by the coaction $\varrho_{k,l}$ if and only if $k = l = 1$ (the quantum Hopf fibration).

Quantum lens spaces

- **[JH Hong & W Szymański '03]** The *quantum lens space* $\mathcal{O}(L_q(l;1,l))$ is a $*$-algebra generated by c and d:

$$cd = q^l dc, \qquad cd^* = q^l d^* c, \quad dd^* = d^* d,$$

$$cc^* = \prod_{m=0}^{l-1} (1 - q^{2m} dd^*), \quad c^* c = \prod_{m=1}^{l} (1 - q^{-2m} dd^*).$$

- $\mathcal{O}(L_q(l;1,l))$ is a right comodule algebra over $\mathcal{O}(U(1))$,

$$\varrho_l : c \mapsto c \otimes v, \qquad d \mapsto d \otimes v^*.$$

- $\mathcal{O}(L_q(l;1,l))^{co\mathcal{O}(U(1))} \simeq \mathcal{O}(\mathbb{WP}_q(1,l))$, $(a = cd,\ b = dd^*)$.

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The principality

Theorem

The coordinate algebra of the quantum lens space $\mathcal{O}(L_q(l;1,l))$ is a (non-cleft) principal $\mathcal{O}(U(1))$-comodule algebra over $\mathcal{O}(\mathbb{WP}_q(1,l))$.

A strong connection is defined recursively: $\omega(1) = 1 \otimes 1$, and then, for all $n > 0$,

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