Holographic fluids and vorticity in 2 + 1 *dimensions*

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Highlights

Motivations

Holographic fluids

AdS Kerr & Taub–NUT backgrounds

Alternative interpretations

Outlook

Framework

$AdS/CFT \rightarrow QCD \& plethora of strongly coupled systems$

- Superconductors and superfluids [Hartnoll, Herzog, Horowitz '08]
- Strange metals [e.g. Faulkner et al. '09]
- Quantum-Hall fluids [e.g. Dolan et al. '10]

Holography also applied to hydrodynamics i.e. to a regime of local thermodynamical equilibrium for the boundary theory

- ► Universal bound $\eta/s \ge \hbar/4\pi k_{\rm B}$ saturated in holographic fluids (nearly-perfect) [Policastro, Son, Starinets '01, Baier *et al.* '07, Liu *et al.* '08]
- ► More systematic description of fluid dynamics [Minwalla et al. '08]

Why vorticity?

Developments in ultra-cold-atom physics: new twists in the physics of near-perfect neutral fluids fast rotating in normal or superfluid phase \rightarrow new challenges in strong-coupling regimes

Dilute rotating Bose gases in harmonic traps – potentially fractional-quantum-Hall liquids or topological (anyonic) superfluids [e.g. Cooper et al. '10, Chu et al. '10, Dalibard et al. '11]



Figure: Trap, rotation and Landau levels – toward a strongly coupled FQH phase for small filling factor ($\nu = \text{particles/vortices} \approx 1$)

Strongly interacting Fermi gases above BEC behave like near-perfect fluids with very low η/s [Shaefer et al. '09, Thomas et al. '09]



Figure: Irrotational elliptic flow in very small η/s rotating fluid – rotates *faster* as it expands due to *inertia moment quenching*

Foreseeable progress in the measurement of transport coefficients calls for a better understanding of the strong-coupling dynamics of vortices Developments in analogue-gravity systems for the description of sound/light propagation in moving media – sometimes in super sonic/superluminal vortex flows [Leonhardt et al. '00, Barcelo et al. '05, Cacciatori et al. '10]



Figure: White hole's horizon in analogue gravity

- Optical & acoustic black holes and Hawking radiation
- Vortices and Aharonov–Bohm effect

Use AdS/CFT to describe rotating fluids viewed

- either as rotating near-perfect Bose or Fermi gases
- or as analogue-gravity set ups for acoustics/optics in rotating media [see also Schüfer et al. '09, Das et al. '10]

Here

Starting from a 3 + 1 dim asymptotically AdS background a 2 + 1 dim holographic dual appears as a set of boundary data

- boundary frame
- boundary stress tensor

Within hydrodynamics, data interpreted as a 2 + 1 dim fluid moving in a background – generically with vorticity

- Kerr AdS
- Taub–NUT AdS

will serve to illustrate various properties

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Applied beyond the original framework – maximal susy YM in D = 4 – usually in the classical gravity approximation without backreaction

- Bulk with $\Lambda = -3k^2$: asymptotically AdS $d = D + 1 \dim \mathcal{M}$
- ▶ Boundary at $r \rightarrow \infty$: asymptotic coframe $E^{\mu} \mu = 0, ..., D-1$

$$ds^{2} \approx \frac{dr^{2}}{k^{2}r^{2}} + k^{2}r^{2}\eta_{\mu\nu}E^{\mu}E^{\nu} = \frac{dr^{2}}{k^{2}r^{2}} + k^{2}r^{2}g_{(0)\mu\nu}dx^{\mu}dx^{\nu}$$

Holography: determination of $\langle \mathcal{O} \rangle_{\text{bry. F.T.}}$ as a response to a boundary source perturbation $\delta \phi_{(0)}$ (momentum vs. field in Hamiltonian formalism – related via some regularity condition)

Pure gravity

Holographic data

- ► Field $g_{rr}, g_{\mu\nu} \rightarrow g_{(o)\mu\nu}$: boundary metric source
- Momentum T_{rr} , $T_{\mu\nu} \rightarrow T_{(o)\mu\nu}$: $\langle T_{(o)\mu\nu} \rangle$ response

Palatini formulation and 3 + 1 split [Leigh, Petkou '07, Mansi, Petkou, Tagliabue '08]

- $heta^a$: orthonormal coframe d $s^2 = \eta_{ab} \theta^a \theta^b$ ($\eta: +-++$)
 - ► Vierbein: $\theta^r = N \frac{dr}{kr}$ $\theta^\mu = N^\mu dr + \tilde{\theta}^\mu$ $\mu = 0, 1, 2$
 - Gauge choice: N = 1 and $N^{\mu} = 0 \rightarrow \tilde{\theta}^{\mu} = \theta^{\mu}$

Holography: Hamiltonian evolution from data on the boundary – captured in Fefferman–Graham expansion for large r [Fefferman, Graham '85]

$$\theta^{\mu}(r,x) = kr \, E^{\mu}(x) + \frac{1}{kr} F^{\mu}_{[2]}(x) + \frac{1}{k^2 r^2} F^{\mu}(x) + \cdots$$

Independent 2 + 1 boundary data: vector-valued 1-forms E^{μ} and F^{μ}

- ► E^{μ} : boundary orthonormal coframe allows to determine $ds_{bry.}^2 = g_{(0)\mu\nu} dx^{\mu} dx^{\nu} = \eta_{\mu\nu} E^{\mu} E^{\nu}$
- ► F^{μ} : stress-tensor current one-form allows to construct the boundary stress tensor ($\kappa = \frac{3k}{8\pi G}$) $T = \kappa F^{\mu} e_{\mu} = T^{\mu}_{\ \nu} E^{\nu} \otimes e_{\mu}$

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AdS Kerr: the solid rotation

The bulk data

$$ds^{2} = (\theta^{r})^{2} - (\theta^{t})^{2} + (\theta^{\theta})^{2} + (\theta^{\varphi})^{2}$$

$$= \frac{d\tilde{r}^{2}}{V(\tilde{r},\theta)} - V(\tilde{r},\theta) \left[dt - \frac{a}{\Xi}\sin^{2}\theta d\phi\right]^{2}$$

$$+ \frac{\rho^{2}}{\Delta_{\theta}}d\theta^{2} + \frac{\sin^{2}\theta\Delta_{\theta}}{\rho^{2}} \left[a dt - \frac{r^{2} + a^{2}}{\Xi}d\phi\right]^{2}$$

 $V(\tilde{r}, \vartheta) = \Delta_r / \rho^2$ with

$$\begin{array}{ll} \Delta_r &= \left(\tilde{r}^2 + a^2\right) \left(1 + k^2 \tilde{r}^2\right) - 2M\tilde{r} \\ \rho^2 &= \tilde{r}^2 + a^2 \cos^2 \vartheta \\ \Delta_\vartheta &= 1 - k^2 a^2 \cos^2 \vartheta \\ \Xi &= 1 - k^2 a^2 \end{array}$$

The boundary metric – following FG expansion

$$ds_{\text{bry.}}^{2} = \eta_{\mu\nu} E^{\mu} E^{\nu} = g_{(0)\mu\nu} dx^{\mu} dx^{\nu}$$
$$= -\left(dt - \frac{a\sin^{2}\vartheta}{\Xi}d\varphi\right)^{2} + \frac{1}{k^{2}\Delta_{\vartheta}}\left(d\vartheta^{2} + \left(\frac{\Delta_{\vartheta}\sin\vartheta}{\Xi}\right)^{2}d\varphi^{2}\right)$$

• conformal to Einstein universe in a *rotating frame* (requires $(\vartheta, \varphi) \rightarrow (\vartheta', \varphi')$) – spatial section: squashed 2-sphere

• $\nabla_{\partial_t} \partial_t = 0$: observers at rest are *inertial*

The boundary stress tensor $\kappa F^{\mu} e_{\mu}$ [see also Caldarelli, Dias, Klemm '08]

$$T = T_{\mu\nu} E^{\mu} E^{\nu} = \frac{\kappa M k}{3} \left(2(E^{t})^{2} + (E^{\vartheta})^{2} + (E^{\varphi})^{2} \right)$$

perfect-fluid-like ($T = (\varepsilon + p)u \otimes u + p\eta_{\mu\nu}E^{\mu} \otimes E\nu$)

- traceless: conformal fluid with $\varepsilon = 2p = \frac{2\kappa Mk}{3}$
- velocity one-form: $u = -E^t = -dt + b$
- velocity field $\mathbf{u} = e_t = \partial_t$: comoving & inertial

Fluid without expansion and shear but with vorticity

$$\omega = \frac{1}{2} \mathrm{d}u = \frac{1}{2} \mathrm{d}b = \frac{a \cos \vartheta \sin \vartheta}{\Xi} \mathrm{d}\vartheta \wedge \mathrm{d}\varphi = k^2 a \cos \vartheta E^\vartheta \wedge E^\varphi$$

Reminder: $u \to \nabla_{\mu} u_{\nu} \to \{\sigma_{\mu\nu}, \Theta, \omega_{\mu\nu}\}$

How is vorticity i.e. rotation observed?

Boundary geometries are stationary of Randers form [Randers '41]

$$\mathrm{d}s^2 = -\left(\mathrm{d}t - b\right)^2 + a_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

and the fluid is at rest: $\mathbf{u} = \partial_t$

- $\nabla_{\partial_t} \partial_t = 0$: the fluid is inertial
- ► $\omega = \frac{1}{2} db$ and $\nabla_{\partial_t} \partial_i = \omega_{ij} a^{jk} (\partial_k + b_k \partial_t)$: frame and fluid dragging

Other privileged frames exist where the observers experience differently the rotation of the fluid - e.g. Zermelo dual frame

AdS Taub–NUT: the nut charge

The bulk data [Taub '51, Newman, Tamburino, Unti '63]

$$ds^{2} = (\theta^{r})^{2} - (\theta^{t})^{2} + (\theta^{\vartheta})^{2} + (\theta^{\varphi})^{2}$$
$$= \frac{d\tilde{r}^{2}}{V(\tilde{r})} - V(\tilde{r}) \left[dt - 2n\cos\vartheta \, d\varphi \right]^{2} + \rho^{2} \left[d\vartheta^{2} + \sin^{2}\vartheta \, d\varphi \right]^{2}$$
$$/(\tilde{r}) = \Delta_{r}/\rho^{2} \text{ with}$$

$$\Delta_{r} = (\tilde{r}^{2} - n^{2}) (1 + k^{2} (\tilde{r}^{2} + 3n^{2})) + 4k^{2} n^{2} \tilde{r}^{2} - 2M\tilde{r}$$

$$\rho^{2} = \tilde{r}^{2} + n^{2}$$

One of the most peculiar solutions to Einstein's Eqs. [Misner '63]

Parenthesis: Kerr vs. Taub–NUT (Lorentzian time)

Taub–NUT: rich geometry – foliation over squashed 3-spheres with $SU(2) \times U(1)$ isometry (homogeneous and axisymmetric)

- ► horizon at $r = r_+ \neq n$: 2-dim fixed locus of $-2n\partial_t \rightarrow bolt$ (Killing becoming light-like)
- extra fixed point of $\partial_{\varphi} 4n\partial_t$ on the horizon at $\vartheta = \pi$

nut at $r = r_+$, $\vartheta = \pi$ from which departs a *Misner string* (coordinate singularity if $t \ncong t + 8\pi n$) [Misner '63]

Kerr: stationary (rotating) black hole

- ▶ horizon at $r = r_+$: fixed locus of $\partial_t + \Omega_H \partial_{\varphi} \rightarrow \text{bolt}$
- ▶ pair of nut-anti-nut at r = r₊, θ = 0, π (fixed points of ∂_φ) connected by a Misner string

Pictorially: nuts and Misner strings



Figure: Kerr vs. Taub-NUT

Back to Taub–NUT

Following $FG \rightarrow boundary$ metric and stress tensor

$$ds_{\text{bry.}}^{2} = \eta_{\mu\nu} E^{\mu} E^{\nu} = g_{(0)\mu\nu} dx^{\mu} dx^{\nu}$$

= $- (dt + 2n(1 - \cos \vartheta) d\varphi)^{2} + \frac{1}{k^{2}} (d\vartheta^{2} + \sin^{2} \vartheta d\varphi^{2})$
 $T = T_{\mu\nu} E^{\mu} E^{\nu} = \frac{\kappa M k}{3} \left(2(E^{t})^{2} + (E^{\vartheta})^{2} + (E^{\varphi})^{2} \right)$

Fluid interpretation: perfect-like stress tensor

- conformal with $\varepsilon = 2p = \frac{2\kappa Mk}{3}$
- ▶ velocity field $\mathbf{u} = e_t = \partial_t$: comoving & inertial

Same fluid: no expansion, no shear but vorticity

The vorticity on the boundary of AdS Taub–NUT

$$b = -2n(1 - \cos \vartheta) d\varphi$$

$$\omega = -n\sin \vartheta \, d\vartheta \wedge d\varphi = -nk^2 E^\vartheta \wedge E^\varphi$$

Dirac-monopole-like vortex

around the Misner string created by the nut charge

$$n=-\frac{1}{4\pi}\int_{S^2}\omega$$

Kerr produces a dipole without nut charge: $\int \omega = 0$ – *solid rotation Taub–NUT is well designed to describe "monopolar" vortices*

∃ multipolar b yet to be unravelled in the bulk [Weyl 1919]

Reminder

Rotation in flat space (spherical coordinates)

Data:
$$\vec{v}$$
 $\vec{\omega} = 1/2\vec{\nabla} \times \vec{v}$

- Solid rotation ($\ell = 2$):
 - $\vec{v} = \Omega \partial_{\varphi}$
 - $\vec{\omega} = \Omega \cos \vartheta \partial_r \frac{\Omega \sin \vartheta}{r} \partial_\vartheta = \Omega \partial_z$ (parallel to Oz)
- Dirac-monopole vortex ($\ell = 1$):

•
$$\vec{v} = \alpha \frac{1 - \cos \vartheta}{r^2 \sin^2 \vartheta} \partial_{\varphi}$$

• $\vec{\omega} = \frac{\alpha}{2r^2} \partial_r$ (hedgehog

• Ordinary vortex ($\ell = 0$):

•
$$\vec{v} = \frac{\beta}{r^2 \sin^2 \vartheta} \partial \varphi$$

• $\vec{\omega} = 0$ (irrotational) – up to δ -function contributions

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Randers forms and Zermelo metrics [Zermelo'31, Randers '41]

The boundary geometries are stationary metrics of the Randers form

$$\mathrm{d}s^2 = -\left(\mathrm{d}t - b\right)^2 + a_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

Properties: magnetic paradigm and CTCs

- ► The projection of geodesics onto the base space with metric dℓ² = a_{ij}dxⁱdx^j provides trajectories for a non-relativistic charged particle in a magnetic field F̃ = db
- CTCs can appear for $1 b^2 < 0$ $(b^2 = a^{ij}b_ib_j)$
 - Kerr: none
 - ► Taub-NUT: \exists CTCs \rightarrow horizon around the vortex

Equivalently recast as Zermelo metrics $(a, b) \leftrightarrow (W, h)$

$$\mathrm{d}s^{2} = \frac{1}{c^{2} - W^{2}} \left(-c^{2} \mathrm{d}t^{2} + h_{ij} \left(\mathrm{d}x^{i} - W^{i} \mathrm{d}t \right) \left(\mathrm{d}x^{j} - W^{j} \mathrm{d}t \right) \right)$$

- Originally: navigation in a drift current
- Here: analogue-gravity geometries originating from bulk solutions of Einstein's equations via holography
- Zermelo metrics are acoustic: null geodesics describe sound propagation in non-relativistic fluids moving on geometries h_{ij}dxⁱdx^j with velocity field W = Wⁱ∂_i [see e.g. Visser '97]
- CTCs capture physical effects: sound propagation in supersonic-flow regions (W² > c²) → horizons

Similar approaches exist for light propagation in moving media

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Class of bulk solutions describing conformal fluids in 2 + 1 dim with vorticity (still to be unravelled for $\ell > 2$)

- ► spectrum of bulk excitations → anyons on the boundary like in exotic BEC phases (under experimental investigation)
- compute transport coefficients like shear viscosity (nearly-perfect fluids)
- ► combine Kerr and nut charge in AdS Kerr Taub–NUT thermodynamics: M → temperature, {a, n} → rotation
 - study the potential transition as nut-anti-nut dissociation
 - analyse the phase diagramme, identify the order parameter
 - add a U(1) and a scalar field
- understand the analogue-gravity interpretation

Formation of a vortex: nut-anti-nut dissociation



Figure: high-T vs. low-T stable phase

Highlights

Holography in a nutshell

More on AdS Taub–NUT

Sailing in a drift current

Randers vs. Zermelo pictures and analogue gravity

Holography

Applied beyond the original framework – maximal susy YM in D = 4 – usually in the classical gravity approximation without backreaction

• Bulk: "asymptotically AdS" d-dim \mathcal{M} (d = D + 1)

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{k^2r^2} + k^2r^2H(kr)\left(-\mathrm{d}t^2 + \mathrm{d}x^2\right)$$

- Boundary at $r \to \infty$: $ds^2 \approx \frac{dr^2}{k^2r^2} + k^2r^2g_{(0)\mu\nu}(x)dx^{\mu}dx^{\nu}$
- Dynamical field ϕ with action $I[\phi]$ and boundary value $\phi_{(0)}(x)$

The basic relation

 $Z_{
m bulk}[\phi] = \langle 1
angle_{
m bry. \ F.T.}$

gives access to the data of the boundary theory

$$\left\langle \exp i \int_{\partial \mathcal{M}} \mathrm{d}^{D} x \sqrt{-g_{(0)}} \delta \phi_{(0)} \mathcal{O} \right\rangle_{\mathrm{bry. \ F.T.}} = Z_{\mathrm{bulk}} [\phi + \delta \phi_{(0)}]$$

- $\delta \phi_{(0)}$: boundary perturbation \rightarrow source
- \mathcal{O} : observable functional of $\phi_{(0)} \rightarrow$ response
- $\phi_{(0)} \leftrightarrow \mathcal{O}$: conjugate variables

Semi-classically around a classical solution ϕ_{\star}

$$Z_{\text{bulk}}[\phi] = \exp - I_{\mathsf{E}} \left[\phi_{\star} \right]$$
$$\langle \mathcal{O} \rangle = \left. \frac{\delta I}{\delta \phi_{(0)}} \right|_{\phi_{\star}}$$

Hamiltonian interpretation of $\langle \mathcal{O} \rangle$

$$= \pi = \frac{\partial \mathcal{L}}{\partial \partial_r \phi} \Rightarrow I = \int \mathrm{d}r \int \mathrm{d}^D x \left[\pi \partial_r \phi - \mathcal{H}(\pi, \phi, \partial_\mu \phi) \right]$$

on-shell variation

$$\delta I|_{\phi_{\star}} = \int_{\partial \mathcal{M}} \mathrm{d}^{D} x \, \pi_{(0)} \, \delta \phi_{(0)} \Rightarrow \langle \mathcal{O} \rangle = \pi_{(0)}$$

What is holography? How do we get $\pi_{(0)} = \pi_{(0)} \left[\phi_{(0)} \right]$ *?*

$$\partial \mathcal{M} = egin{cases} \mathsf{boundary} \ r o \infty \ \mathsf{horizon} \ r_\mathsf{H} \end{cases}$$

▶ $\phi_{(0)}(x)$ and $\pi_{(0)}(x)$ are *independent* data set at large *r*

$$\phi(r) = r^{\Delta-d}\phi_{(0)}(x) + \frac{r^{-\Delta}}{k(2\Delta-D)}\pi_{(0)}(x) + \cdots$$

(non-normalizable and normalizable modes)

become related if a regularity condition is imposed at r_H

$$\langle \mathcal{O}
angle = \pi_{(0)} \left[\phi_{(0)} \right]$$

In summary

Holography: computation of $\langle \mathcal{O} \rangle_{\text{bry. F.T.}}$ as a response to a boundary source perturbation $\delta \phi_{(0)}$

- Dynamical field ϕ with action $I[\phi]$ and boundary value $\phi_{(0)}(x)$
- Momentum $\pi(r, x)$ with boundary value $\pi_{(0)}(x)$
- On-shell variation

$$\delta I|_{\phi_{\star}} = \int_{\partial \mathcal{M}} \mathrm{d}^{D} x \, \pi_{(0)} \, \delta \phi_{(0)}$$

► Holography: regularity on $r_{\rm H} \Rightarrow \pi_{(0)} = \pi_{(0)} \left[\phi_{(0)} \right] \longrightarrow$

$$\left< \mathcal{O} \right> = \pi_{(0)} \left[\phi_{(0)} \right]$$

Examples

Electromagnetic field in d = 4, D = 3

- ▶ Field A_r , $A_\mu \rightarrow A_{(o)\mu}$: boundary electromagnetic field source
- Momentum $\mathcal{E}_{\mu} \to \mathcal{E}_{(0)\mu}$: $\langle \varrho \rangle$, $\langle j_i \rangle$ response
- ► Bulk gauge invariance → continuity equation

Gravitation in d = D + 1

- ▶ Field $g_{rr}, g_{\mu\nu} \rightarrow g_{(o)\mu\nu}$: boundary metric source
- Momentum $T_{\mu\nu} \rightarrow T_{(o)\mu\nu}$: $\langle T_{(o)\mu\nu} \rangle$ response
- ► Bulk diffeomeorphism invariance → conservation equation

Gravity in d = 4

Palatini formulation and 3 + 1 split [Leigh, Petkou '07, Mansi, Petkou, Tagliabue '08]

$$I_{\mathsf{EH}} = -\frac{1}{32\pi G} \int_{\mathcal{M}} \epsilon_{\mathsf{abcd}} \left(\mathcal{R}^{\mathsf{ab}} - \frac{\Lambda}{6} \theta^{\mathsf{a}} \wedge \theta^{\mathsf{b}} \right) \wedge \theta^{\mathsf{c}} \wedge \theta^{\mathsf{d}}$$

 $heta^a$ an orthonormal frame d $s^2=\eta_{ab} heta^a heta^b$ $(\eta:+-++)$

► Vierbein: $\theta^r = N \frac{dr}{kr}$ $\theta^\mu = N^\mu dr + \tilde{\theta}^\mu$ $\mu = 0, 1, 2$

$$\mathrm{d}s^{2} = N^{2}\frac{\mathrm{d}r^{2}}{k^{2}r^{2}} + \eta_{\mu\nu}\left(N^{\mu}\mathrm{d}r + \tilde{\theta}^{\mu}\right)\left(N^{\nu}\mathrm{d}r + \tilde{\theta}^{\nu}\right)$$

• Connection: $\omega^{r\mu} = q^{r\mu} dr + \mathcal{K}^{\mu} \quad \omega^{\mu\nu} = -\epsilon^{\mu\nu\rho} \left(Q_{\rho} \frac{dr}{kr} + \mathcal{B}_{\rho} \right)$ (note: $\Lambda = -3k^2$) *Aim: Hamiltonian evolution from data on the boundary* $r \rightarrow \infty$ *Question: what are the field and momentum variables?*

• Gauge choice: N=1 and $N^{\mu}=q^{r\mu}=Q_{
ho}=0$

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{k^2r^2} + \eta_{\mu\nu}\tilde{\theta}^\mu\tilde{\theta}^\nu$$

• Fields and momenta: $\tilde{\theta}^{\mu}$, \mathcal{K}^{μ} , \mathcal{B}_{ρ} one-forms

What are the independent boundary data? Answer in asymptotically AdS: Fefferman–Graham expansion for large r [Fefferman, Graham '85]

$$\begin{aligned} \tilde{\theta}^{\mu}(r, x) &= kr \, E^{\mu}(x) + \frac{1}{kr} F^{\mu}_{[2]}(x) + \frac{1}{k^2 r^2} F^{\mu}(x) + \cdots \\ \mathcal{K}^{\mu}(r, x) &= -k^2 r \, E^{\mu}(x) + \frac{1}{r} F^{\mu}_{[2]}(x) + \frac{2}{kr^2} F^{\mu}(x) + \cdots \\ \mathcal{B}^{\mu}(r, x) &= B^{\mu}(x) + \frac{1}{k^2 r^2} B^{\mu}_{[2]}(x) + \cdots \end{aligned}$$

Independent 2 + 1 boundary data: E^{μ} and F^{μ}

Upon canonical transformations (i.e. boundary terms or holographic renormalization)

$$\delta I_{\mathsf{EH}}|_{\mathsf{on-shell}} = \int_{\partial \mathcal{M}} T^{\mu} \wedge \delta \Sigma_{\mu}$$

•
$$\Sigma_{\mu} = \frac{1}{2} \epsilon_{\mu\nu\rho} E^{\nu} \wedge E^{\rho}$$
: field – source

• $T^{\mu} = \kappa F^{\mu}$: momentum – response

Application: Schwartzschild AdS

The bulk data

$$\mathrm{d}s^2 = \frac{\mathrm{d}\tilde{r}^2}{V(\tilde{r})} - V(\tilde{r})\mathrm{d}t^2 + \tilde{r}^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right)$$

$$V(r) = 1 + k^2 \tilde{r}^2 - \frac{2M}{\tilde{r}}$$

$$\theta^r = \frac{d\tilde{r}}{\sqrt{V(\tilde{r})}} = \frac{dr}{kr}$$

The Fefferman–Graham expansion

$$\begin{array}{rcl} \theta^t &=& \sqrt{V(\tilde{r})} \mathrm{d}t = \left(kr + \frac{1}{4kr} - \frac{2M}{3kr^2} + \mathrm{O}\left(\frac{1}{r^3}\right)\right) \mathrm{d}t \\ \theta^\vartheta &=& \tilde{r} \, \mathrm{d}\vartheta = \left(r - \frac{1}{4k^2r} + \frac{M}{3k^2r^2} + \mathrm{O}\left(\frac{1}{r^3}\right)\right) \mathrm{d}\vartheta \\ \theta^\varphi &=& \tilde{r} \sin\vartheta \, \mathrm{d}\varphi = \left(r - \frac{1}{4k^2r} + \frac{M}{3k^2r^2} + \mathrm{O}\left(\frac{1}{r^3}\right)\right) \sin\vartheta \, \mathrm{d}\varphi \end{array}$$

The boundary data

- coframe: $E^t = dt$ $E^{\vartheta} = \frac{d\vartheta}{k}$ $E^{\varphi} = \frac{\sin \vartheta \, d\varphi}{k}$
- ► stress current: $F^t = -\frac{2Mk}{3} dt$ $F^{\vartheta} = \frac{M}{3} d\vartheta$ $F^{\varphi} = \frac{M}{3} \sin \vartheta d\varphi$

The boundary metric

$$ds_{\text{bry.}}^{2} = \eta_{\mu\nu} E^{\mu} E^{\nu} = g_{(0)\mu\nu} dx^{\mu} dx^{\nu}$$

= $-dt^{2} + \frac{1}{k^{2}} (d\vartheta^{2} + \sin^{2}\vartheta d\varphi^{2})$

- Einstein universe
- $\blacktriangleright e_t = \partial_t$
- $\nabla_{e_t} e_t = 0$: observers at rest are inertial

The boundary stress tensor $\kappa F^{\mu} e_{\mu}$

$$T = T_{\mu\nu} E^{\mu} E^{\nu} = \frac{\kappa M k}{3} \left(2(E^{t})^{2} + (E^{\vartheta})^{2} + (E^{\varphi})^{2} \right)$$

- traceless: conformal fluid with $\varepsilon = 2p = \frac{2\kappa Mk}{3}$
- ▶ velocity field $\mathbf{u} = e_t = \partial_t$: comoving & inertial
- velocity one-form: $u = -E^t = -dt$

Static fluid without expansion, shear or vorticity

More general examples

We can exhibit backgrounds with stationary boundaries and fluids

 $T = (\varepsilon + p)\mathbf{u} \otimes \mathbf{u} + p\eta^{\mu\nu} e_{\mu} \otimes e_{\nu}$

- $\varepsilon = 2p$: conformal
- $\nabla_{\mathbf{u}}\mathbf{u} = 0$: inertial
- $\mathbf{u} = e_0$: at rest (comoving)

Note: the fluid may be perfect or not

$$\mathcal{T}_{\mathsf{visc}} = -\left(2\eta\sigma^{\mu
u} + \zeta h^{\mu
u}\Theta
ight) oldsymbol{e}_{\mu}\otimesoldsymbol{e}_{
u}$$

 $T_{\text{visc}} = 0$ if the geodesic congruence is shear- and expansion-less A shear- and expansion-less fluid is geodesic if [Caldarelli, Dias, Emparan, Klemm '08]

 $\nabla_{\mathbf{u}}\varepsilon = 0$ $\nabla p + u \nabla_{\mathbf{u}} p = 0$

fulfilled here with ε , p csts.

On vector-field congruences [Ehlers '61]

Vector field **u** *with* $u_{\mu}u^{\mu} = -1$ *and space–time variation* $\nabla_{\mu}u_{\nu}$

$$\nabla_{\mu}u_{\nu} = -u_{\mu}a_{\nu} + \sigma_{\mu\nu} + \frac{1}{D-1}\Theta h_{\mu\nu} + \omega_{\mu\nu}$$

h_{µν} = u_µu_ν + g_{µν}: projector/metric on the orthogonal space
 a_µ = u^ν∇_νu_µ: acceleration - transverse

• $\sigma_{\mu\nu}$: symmetric traceless part – shear

•
$$\Theta = \nabla_{\mu} u^{\mu}$$
: trace – expansion

• $\omega_{\mu\nu}$: antisymmetric part – vorticity

$$\omega = rac{1}{2}\omega_{\mu
u}\mathsf{d}x^{\mu}\wedge\mathsf{d}x^{
u} = rac{1}{2}(\mathsf{d}u+u\wedge \mathsf{a})$$

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Sailing in a drift current

Randers vs. Zermelo pictures and analogue gravity

AdS Taub–NUT: the nut charge

V

Reminder: the bulk data [Taub '51, Neroman, Tamburino, Unti '63]

$$ds^{2} = \frac{d\tilde{r}^{2}}{V(\tilde{r})} - V(\tilde{r}) \left[dt - 2n\cos\vartheta \, d\varphi \right]^{2} + \rho^{2} \left[d\vartheta^{2} + \sin^{2}\vartheta \, d\varphi \right]^{2}$$

$$V(\tilde{r}) = \Delta_{r}/\rho^{2} \text{ with}$$

$$\Delta_{r} = \left(\tilde{r}^{2} - n^{2} \right) \left(1 + k^{2} \left(\tilde{r}^{2} + 3n^{2} \right) \right) + 4k^{2}n^{2}\tilde{r}^{2} - 2M\tilde{r}$$

$$\rho^{2} = \tilde{r}^{2} + n^{2}$$

The Fefferman–Graham expansion with $r s.t. dr/kr = d\tilde{r}/\sqrt{V(\tilde{r})}$

boundary coframe and frame

$$E^{t} = dt - b \quad E^{\vartheta} = \frac{d\vartheta}{k} \quad E^{\varphi} = \frac{\sin\vartheta\,d\varphi}{k}$$

$$e_{t} = \partial_{t} \quad e_{\vartheta} = k\,\partial_{\vartheta} \quad e_{\varphi} = -\frac{2kn(1-\cos\vartheta)}{\sin\vartheta}\partial_{t} + \frac{k}{\sin\vartheta}\partial_{\varphi}$$

$$b = -2n(1-\cos\vartheta)d\varphi$$

boundary stress current

$$F^{t} = -\frac{2Mk}{3}E^{t} \quad F^{\theta} = \frac{Mk}{3}E^{\theta} \quad F^{\varphi} = \frac{Mk}{3}E^{\phi}$$

For comparison: AdS Kerr

The Fefferman–Graham expansion of θ^t , θ^{ϑ} , θ^{φ}

boundary orthonormal coframe and frame

$$E^{t} = dt - b \quad E^{\theta} = \frac{d\theta}{k\sqrt{\Delta_{\theta}}} \qquad E^{\varphi} = \frac{\sqrt{\Delta_{\theta}}\sin\theta \,d\varphi}{k\Xi}$$

$$e_{t} = \partial_{t} \qquad e_{\theta} = k\sqrt{\Delta_{\theta}} \,\partial_{\theta} \quad e_{\varphi} = \frac{ka\sin\theta}{\sqrt{\Delta_{\theta}}} \partial_{t} + \frac{k\Xi}{\sin\theta\sqrt{\Delta_{\theta}}} \partial_{\varphi}$$

$$b = \frac{a\sin^{2}\theta}{\Xi} d\varphi$$

boundary stress current

$$F^t = -\frac{2Mk}{3}E^t$$
 $F^{\theta} = \frac{Mk}{3}E^{\theta}$ $F^{\varphi} = \frac{Mk}{3}E^{\phi}$

The boundary metric and stress tensor

$$ds_{\text{bry.}}^{2} = \eta_{\mu\nu}E^{\mu}E^{\nu} = g_{(0)\mu\nu}dx^{\mu}dx^{\nu}$$

= $-(dt + 2n(1 - \cos\vartheta)d\varphi)^{2} + \frac{1}{k^{2}}(d\vartheta^{2} + \sin^{2}\vartheta d\varphi^{2})$
 $T = T_{\mu\nu}E^{\mu}E^{\nu} = \frac{\kappa Mk}{3}\left(2(E^{t})^{2} + (E^{\vartheta})^{2} + (E^{\varphi})^{2}\right)$

Fluid interpretation: perfect-like stress tensor

- conformal fluid with $\varepsilon = 2p = \frac{2\kappa Mk}{3}$
- velocity field $\mathbf{u} = e_t = \partial_t$: comoving & inertial

Fluid without expansion and shear but with vorticity

$$\omega = \frac{1}{2} \mathrm{d}b = -n \sin \vartheta \mathrm{d}\vartheta \wedge \mathrm{d}\varphi = -k^2 n E^\vartheta \wedge E^\varphi$$

AdS Taub–NUT: more on the boundary

Homogenous boundary space-time: Lorentzian squashed 3-sphere

$$\mathsf{d}s_{\mathsf{bry}.}^{2} = \frac{1}{k^{2}} \left((\sigma^{1})^{2} + (\sigma^{2})^{2} - 4n^{2} (\sigma^{3})^{2} \right)$$

- Gödel-like space (sourced by dust distribution) [classification in Raychaudhuri et al. '80, Rebouças et al. '83]
- Stationary foliation in 2-spheres with a time fiber
- CTCs of angular opening < 2ϑ₀ (g_{φφ}(ϑ₀) = 0) − no closed time-like geodesics
- Special point: south pole of the 2-sphere track of the Misner string

Around the poles: Som-Raychaudhuri and cosmic spinning string

North pole: Som-Raychaudhuri space – sourced by rigidly rotating charged dust [Som, Raychaudhuri '68]

$$\mathrm{d}s^{2}=-\left(\mathrm{d}t+\Omega\varrho^{2}\mathrm{d}\varphi\right)^{2}+\varrho^{2}\mathrm{d}\varphi^{2}+\mathrm{d}\varrho^{2}$$

 $\Omega = k^2 n \text{ and } \varrho = \vartheta/k$

South pole: spinning cosmic string [vortex in analogue gravity]

$$\mathrm{d}s^2 = -\left(\mathrm{d}t + A\mathrm{d}\varphi\right)^2 + \varrho^2\mathrm{d}\varphi^2 + \mathrm{d}\varrho^2$$

$${\sf A}=4{\sf n}-\Omega arrho^2$$
 and $arrho=\pi-artheta/k$

Around the poles of Kerr: Som–Raychaudhuri with $\Omega = -k^2 a$

Kerr vs. Taub-NUT "rotation" [Dowker '74, Bonnor '75, Hunter '98]

- ► Kerr: rigid rotation with angular momentum and velocity
 - horizon at $r = r_+$: fixed locus of $\partial_t + \Omega_H \partial_{\varphi} \rightarrow \text{bolt}$
 - pair of nut−anti-nut at r = r₊, ϑ = 0, π (fixed points of ∂_φ) connected by a Misner string

asymptotically $\Omega_\infty = -ak^2$

- Taub–NUT: "non-rigid rotation" with angular momentum distribution along the Misner string (vanishing integral) – asymptotically:
 - north pole: angular velocity $\Omega_{\infty} = nk^2$
 - south pole: no angular velocity

Highlights

Holography in a nutshell

More on AdS Taub–NUT

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The Zermelo problem

What is the minimal-time trajectory of a non-relativistic ship sailing on a space with positive-definite metric $dt^2 = h_{ij}dx^i dx^j$ and velocity $U^i = dx^i/dt \ s.t. \|\mathbf{U}\|^2 = 1$?

time functional is

$$T=\int \mathrm{d}t\,\sqrt{h_{ij}\,U^i\,U^j}$$

• minimization is realized with geodesics of dt^2

What happens in the presence of a lateral drifting flow $W = W^i \partial_i$ ("wind" or "tide")? [Zermelo '31]

- velocity: $U^i = dx^i/dt = V^i + W^i$
 - U: vector tangent to the trajectory
 - ▶ **V**: "propelling" velocity with $\|\mathbf{V}\|^2 = 1$
 - no longer aligned with the trajectory
 - instantaneous navigation road velocity of the ship with respect to a local frame dragged by the drifting flow

• norm: $\mathbf{U}^2 = 1 + \mathbf{W}^2 + 2\mathbf{V} \cdot \mathbf{W}$

time functional is

$$T = \int dt \left(\sqrt{\frac{\mathbf{U}^2}{1 - \mathbf{W}^2} + \left(\frac{\mathbf{W} \cdot \mathbf{U}}{1 - \mathbf{W}^2}\right)^2} - \frac{\mathbf{W} \cdot \mathbf{U}}{1 - \mathbf{W}^2} \right)$$
$$= \int dt \left(\sqrt{\left(\frac{h_{ij}}{\lambda} + \frac{W_i W_j}{\lambda^2}\right) U^i U^j} - \frac{W_k U^k}{\lambda} \right)$$

with $\lambda = 1 - \mathbf{W}^2$

 minimization is realized with null geodesics of the Zermelo metric

$$\mathrm{d}s^{2} = \frac{1}{\lambda} \left(-\mathrm{d}t^{2} + h_{ij} \left(\mathrm{d}x^{i} - W^{i} \mathrm{d}t \right) \left(\mathrm{d}x^{j} - W^{j} \mathrm{d}t \right) \right)$$

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Note: the time functional is of Randers type with Finsler Lagrangian

$$T = \int \mathrm{d}t \, F(x^i, \, U^i)$$

with

$$F(x^i, U^i) = \sqrt{a_{ij}U^iU^j + b_iU^i}$$

and

$$\mathsf{a}_{ij} = rac{h_{ij}}{\lambda} + rac{W_i W_j}{\lambda^2} \quad \mathsf{b}_i = -rac{h_{ij} W^j}{\lambda}$$

the data of the Randers form

Equivalently Randers stationary forms are recast as Zermelo metrics

$$\mathrm{d}s^{2} = \frac{1}{\lambda} \left(-\mathrm{d}t^{2} + h_{ij} \left(\mathrm{d}x^{i} - W^{i} \mathrm{d}t \right) \left(\mathrm{d}x^{j} - W^{j} \mathrm{d}t \right) \right)$$

with

$$\begin{array}{ll} h_{ij} &=\lambda\left(a_{ij}-b_{i}b_{j}\right)\\ \lambda &=1-b_{i}b_{j}a^{ij}\\ W^{i} &=-\frac{a^{ij}b_{j}}{\lambda} \end{array}$$

Null geodesics in Zermelo metric are minimal-time curves for sailing in the base space of metric $dt^2 = h_{ij}dx^i dx^j$ under the influence of a drifting "wind" $W = W^i \partial_i$ [Zermelo '31]

Analogue gravity picture

Zermelo metrics are acoustic [see e.g. Visser '97, Chapline, Mazur '04]

$$\mathrm{d}s^{2} = \frac{\varrho}{c_{\mathrm{s}}} \left(-c_{\mathrm{s}}^{2}\mathrm{d}t^{2} + h_{ij} \left(\mathrm{d}x^{i} - W^{i}\mathrm{d}t \right) \left(\mathrm{d}x^{j} - W^{j}\mathrm{d}t \right) \right)$$

Null geodesics describe sound propagation in non-relativistic fluids moving on geometries $h_{ij}dx^i dx^j$ with velocity fields $W = W^i \partial_i$

- inviscid, isolated, barotropic (dh = dp/q)
- local mass density q and pressure p
- local sound velocity $c_{\rm s} = 1/\sqrt{\partial \rho/\partial \rho}$

Alternatively the whole boundary set up could be a gravitational analogue of sound propagating in moving fluids or light in moving dielectrics – acoustic/optical black holes

As such our examples fall in a larger class of backgrounds studied in analogue systems [Gibbons et al. '08] – here equipped with a stress tensor Randers & Zermelo backgrounds address the problems of

- motion of charged particles in magnetic fields
- sailing in the presence of a drift force
- sound propagation in moving media

and are dual to each other

Where are we?

Exploratory tour of some properties of conformal holographic fluids moving in non-trivial gravitational backgrounds

- inertial
- carrying vorticity

Vorticity appears in various fashions

- \blacktriangleright Kerr \rightarrow solid rotation on the boundary: dipole
- Taub-NUT \rightarrow vortex on the boundary: monopole

More general multipoles?

Remark: more general vortices on the boundary

$$\omega = (-1)^\ell lpha \, {\mathcal P}_\ell'(\cos artheta) \sin artheta \, {\mathrm d} artheta \wedge {\mathrm d} arphi$$

▶ for odd ℓ there is indeed a vortex around the track of the Misner string at the south pole with a charge

$$-\frac{1}{4\pi}\int\omega=\alpha$$

► for even ℓ the Misner string does not reach the poles and the total charge vanishes – e.g. Kerr as a dipole with $\alpha = a/3\Xi$

Bulk realization for $\ell \geq 3$: generalization of Weyl multipoles [Weyl, 19]

Alternative analogue interpretation of the boundary backgrounds: propagation of sound/light in moving media (Randers & Zermelo)

- provides holographic AdS/analogue-gravity correspondence
- evades the CTCs caveats within supersonic/superluminal flows

Bulk for general Randers geometries?