Fisher's zeros as Boundary of RG flows in complex coupling space

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work done in part with Alexei Bazavov, Alan Denbleyker, Daping Du, Yuzhi "Louis" Liu, Bugra Oktay, Alex Velytsky and Haiyuan Zou

Content of the talk

- Theoretical background: complex couplings, the convergence of perturbation theory and lattice models with compact field integration.

- Renormalization Group (RG) flows in the complex plane (numerical results for $2D$ $O(N)$ models and hierarchical model + remarks for FRG).

- Digression on (real) continuous RG flows for hierarchical models.

- Fisher's zeros in Lattice Gauge Theory (zeros of the partition function in the $\beta = 2N_c/g^2$ or $1/kT$ plane) .

General Motivations

- The fact that gauge theories are perturbatively renormalizable was a crucial step in the development of the standard model, but the perturbative series are expected to diverge (Dyson). The large order behavior of the series can be estimated using dispersion relations in the complex coupling plane (Bender and Wu, ...).

- In the path-integral formulation, the divergence of the perturbative series can be traced to the large field contributions. Lattice gauge theories with a compact group have a build-in large field cutoff

- We need a computational counterpart of Tomboulis picture of confinement.

Complex RG flows (arXiv:1005.1993; PRL 104 251601)

- Losing conformality=confinement=complex fixed points (Kaplan et al.)
- New picture: Fisher's zeros as boundary and gates for complex RG flows
- $2D$ $O(N)$ nonlinear sigma models (with Haiyuan Zou)
- Ising Hierarchical model ($D=$ 2 and 3) (with Yuzhi "Louis" Liu)
- Fisher's zeros in $U(1)$ and $SU(2)$ LGT (with Alexei Bazavov, Alan Denbleyker and Daping Du)

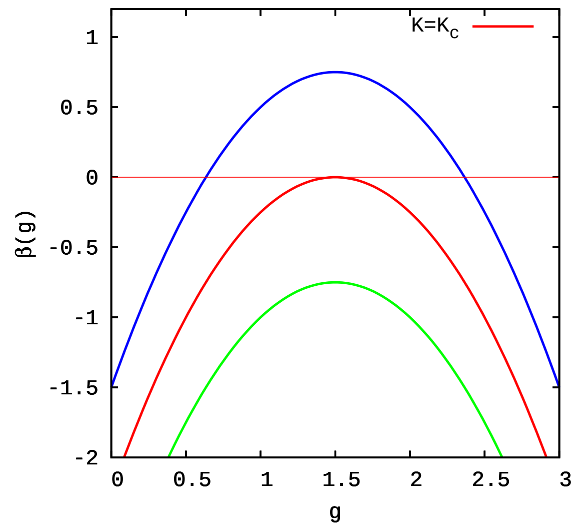

Figure 1: By reducing the constant term in a quadratic β function, the IR and UV fixed points merge and disappear in the complex plane, a mass gap is created, conformality is replaced by confinement (Kaplan, Son and Stephanov, PRD80). The model is integrable in the complex plane (circles, see Moroz and Schmidt, Ann. of Ph. 325).

Fisher's zeros as "gates" for complex RG flows

- Motivated by KSS observation, we studied complex extensions of RG flows in asymptotically free models where the weakly coupled flows reach the strongly coupled fixed point.

- We considered modifications or deformations that may affect that behavior (finite volume, change of dimension, additional pieces in the action).

- In all cases, the Fisher's zeros (of the partition function) seem to govern the global behavior of the flows near the real axis. It is plausible that in the infinite volume limit, these zeros delimit the boundary of the basin of attraction of the strongly coupled fixed point. For confining models, a "gate" remains open.

2D O(N) non-linear sigma model

$Z = \int \prod_{\mathbf{x}} d^N \phi_{\mathbf{x}} \delta(\vec{\phi}_{\mathbf{x}}.\vec{\phi}_{\mathbf{x}} - 1) \mathrm{e}^{-(1/g_0^2)\sum_{\mathbf{x},\mathbf{e}}(1-\vec{\phi}_{\mathbf{x}}.\vec{\phi}_{\mathbf{x}+\mathbf{e}})}$

Notations: $\beta \equiv \frac{1}{(g_0^2 N)}$ (inverse 't Hooft coupling), $M \equiv \frac{m_{gap}}{\Lambda_{UV}}$

Large N: $\beta(M^2) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^2} \frac{1}{2(2-\cos(k_1)-\cos(k_2))+M^2}$

Infinite volume, small coupling (AF): $\beta(M^2) \simeq 1/(4\pi)\ \ln(1/M^2)$

Complex RG I: $m_{gap} = \epsilon \mathrm{e}^{i\theta}$ (small circle around 0), $\Lambda_{UV} \to \Lambda_{UV}/b$

Remarks about complex FRG

- Non relativistic calculations exist (see talks by Moroz, Narain, Birse, ...)

- Complex values can be introduced in the initial conditions or the regulators.

- Complex regulators as a probe? Take your favorite cutoff function $R_k(q)$, for instance, $\frac{q^2}{\mathrm{e}^{q^2/k^2}-1}$. Replace k^2 by $k^2\mathrm{e}^{i\theta}$ in $R_k(q)$. Proceed as usual with k real but Γ_k complex. Patch together the flows in the complex couplings planes for various acceptable θ.

Zeros of the partition function (Y.M. PRD 80)

$\oint_C db(dZ/db)/Z = i2\pi \sum_q n_q(C)$,

where $n_q(C)$ is the number of zeros of order q inside C . For large N,

$\oint_C db(dZ/db)/Z \propto \oint_{C'} dM^2(db/dM^2)(M^2 - 1/b)$

The second term has a pole at $b = 0$, but it is compensated by a pole in M^2. The poles of (db/dM^2) in the cut (the real interval $[-8, 0]$). If the contour C' in the M^2 plane does not cross the cut, then there are no zeros of the partition function inside the corresponding C in the b-plane. We conclude that in the large-N limit, there are no Fisher's zero in the image of the cut M^2 plane. This image limited by four approximate hyperbolas with asymptotes along a cross shaped figure.

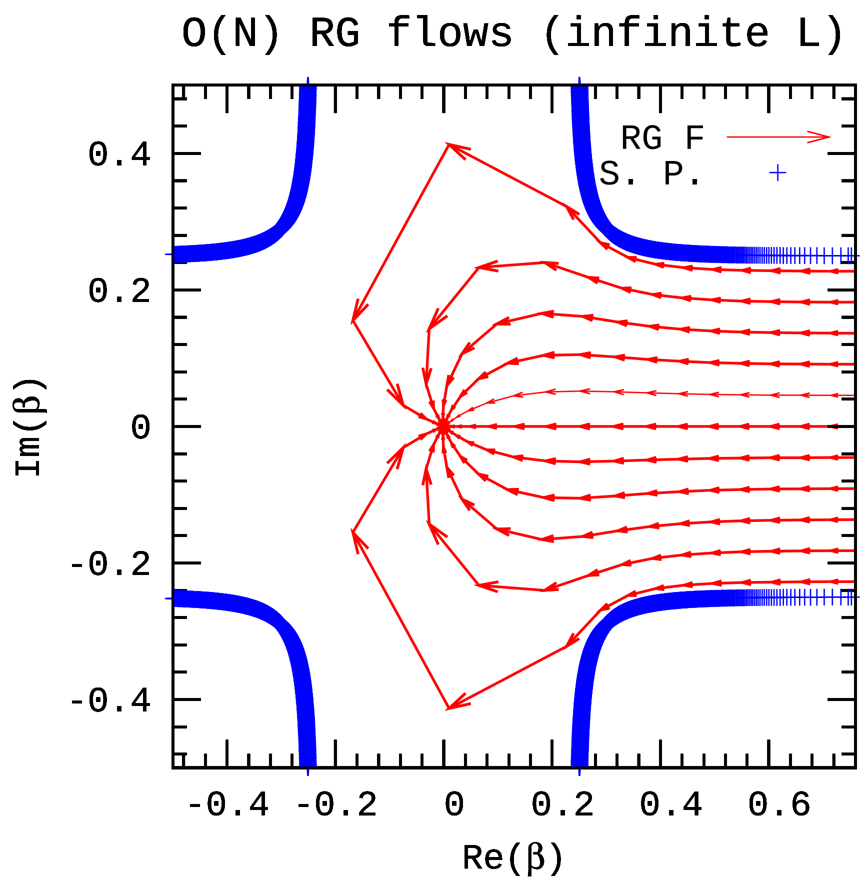

Figure 2: Infinite L RG flows (arrows). The blending blue crosses are the β images of two lines of points located very close above and below the $[-8, 0]$ cut of $\beta(M^2)$ in the M^2 plane.

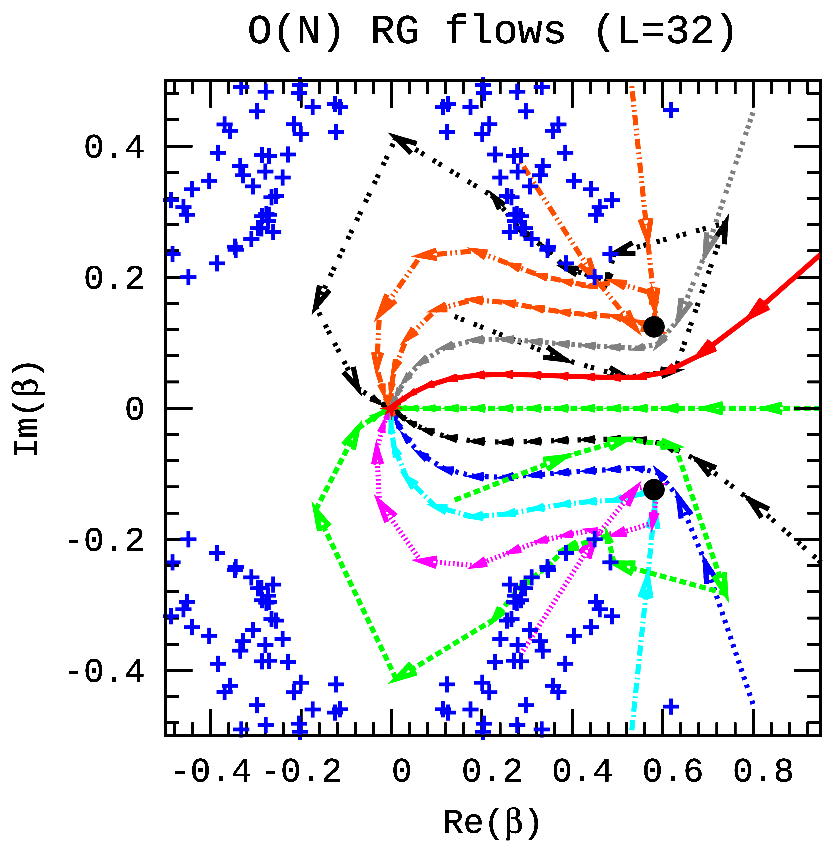

Figure 3: Same procedure and initial conditions but for $L = 32$. $\beta(M^2)$ is now a rational function; the crosses are the images of the singular points. The image of the two singular points closest to 0 appear as two large filled circles.

Complex RG II: Two-lattice matching

We consider the sums of the spins in four $L/2 \times L/2$ blocks B; NB is a nearest neighbor block of B. We define (possibly by reweighting):

$$R(\beta, L) \equiv \frac{\left\langle (\sum_{x\in B} \vec{\phi}_x)(\sum_{y\in NB} \vec{\phi}_y) \right\rangle_\beta}{\left\langle (\sum_{x\in B} \vec{\phi}_x)(\sum_{y\in B} \vec{\phi}_y)) \right\rangle_\beta} .$$

A discrete RG transformation mapping β into β' while the lattice spacing changes from a to $2a$ is obtained by matching: $R(\beta, L) = R(\beta', L/2)$.

Search with Newton's method: ambiguity$\equiv |\beta - \beta_{closest}|/|\beta - \beta_{2d.closest}|$

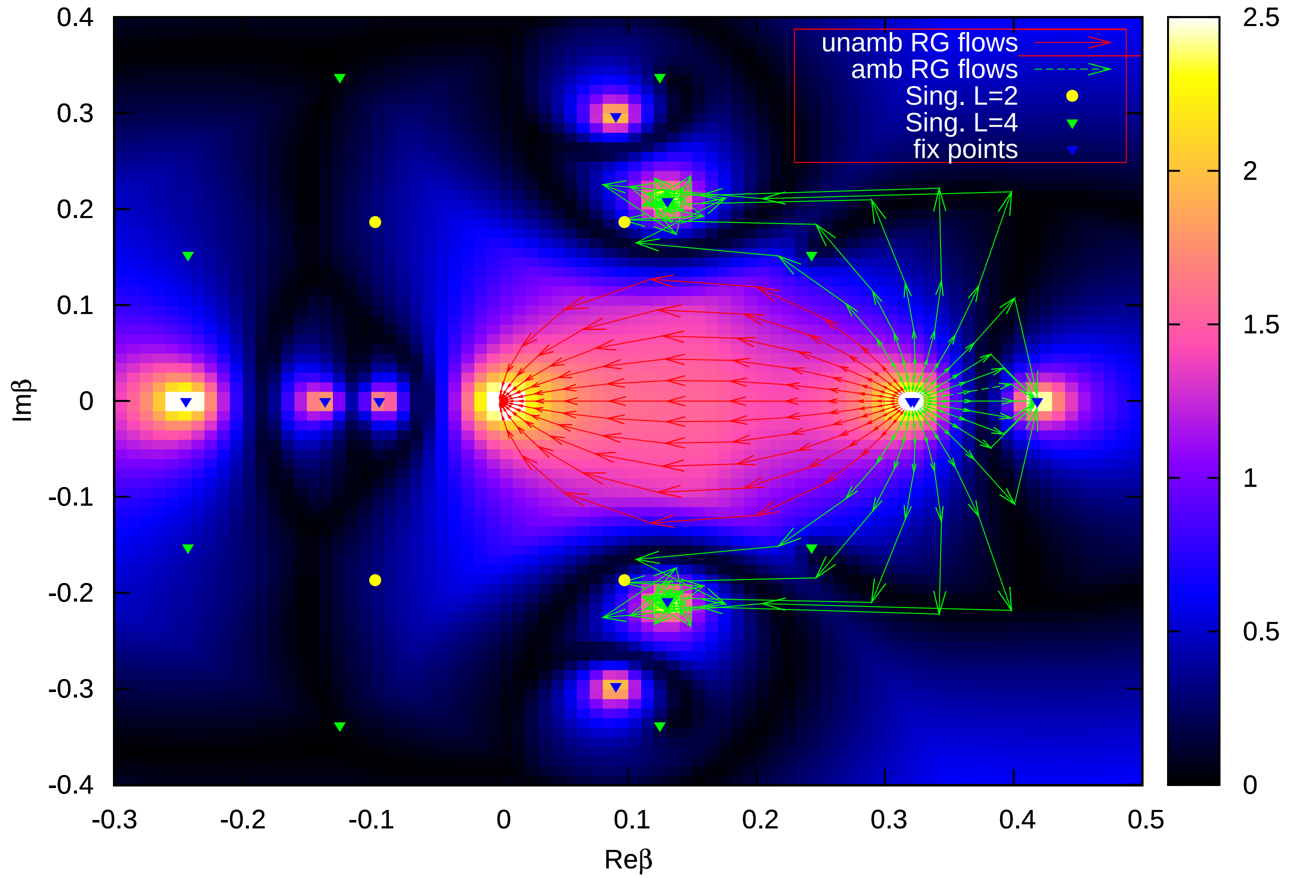

Figure 4: Complex RG flows $L = 4$; Color scale :-Ln(ambiguity)

Hierarchical Model (review in JPA 40)

- It is a lattice model with block interactions depending on the block configurations in a minimal way. The LPA is exact

- Its recursion formula is related to Wilson's approximate recursion formula (that allowed the first numerical RG calculations) but the exponents are different. (JPA 29)

- It is a model on the 2-adic line. The classification of the multiplative characters provides in principle a systematic method of improvement of the hierarchical approximation (YM, Europhysics 93, hep-th/9307128). This has a wavelet translation (Haar system). Analogous to the derivative expansion. Never tried beyond one dimension.

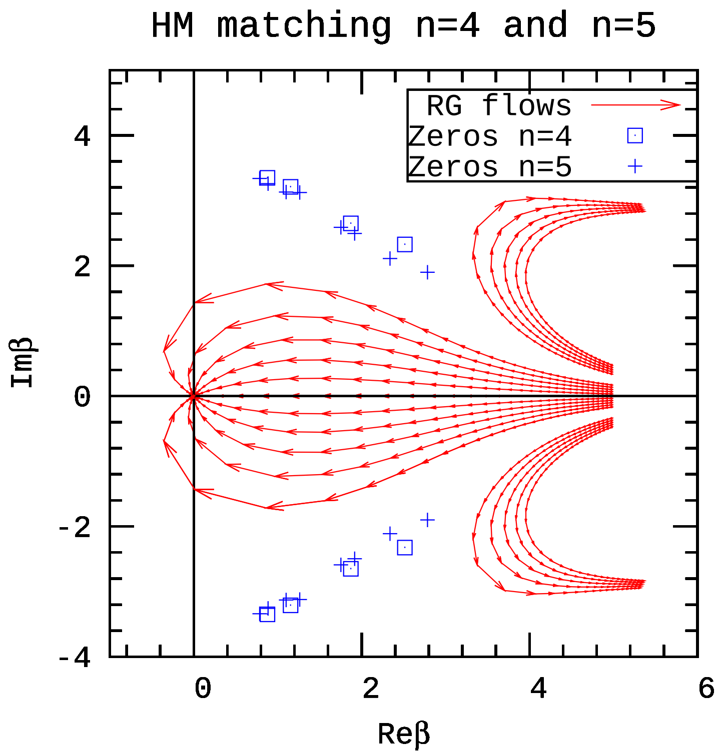

Figure 5: Unambiguous RG flows for the hierarchical model in the complex $\beta = 1/kT$ plane obtained by the two lattice method. The crosses and open boxes are at the Fisher's zeros for 2^4 and 2^5 sites.

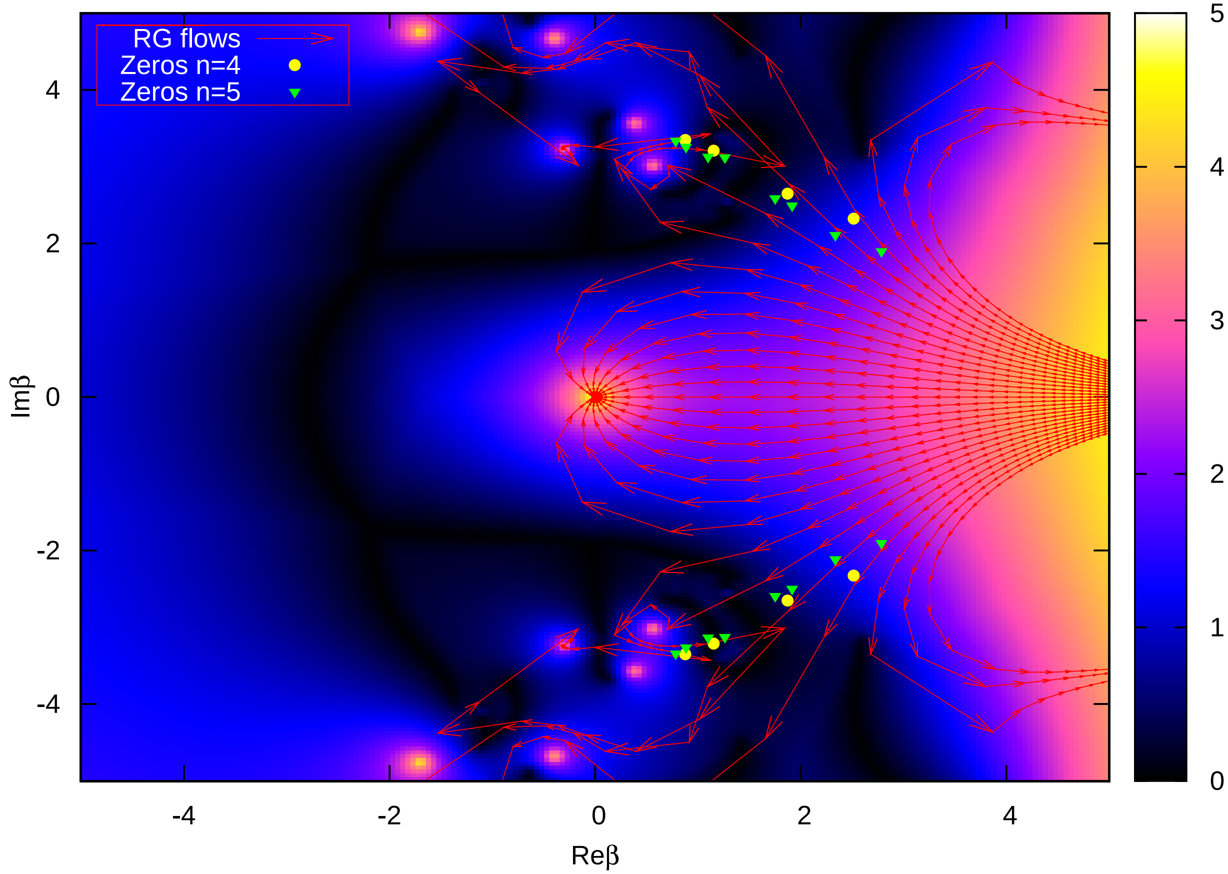

Figure 6: RG flows for the $D = 2$ hierarchical model in the complex β plane obtained by the two lattice method. Circles and triangles are at the Fisher's zeros for 2^4 and 2^5 sites. Darker=more ambiguous.

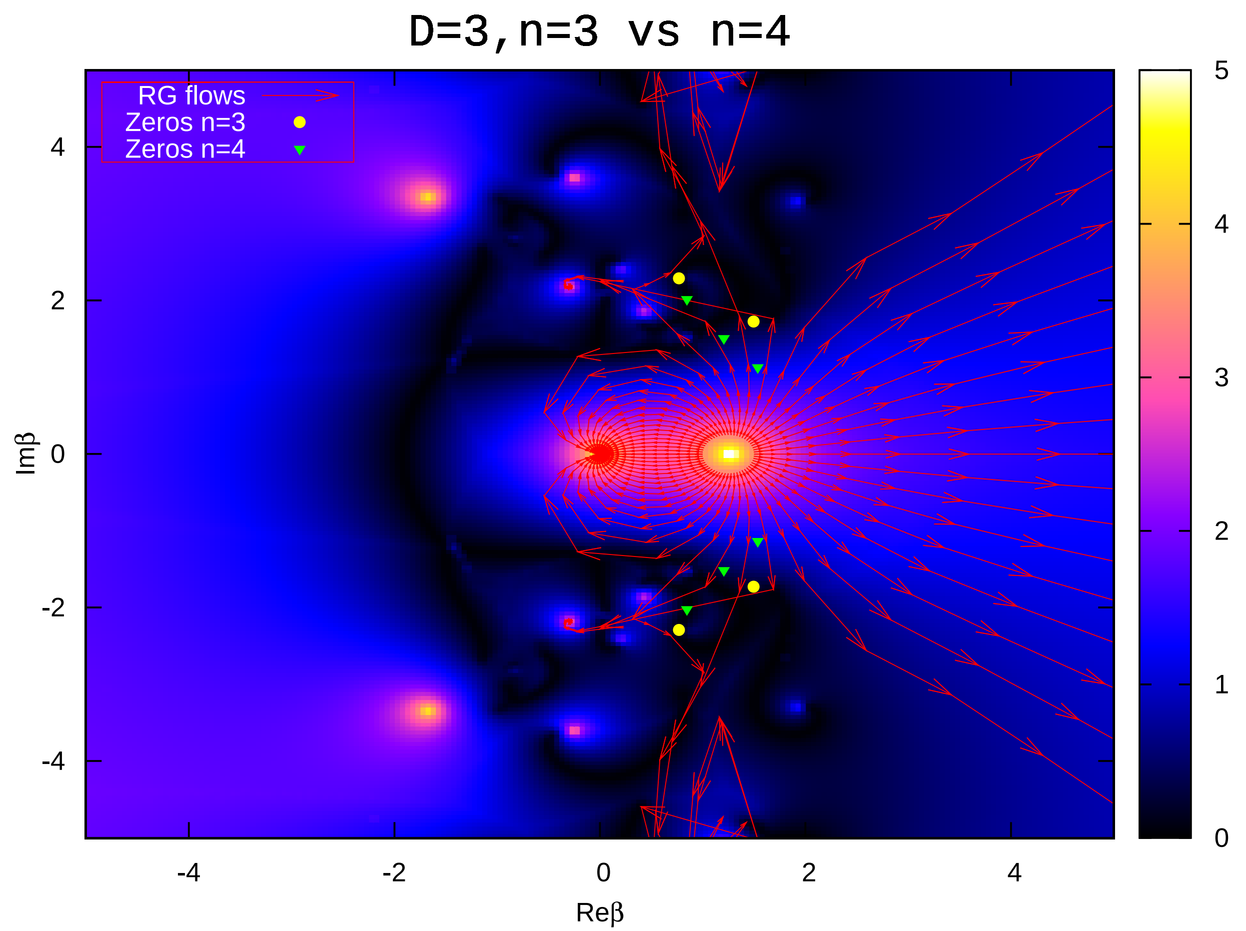

Figure 7: RG flows for the $D = 3$ hierarchical model in the complex β plane obtained by the two lattice method. Circles and triangles are at the Fisher's zeros for 2^3 and 2^4 sites. Darker=more ambiguous.

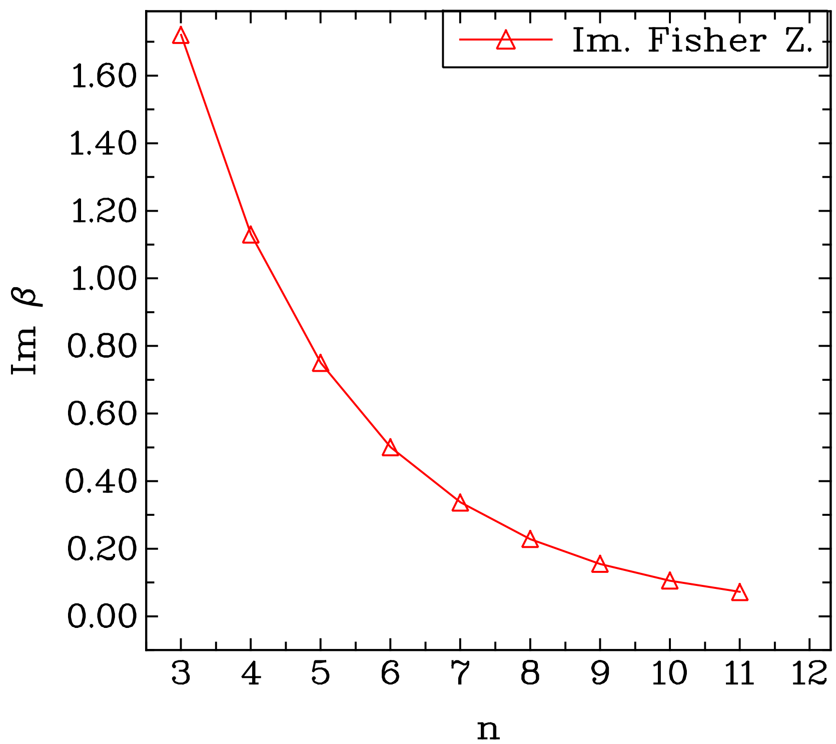

Figure 8: Imaginary part of the lowest Fisher's zero for the $D = 3$ hierarchical model for 2^n sites (the zeros pinch the real axis).

Continuous flows? (real case, with Y. Liu and B. Oktay)

For the hierarchical model, we calculate recursively the Fourier transform $R_n(k)$ of the local measure after n blockings where 2 sites are replaced by 1 site. This is a discrete RG transformation with scale $b = 2^{1/D}$. We can in principle extend to an arbitrary real number of sites being blocked: $2 \rightarrow b^D$ and $\frac{c}{4} \rightarrow b^{-2-D}$ (blocks of size 2^q have a coupling $(c/4)^q$). The recursion formula becomes

$$R_{n+1}(k) = C_{n+1}\ \mathrm{e}^{-\frac{1}{2}\beta\frac{\partial^2}{\partial k^2}} \left(R_n(b^{-(D+2)/2}\ k)\right)^{b^D} ,$$

For b^D integer, polynomial approximations converge rapidly. For b^D noninteger, we need expansions of $ln(R_n(k)) = ln(1 + ak^2 + \dots)$ but no apparent convergence is observed.

Linear analysis near $b^D = 2$

We set $b^D = 2 + \zeta$ for small ζ

$RG_\zeta[R_0^* + \zeta R_1^*] = R_0^* + \zeta R_1^* + O(\zeta^2)$

R_1^* obeys the equation $R_1^* = L[R_1^* + G]$

with $G = -\frac{5}{6} k^2 \frac{\partial R_0^*}{\partial k^2} + \frac{1}{2} R_0^* \ln R_0^*$

Note: $R_0^*(k)$ has zeros!

L is the linear map at $\zeta = 0$: $RG_0[R_0^* + \delta R] = R_0^* + L[\delta R]$

Expansions of $\ln R_0^*$ in terms of the eigenvectors of L fail to converge

Exponents for b^D integer and $D = 3$

The exponents ω and ν for $D = 3$ fall on a curve found by Litim (PRD 76) for a variety of FRG flows in the local potential approximation.

This curve represents values of $N(\nu, \mu) \equiv -log_{10}((\nu - \nu_{opt})^2 + (\omega - \omega_{opt})^2)$ versus ν/ω. A-priori, we would expect no correlations between these two quantities for the large set of models considered.

The values for $b^D = 2,\ 3\ \ldots, 8$ appear to be ordered on that curve. This suggests that the information lost during the RG transformation could be used as a parameter on this curve.

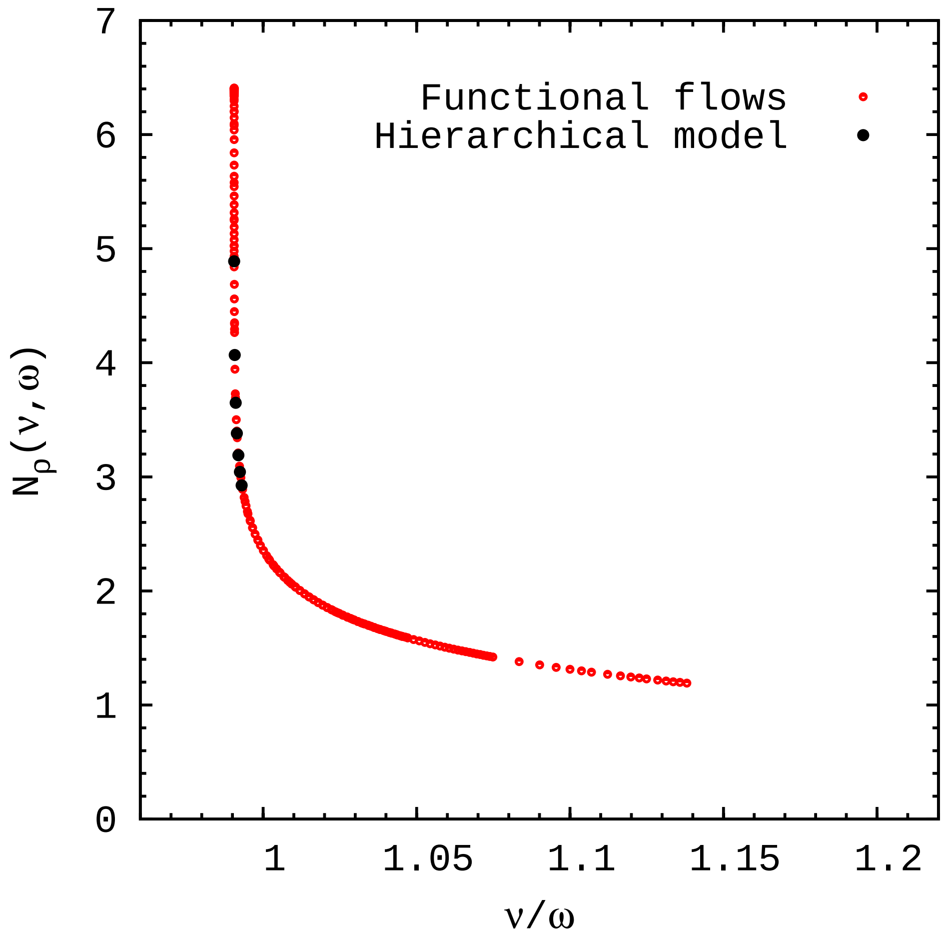

Figure 9: $N(\nu, \mu) \equiv -log_{10}((\nu - \nu_{opt})^2 + (\omega - \omega_{opt})^2)$ versus ν/ω from Litim (red dots) and HM with $b^D = 2,\ 3\ \ldots, 8$ (from L to R, black dots).

Fisher's zeros in $4D$ LGT

Spectral decomposition: $Z = \int_0^{S_{max}} dS n(S) \mathrm{e}^{-\beta S}$

$n(S)$: density of states; $\mathcal{N}$: number of plaquettes.

$n(S)\mathrm{e}^{-\beta \mathcal{N} s} = \mathrm{e}^{\mathcal{N}(f(s)-\beta s)} = \mathrm{e}^{\mathcal{N}(f(s_0)+(1/2)f''(s_0)(s-s_0)^2+\dots)}$

with $s = S/\mathcal{N}$ and $f'(s_0) = \beta$. $f(s)$ is a color entropy density.

If $Ref''(s_0) < 0$, the distribution becomes Gaussian in the infinite volume. Gaussian distributions have no complex zeros. The level curve $Ref''(s_0) = 0$ is the boundary of the region where Fisher's zeros may appear.

In the $U(1)$ case, conjugate pairs pinch the real axis, but for $SU(2)$ a finite gap remains present.

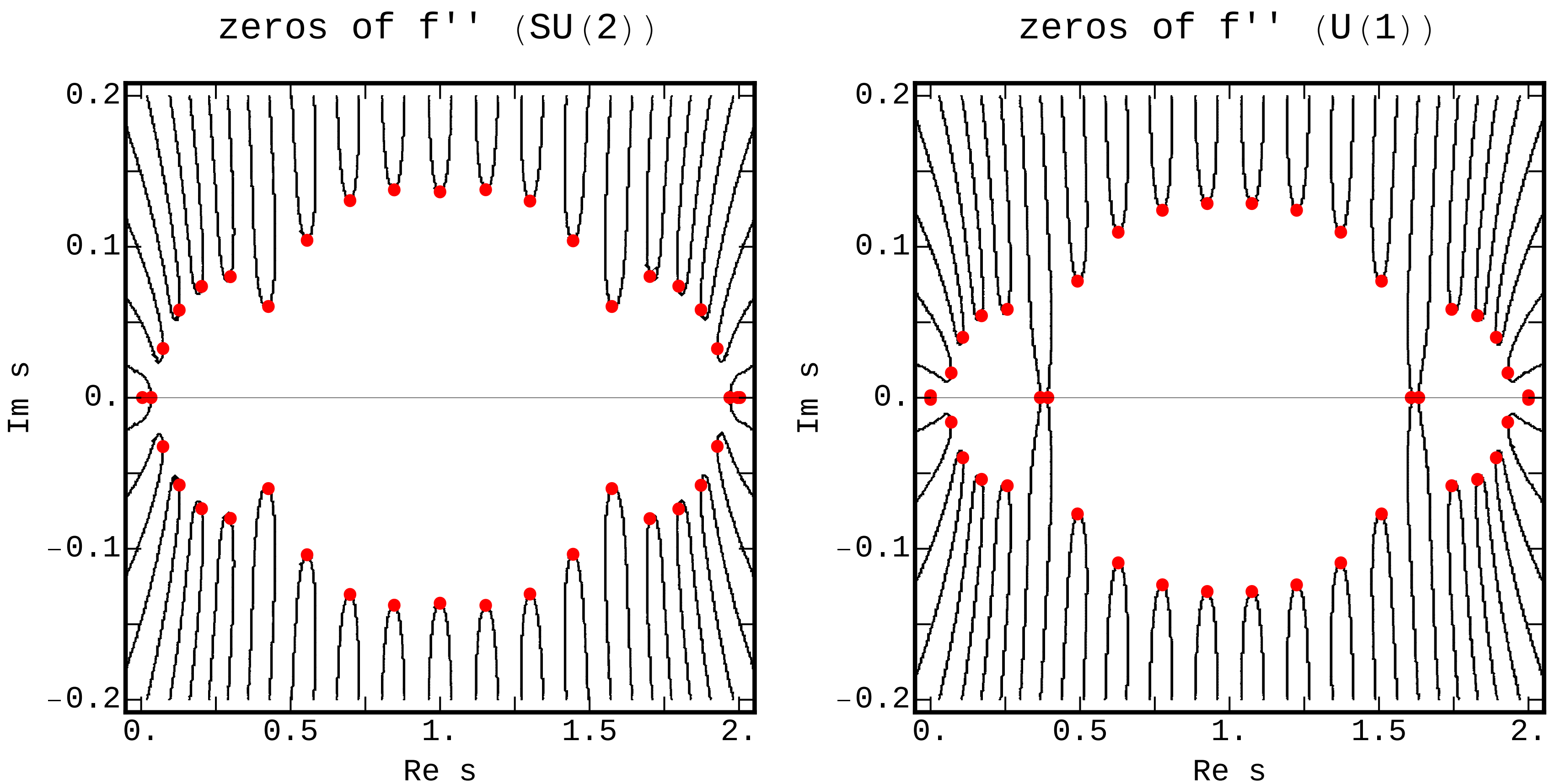

Figure 10: Complex zeros and zeros of the real part of $f''(s)$ in the complex s plane with a Chebyshev (40) on 4^4 for $SU(2)$ (left) and $U(1)$ (right).

Images in the β plane

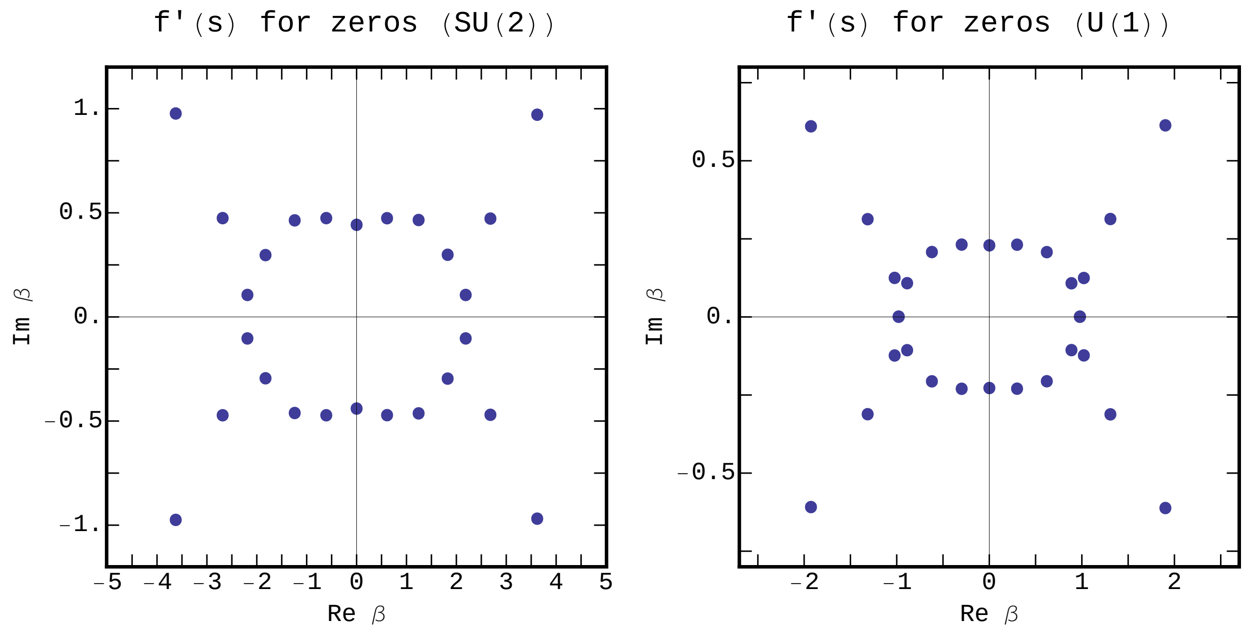

Figure 11: $f'(s)$ evaluated at the complex zeros of $f''(s)$ shown on the previous figure for $SU(2)$ (left) and $U(1)$ (right). $n(S) = \mathrm{e}^{\mathcal{N} f(S/\mathcal{N})}$.

These pictures suggest that Fisher's zeros should appear on approximately vertical linear structures. This is confirmed numerically.

For $U(1)$, naive histogram reweighting works well. δZ can be estimated from $(n_i(S) - < n(S) >)$, where i is an index for independent runs. Zeros can be excluded if $|\delta Z| << |Z|$.

For $SU(2)$, the imaginary part of Fisher's zeros are too large to use simple reweighting methods. By using Chebyshev interpolation for $f(s)$ and monitoring the numerical stability of the integrals with the residue theorem, it is possible to obtain reasonably stable results. Unlike the $U(1)$ case, the imaginary part of the lowest zeros does not decrease as the volume increases, but their linear density increases at a rate compatible with L^{-4}.

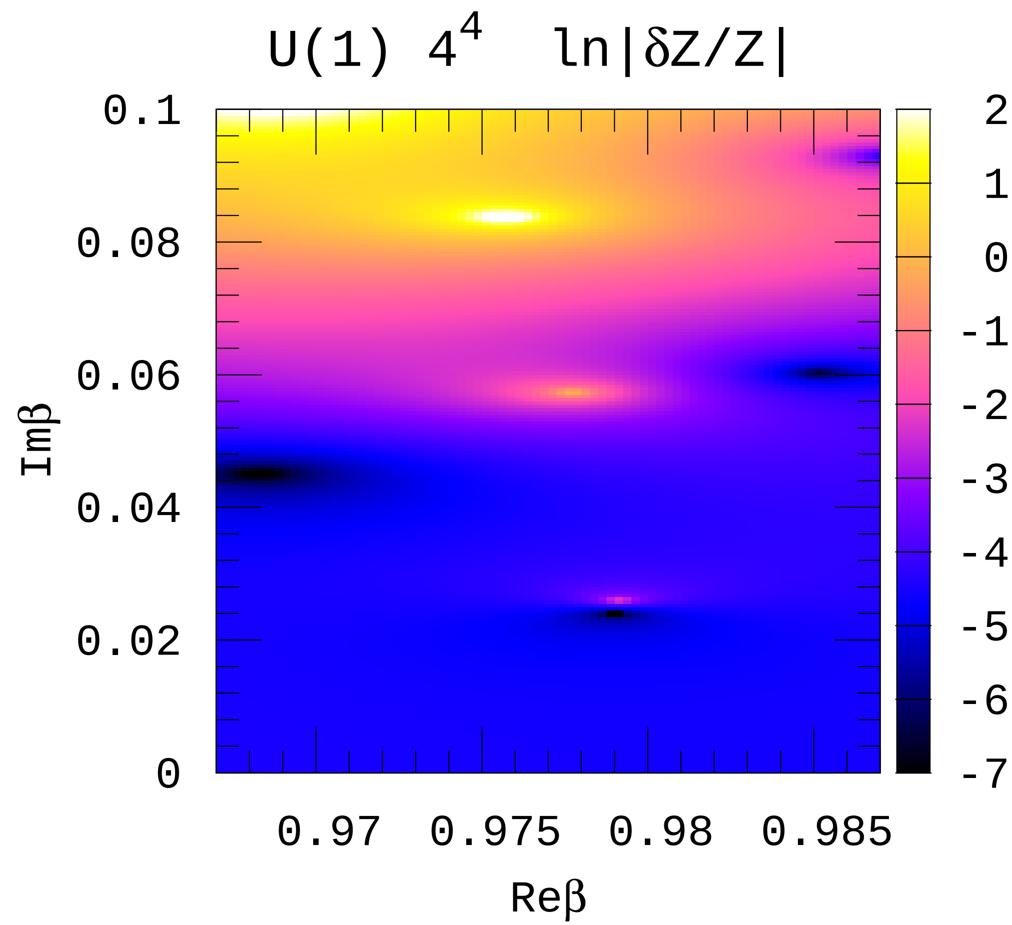

Figure 12: $|\delta Z/Z|$ for $U(1)$ on 4^4.

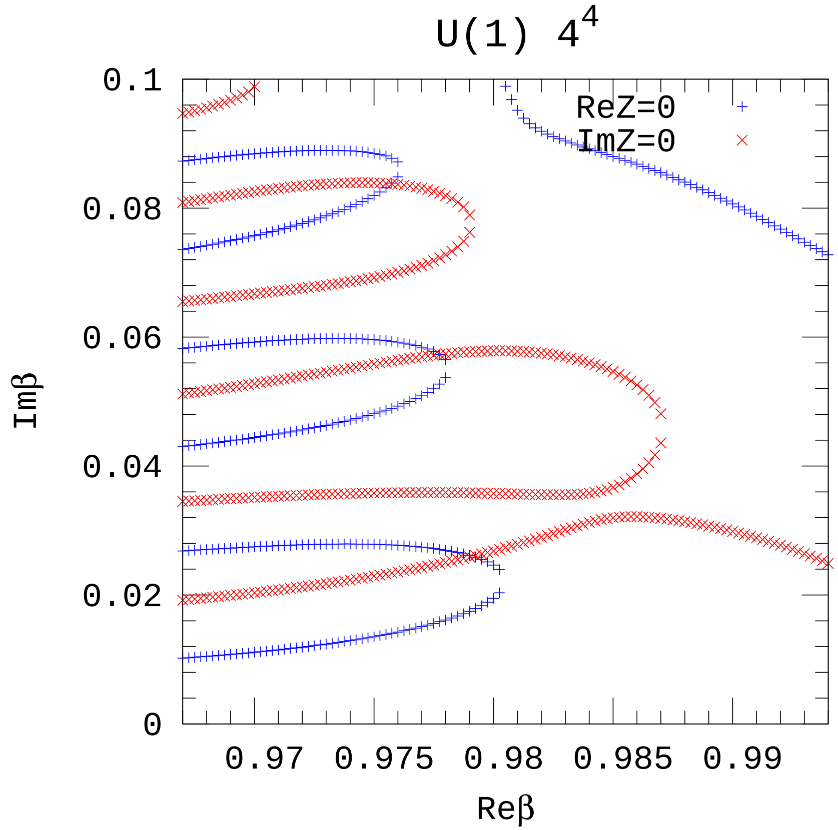

Figure 13: Zeros of the Re (blue) and Im (red) part of Z for $U(1)$ using the density of states for 4^4.

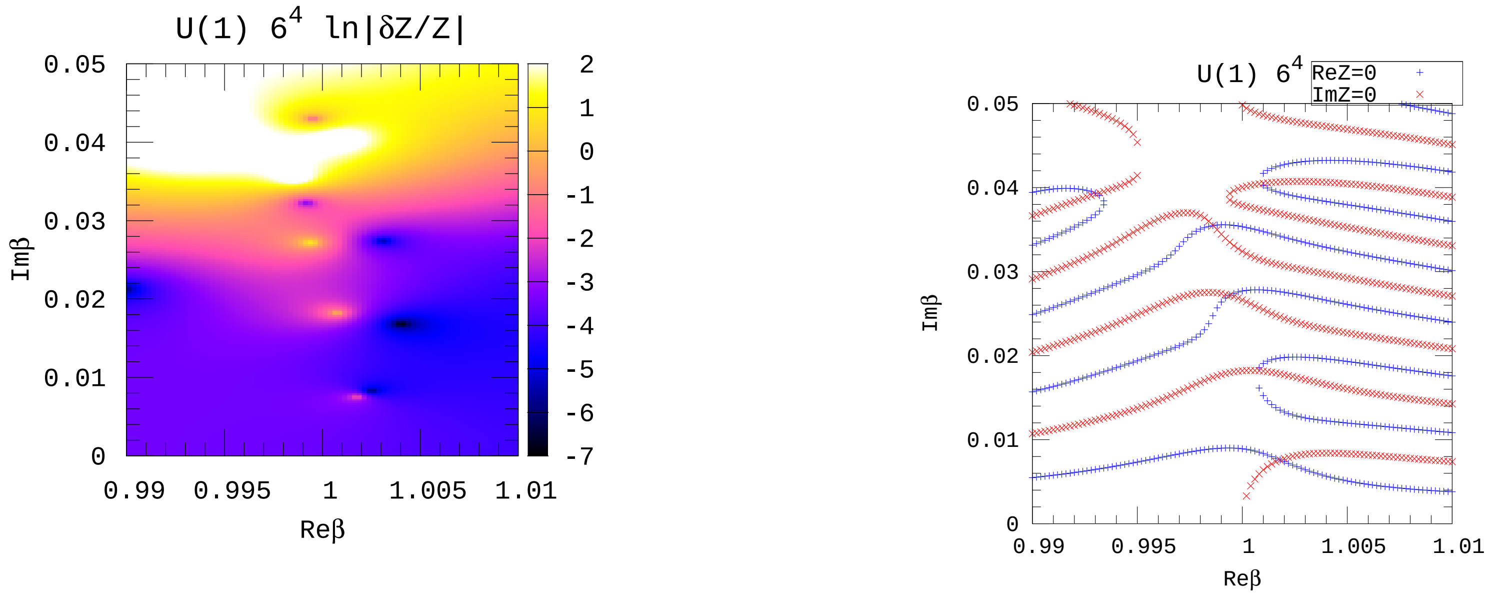

Figure 14: Same figures for a 6^4 lattice.

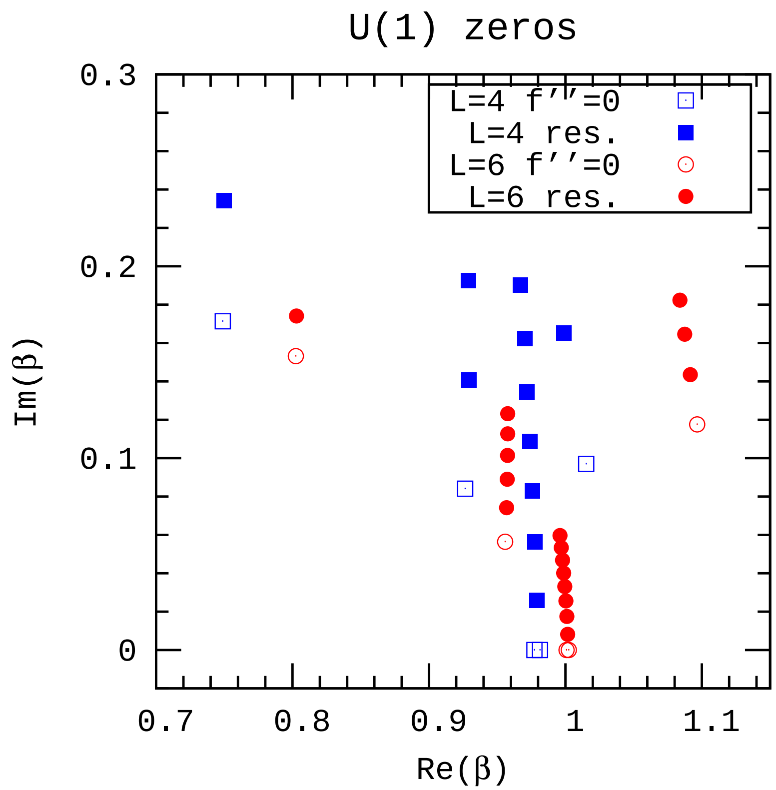

Figure 15: Images of the zeros of $f''(s)$ in the β plane (open symbols) and Fisher's zeros (filled symbols) for $U(1)$ on 4^4 (squares) and 6^4 (circles) lattices.

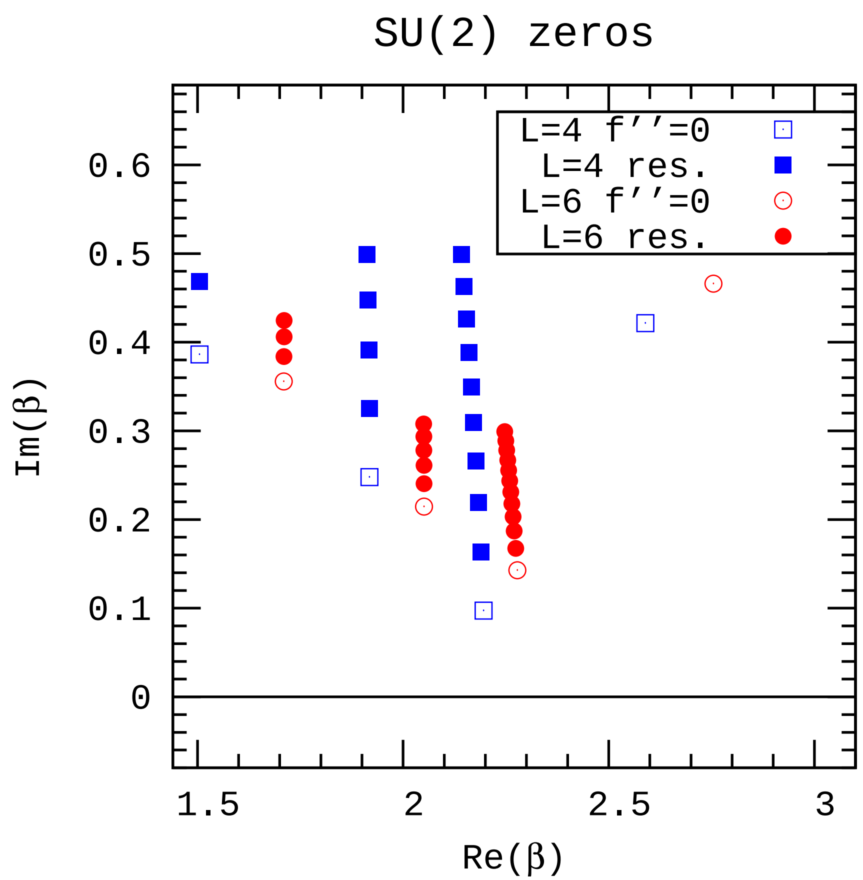

Figure 16: Images of the zeros of $f''(s)$ in the β plane (open symbols) and Fisher's zeros (filled symbols) for $SU(2)$ on 4^4 (squares) and 6^4 (circles) lattices.

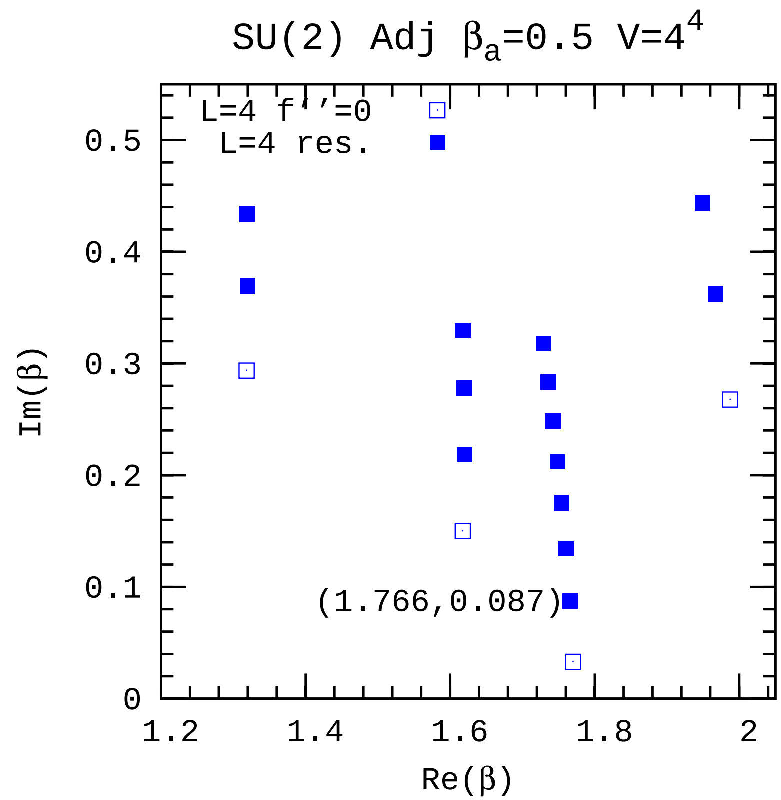

Figure 17: Effect of an adjoint term (+0.5), the lowest zero goes down by about 40 percent.

Conclusions

- It is possible to extend various RG flows to the complex β plane.
- When the size of the system is comparable to the Compton wavelength of the gap, there is a strong scheme dependence.
- Fisher's zeros control the global behavior of the RG flows.
- Confinement="open gate".
- Plans: QED, $SU(3)$ with various N_f.
- $\epsilon\upsilon\chi\alpha\rho\iota\sigma\tau\omega$!