Alternative models of noncommutativity

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Space-time is nonlocal on very short distances.

S. Doplicher, K. Fredenhagen and J. Roberts, PLB 331, 39

In general case:

$$[\hat{x}^\mu, \hat{x}^\nu] = \hat{\omega}^{\mu\nu}.$$

Noncommutativity in Quantum Mechanics

1. Noncommutativity with mixed spatial and spin degrees of freedom

$$\left[\hat{x}^i, \hat{x}^j\right] = i\theta^2 \varepsilon^{ijk} \hat{s}^k.$$

2. Position-dependent noncommutativity in quantum mechanics

$$[\hat{x}^\mu, \hat{x}^\nu] = \theta\omega^{\mu\nu}(\hat{x}),$$

e.g.,

$$[\hat{x}, \hat{y}] = \frac{i\theta}{1 + \theta\alpha\left(\hat{x}^2 + \hat{y}^2\right)}.$$

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F. Berezin, M. Marinov, Ann.Phys.104, 336

Phase superspace: $\left(x^i, p_i, \xi^i\right)$, $i = 1, 2, 3$, $\xi^i \xi^j + \xi^j \xi^i = 0$. The Poisson bracket:

$$\{f(\xi), g(\xi)\} = -i \left(f \overleftarrow{\partial_k}\right) \left(\overrightarrow{\partial_k} g\right).$$

For the canonical variables:

$$\left\{\xi^k, \xi^l\right\} = -i\delta^{kl}, \quad \left\{x^k, p_l\right\} = \delta^k_l.$$

The rotation group in the Grassmann subspace:

$$S^i = -\frac{i}{2}\varepsilon^{ijk}\xi^j\xi^k, \quad \left\{S^i, \xi^j\right\} = \varepsilon^{ijk}\xi^k, \quad \left\{S^i, S^j\right\} = \varepsilon^{ijk}S^k.$$

The orbital angular momentum:

$$L^i = \varepsilon^{ikl}x^k p^l, \quad \left\{L^i, x^j\right\} = \varepsilon^{ijk}x^k, \quad \left\{L^i, L^j\right\} = \varepsilon^{ijk}L^k.$$

The complete angular momentum:

$$\mathbf{J} = \mathbf{L} + \mathbf{S}, \quad \left\{J^i, J^j\right\} = \varepsilon^{ijk}J^k.$$

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The classical Hamiltonian action of the model reads

$$S_0 = \int dt \left[\mathbf{p}\dot{\mathbf{x}} - \frac{i}{2}\xi\dot{\xi} - H(x,p,\xi) \right],$$

where $H(x,p,\xi) = \mathbf{p}^2/2 + V_0(x) + (\mathbf{LS})\, V_1(x) + \mathbf{SB}(x)$.

The simplest way to obtain nonvanishing PB is Bopp shift:

$$x^i \to x^i_{NC} = x^i - \frac{1}{2}\theta^{ij} p_j, \; \left\{ x^i_{NC}, x^j_{NC} \right\} = \theta^{ij},$$

but it breaks symmetries of the system.

To preserve rotational symmetry, consider a following deformation

$$x^i \to \tilde{x}^i = x^i + \theta S^i,$$

New coordinates $\tilde{x}^i$ are even elements of the Grassmann algebra, transform like a vector

$$\left\{ J^i, \tilde{x}^j \right\} = \varepsilon^{ijk} \tilde{x}^k.$$

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Nonvanishing Poisson Barckets are

$$\{\tilde{x}^i, \tilde{x}^j\} = \theta^2 \varepsilon^{ijk} S^k, \ \ \{\tilde{x}^i, p_j\} = \delta^i_j,$$

$$\{\tilde{x}^i, \xi^j\} = \theta \varepsilon^{ijk} \xi^k, \ \ \left\{\xi^k, \xi^l\right\} = -i\delta^{kl}.$$

In course of quantization these PB determine

$$[\hat{x}^i, \hat{x}^j] = i\theta^2 \varepsilon^{ijk} \hat{s}^k, \ \ [\hat{x}^i, \hat{p}_j] = i\delta^i_j,$$

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Commutation relations involving $\hat{s}^i = -\frac{i}{2}\varepsilon^{ijk}\hat{\xi}^j\hat{\xi}^k$ are

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H. Falomir, et al, PLB 680, 384.

Quantization

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Quantization

Renormalizing $\hat{\xi}^i = \hat{\sigma}^i/\sqrt{2}$, one gets the Clifford algebra,

$$\left[\hat{\sigma}^i, \hat{\sigma}^j\right]_+ = 2\delta^{ij},$$

realized by Pauli matrices σ^i.

Representation: $\hat{\xi}^i = \sigma^i/\sqrt{2}$, $\hat{s}^i = -i/2\varepsilon^{ijk}\hat{\xi}^j\hat{\xi}^k = \sigma^i/2$,

$$\hat{x}^i = x^i \mathbf{I} + \theta\sigma^i/2, \;\; \hat{p}_i = -i\partial_i \mathbf{I},$$

Modified Pauli equation:

$$i\partial_t\varphi = \hat{H}\left(\hat{x}, \hat{p}, \hat{\xi}\right)\varphi.$$

Nonlocality:

$$\Delta x^i \Delta x^j \geq \theta^2 \varepsilon^{ijk} \left| \langle\Psi| \hat{s}^k |\Psi\rangle \right|.$$

Let $|\Psi\rangle$ be eigenstate for $\hat{s}_z$, $\hat{s}_z |\Psi\rangle = s_z |\Psi\rangle$, $s_z = -s, ..., s$.

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Relativistic generalization

Hamiltonian form of the Berezin-Marinov action is

$$S=\int_{\tau_i}^{\tau_f}\left[p_\mu\dot{x}^\mu-\frac{i}{2}\xi_\mu\dot{\xi}^\mu+\frac{i}{2}\xi^5\dot{\xi}^5-\frac{i}{2}\chi T_1-\lambda T_2\right]d\tau,$$

$$T_1=\xi^\mu\left(p_\mu+eA_\mu\right)+m\xi^5,\;\; T_2=\left(p_\mu+eA_\mu\right)^2-m^2+ieF_{\mu\nu}\xi^\mu\xi^\nu,$$

Poisson brackets between the canonical variables are

$$\{x^\mu,p^\nu\}=g^{\mu\nu},\;\;\{\xi^\mu,\xi^\nu\}=-ig^{\mu\nu},\;\;\left\{\xi^5,\xi^5\right\}=i,$$

where $g^{\mu\nu}$ =diag$(1,-1,-1,-1)$. Two first-class constraints are

$$T_1=0,\;\; T_2=0.$$

Generators of the Lorentz group are: $J^{\mu\nu}=L^{\mu\nu}+S^{\mu\nu}$,

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Noncommutative deformation is:

$$x^\mu \to \tilde{x}^\mu = x^\mu + \theta W^\mu,$$

$$W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} p_\nu J_{\rho\sigma} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} p_\nu S_{\rho\sigma}.$$

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Relativistic generalization

Observe that $T_1 = \xi^\mu (p_\mu + eA_\mu) + m\xi^5$ is an odd element of the Grassmann algebra, therefore

$$\{T_1, T_1\} = -iT_2 = -i\left[(p_\mu + eA_\mu)^2 - m^2 + ieF_{\mu\nu}\xi^\mu\xi^\nu\right].$$

T_2 is even, so that $\{T_2, T_2\} = 0$, and

$$\{T_2, T_1\} = i\{\{T_1, T_1\}, T_1\} \equiv 0.$$

We postulate the form of the first constraint as

$$\tilde{T}_1 = \xi^\mu (p_\mu + eA_\mu(\tilde{x})) + m\xi^5 = 0.$$

We determine the second constrained as $\tilde{T}_2 = i\left\{\tilde{T}_1, \tilde{T}_1\right\} = 0$,

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Observe that $T_1 = \xi^\mu (p_\mu + eA_\mu) + m\xi^5$ is an odd element of the Grassmann algebra, therefore

$$\{T_1, T_1\} = -iT_2 = -i\left[(p_\mu + eA_\mu)^2 - m^2 + ieF_{\mu\nu}\xi^\mu\xi^\nu\right].$$

T_2 is even, so that $\{T_2, T_2\} = 0$, and

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The operators $\hat{\xi}^\mu$, $\hat{\xi}^5$ are generators of the Clifford algebra C_5,

$$\hat{\xi}^\mu = i\gamma^5\gamma^\mu/\sqrt{2}, \ \ \hat{\xi}^5 = i\gamma^5/\sqrt{2},$$

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The representation for the operators of coordinates and momenta

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First-class constraints become conditions on the physical states

$$\hat{T}_1 \psi = 0, \quad \hat{T}_2 \psi = 0.$$

The first equation reads

$$\left[i\gamma^\mu \left(\partial_\mu + ieA_\mu \left(x^\mu \mathbf{I} - \frac{i\theta}{2}\gamma^5 \sigma^{\mu\nu} \partial_\nu \right) \right) - m \right] \psi(x) = 0,$$

where some ordering should be specified. The second eq. is a consequence of the first one, since

$$\hat{T}_2 = \left(\hat{T}_1 \right)^2 .$$

The action of operator $f\left(x^\mu \mathbf{I} - \frac{i\theta}{2}\gamma^5\sigma^{\mu\nu}\partial_\nu\right)$ on spinor $\psi(x)$ can be represented as:

$$f\left(x^\mu \mathbf{I} - \frac{i\theta}{2}\gamma^5 \sigma^{\mu\nu}\partial_\nu \right)\psi(x) = f \tilde{\star} \psi = f(x) \exp\left\{ -\frac{i\theta}{2} \overleftarrow{\partial_\mu} \gamma^5 \sigma^{\mu\nu} \overrightarrow{\partial_\nu} \right\} \psi(x)$$

$$= f\psi - \frac{i\theta}{2}\partial_\mu f \gamma^5 \sigma^{\mu\nu}\partial_\nu \psi - \frac{\theta^2}{8}\partial_{\mu_1}\partial_{\mu_2} f \gamma^5 \sigma^{\mu_1\nu_1}\gamma^5\sigma^{\mu_2\nu_2}\partial_{\nu_1}\partial_{\nu_2}\psi + \ldots$$

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The above equation can be represented in the form

$$\left[i\gamma^{\mu}\left(\partial_{\mu}+ieA_{\mu}\left(x\right)\right)-m\right]\tilde{\star}\psi=0.$$

This equation is covariant under the Lorentz transformation

$$x^{\mu}\rightarrow x'^{\mu}=\Lambda^{\mu}_{\nu}x^{\nu},\ \psi\rightarrow\psi'\left(x'\right)=S\left(\Lambda\right)\psi\left(x\right),$$

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Introduce modified multiplication between two spinors

$$\bar{\varphi}\tilde{\star}\psi=\bar{\varphi}\exp\left\{-\frac{i\theta}{2}\overleftarrow{\partial_{\mu}}\gamma^{5}\sigma^{\mu\nu}\overrightarrow{\partial_{\nu}}\right\}\psi,$$

with properties

$$\left(\bar{\varphi}\tilde{\star}\psi\right)^{*}=\bar{\psi}\tilde{\star}\varphi,\ \ \int d^{4}x\bar{\varphi}\tilde{\star}\psi=\int d^{4}x\bar{\varphi}\psi.$$

Noncommutative Dirac equation

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Conclusions and questions

Action functional for noncommutative Dirac field is:

$$S = \int d^4x \left[\frac{i}{2}\bar{\psi}\tilde{\star}\gamma^\mu \partial_\mu \psi + \frac{i}{2}\partial_\mu \bar{\psi}\gamma^\mu \tilde{\star}\psi - m\bar{\psi}\tilde{\star}\psi + e\bar{\psi}\tilde{\star}\gamma^\mu A_\mu \tilde{\star}\psi \right].$$

Variation of this action over $\bar{\psi}$ gives

$$\left[i\gamma^\mu \left(\partial_\mu + ieA_\mu(x) \right) - m \right] \tilde{\star}\psi = 0.$$

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Questions:

- Unitarity
- Gauge invariance
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Suppose that noncommutativity $[\hat{x}^i, \hat{x}^j] = \omega^{ij}(\hat{x})$ is given, e.g.,

$$[\hat{x}, \hat{y}] = \frac{i\theta}{1 + \theta\alpha\left(\hat{x}^2 + \hat{y}^2\right)}$$

• The aim is to construct consistent quantum mechanics on such noncommutative space.

The problem is to complete the algebra: $[\hat{x}^i, \hat{p}_j] = ?$, $[\hat{p}_i, \hat{p}_j] = ?$

$$[\hat{p}_k, [\hat{x}^i, \hat{x}^j]] + [\hat{x}^j, [\hat{p}_k, \hat{x}^i]] + [\hat{x}^i, [\hat{x}^j, \hat{p}_k]] = 0 \ ,$$

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Now if $\omega^{ij}(\hat{x}) = f^{ij}_l \hat{x}^l$, we have $f^{ij}_k = 0$.

• So, $[\hat{x}^i, \hat{p}_j] = i\delta^i_j(\hat{x})$, $[\hat{p}_i, \hat{p}_j] = \varpi_{ij}(\hat{x})$.

To answer the question consider the quantization of the model:

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The Dirac brackets for canonical variables are:

$$\left\{x^i, x^j\right\}_D = \theta d(x)\varepsilon^{ij}, \ \left\{p_i, p_j\right\}_D = \theta\left(\partial_2 B_2 \partial_1 B_1 - \partial_1 B_2 \partial_2 B_1\right) d\varepsilon_{ij},$$

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From the equation

$$\frac{\varepsilon^{ij}}{\left[1+\theta\left(\partial_1 B_2 - \partial_2 B_1\right)\right]} = \omega^{ij}(x),$$

one can determine B_i and complete the algebra.

Let $B_i = \varepsilon^{ij}\partial_j\phi$, and

$$\omega^{ij} = \varepsilon^{ij}/\{1+\theta f\left[\alpha\left(x^2+y^2\right)\right]\}, \quad f(0) = const.$$

In this case

$$B_1 = y\frac{F\left(\alpha\left(x^2+y^2\right)\right) - F(0)}{2\alpha\left(x^2+y^2\right)}, \ B_2 = -x\frac{F\left(\alpha\left(x^2+y^2\right)\right) - F(0)}{2\alpha\left(x^2+y^2\right)},$$

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Local noncommutativety

$$[\hat{x},\hat{y}]=\frac{i\theta}{1+\theta\alpha\left(\hat{x}^2+\hat{y}^2\right)},\ [\hat{p}_x,\hat{p}_y]=\frac{3\theta\alpha^2}{16}\left(\hat{x}^2+\hat{y}^2\right)^2 d\ ,$$

$$[\hat{x},\hat{p}_x]=\left[1+\frac{\alpha\theta}{4}\left(\hat{x}^2+3\hat{y}^2\right)\right]d\ ,\ [\hat{x},\hat{p}_y]=-\frac{\alpha\theta}{2}\hat{x}\hat{y}d\ ,$$

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Perspectives

- Scattering of plane waves on local noncommutativity

$$\hat{H} = \frac{1}{2}\hat{p}^2$$

- Relativistic generalization $d = 2$:

$$L = p_\mu \dot{x}^\mu + \lambda[p^2 - m^2] + \frac{\theta}{2}\left(p_\mu + B_\mu(x,\alpha)\right)\varepsilon^{\mu\nu}\left(\dot{p}_\nu + \dot{B}_\nu(x,\alpha)\right).$$

After quantization one gets

$$[\hat{x}^\mu, \hat{x}^\nu] = \theta\varepsilon^{\mu\nu}d(\hat{x}), \quad [\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu(\hat{x}), \quad [\hat{p}_\mu, \hat{p}_\nu] = \varepsilon_{\mu\nu}\varpi(\hat{x}),$$

$$[\hat{p}^2 - m^2]\Phi = 0.$$

- Relativistic wave equations on curved noncommutative space.

$$\hat{x}^\mu = x^\mu + O(\theta), \quad \hat{p}_\mu = -i\partial_\mu + O(\theta),$$

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