Quantum corrections in the GFT formulation of the EPRL/FK models

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Quantum Corrections in the Group Field Theory Formulation of the EPRL/FK Models

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Plan

- Motivation for GFT
- GFT
- GFT for BF
- GFT for EPR
- saddle point method for GFT diagrams and power counting

Motivation for GFT

Einstein-Hilbert-Palatini-Holst action

$$\begin{aligned} S &= \frac{1}{8\pi G}\int_M \epsilon_{ABCD} e^A \wedge e^B \wedge R^C D[\omega] \\ &+ \frac{1}{\gamma} e^A \wedge e^B \wedge R_{AB}[\omega] \end{aligned}$$

with $e \in \Omega^1(M)$, $\omega \in \Omega^1(M) \times so(3,1), R \in \Omega^2(M) \times so(3,1)$

second term classically irrelevant if torsion is zero

$\Longrightarrow$ BF theory with constraints

$$S = \int_M B \wedge R[\omega] + \lambda^\alpha C_\alpha[B]$$

without constraints it is topological. It gives ω flat and $D_\omega B = 0$

Plebanski simplicity constraints

$$\epsilon^{abcd} B_{ab}^{AB} \wedge B_{cd}^{CD} \propto \epsilon^{ABCD}$$

Discretization for BF (2+1)

$$e^A \to E^A = \int_\ell e^A$$

$$\omega \to \int_{\ell_\star} \omega = \log h_{\ell_\star} \in \mathcal{L}$$

$$\Omega_\ell = \log h_{f_*} = \log \prod_{\ell_\star \subset \partial f_\star} h_{\ell_\star}$$

$$S(e, \omega) \to S(E, \Omega) = \sum_{\ell \in L} \mathrm{Tr} E_\ell \Omega_\ell$$

$$Z(T) = \int_{\mathcal{L}} \prod_{\ell \in L} dE_\ell \int_G \prod dh_{\ell_\star} \exp\left(i \sum_\ell \mathrm{Tr} E_\ell \Omega_\ell \right) = \int_G \prod dh \delta(\prod_{f_\star} h_{f_\star})$$

This is a spin foam amplitude, that is the amplitude we obtain discretizing 3-d BF and associating to space-time triangulation with tetraedra graphs in the dual triangulation.

In 3d easier

- tetraedron$\rightarrow$point
- triangles (faces) $f \rightarrow$ edges $\ell_\star$
- edges ℓ $\rightarrow$faces $f_\star$

This result can be obtained by a field theory

Group field theory [Boulatov, Freidel, Oriti ...]

In GFT, the field arguments live on products of Lie groups. Feynman amplitudes of GFT give back spin-foams (two-complexes with vertices, stranded lines (propagators) and faces (closed circuits of strands)

Technically a generalization of matrix models (the fields are higher order tensors)

- Fields

$$\phi : (g_1, ...g_D) \in [SO(D)]^D \to \phi(g_1, ...g_D)$$

D is spacetime dimension.

• Propagator

$C : \phi \to C\phi$ Hermitian

with Hermitian kernel $C(g_1, \ldots, g_D; g'_1, \ldots, g'_D)$:

$$[C\phi](g_1, \ldots, g_D) = \int dg'_1 \ldots dg'_D C(g_1, \ldots, g_D; g'_1, \ldots, g'_D)\phi(g'_1, \ldots, g'_D)$$

$\Longrightarrow$ propagators are stranded lines with D strands (figure).

The precise form of C defines the different models.

• Vertices

Simple vertex joining $2p$ strands: its kernel is a product of p delta functions matching strand arguments, so that each delta function joins two strands in two different lines.

It is the same for all models.

while

in the spin-foam literature the term "vertex" refers to the vertex *together with the square roots of its dressing propagators*.

The usual vertex for $D-$dimensional GFT is a ϕ^{D+1} simple vertex (picture)

For instance the SU(2) BF vertex in 3 dimensions is simple (with $p = 6$)

$$S_{\text{int}}[\phi] = \frac{\lambda}{4}\int \left(\prod_{i=1}^{12} dg_i\right) \phi(g_1, g_2, g_3)\phi(g_4, g_5, g_6)\phi(g_7, g_8, g_9) \phi(g_{10}, g_{11}, g_{12})\, K(g_1, ..g_{12}),$$

with

$$K(g_1, ..g_{12}) = \delta(g_3 g_4^{-1})\delta(g_2 g_8^{-1})\delta(g_6 g_7^{-1})\delta(g_9 g_{10}^{-1})\delta(g_5 g_{11}^{-1})\delta(g_1 g_{12}^{-1})$$

- Graphs

A stranded graph is called *regular* if it has no *tadpoles* (hence any line ℓ joins two distinct vertices) and no *tadfaces* (hence each face f goes at most once through any line of the graph).

It is convenient to introduce orientations

- the ordinary incidence matrix $\epsilon_{\ell v}$ which has value 1 (-1) if the edge ℓ enters (exits) the vertex v, 0 otherwise.

- the incidence matrix $\eta_{f\ell}$ which has value $+1$ if the face f goes through edge ℓ in the same direction, -1 if the face f goes through edge ℓ in the opposite direction, 0 otherwise.

GFT for 3D BF [Boulatov]

• Propagator in direct space

It is just the projection on gauge invariant fields,

$$\mathbb{P}(\phi) = \int_{SO(D)} dh \phi(g_1 h, \ldots, g_D h),$$

$\mathbb{P}^2 = \mathbb{P}$ so that the only eigenvalues are 0 and 1 (which means that the BF theory has no dynamics).

$\mathbb{P}$ is Hermitian with kernel

$$\mathbb{P}(g_1, ..., g_D; g'_1, ...g'_D) = \int dh \prod_{i=1}^{D} \delta(g_i h (g'_i)^{-1}).$$

- Amplitudes in direct space

Choose an arbitrary orientation of the lines and faces of a graph $\mathcal{G}$ (which for simplicity has no external legs).

Combining the vertex and the propagator, integration over all g leads to

$$A_{\mathcal{G}} = \int \prod_{\ell \in L_{\mathcal{G}}} dh_\ell \prod_{f \in \mathcal{F}_{\mathcal{G}}} \delta \left(\vec{\prod}_{\ell \in f} h_\ell^{\eta_{\ell f}} \right) ,$$

this amplitude neither depends on the arbitrary orientation of the lines, nor on those of the faces.

- Amplitudes in the angular momentum basis $|j, m>$

$$1_j = \sum_m |j, m\rangle\langle j, m|,$$

in 3D

$$A_{\mathcal{G}} = \prod_f \sum_{j_f} d_{j_f} \prod_v \{6j\}$$

In 4D $\qquad g = (g_+, g_-) \in SU(2) \times SU(2) \quad j \equiv (j_+, j_-)$

$$A_{\mathcal{G}} = \prod_f \sum_{j_{f+}, j_{f-}} d_{j_{f+}} d_{j_{f-}} \prod_v \{15 j_+\}\{15 j_-\}.$$

Coherent states representation

• In 3D we use $SU(2)$ coherent states

$$|j, g\rangle \equiv g|j, j\rangle = \sum_m |j, m\rangle [R^{(j)}]^m_j(g).$$

$$1_j = \mathsf{d}_j \int_{\mathsf{SU(2)}} dg\, |j, g\rangle\langle j, g| = \mathsf{d}_j \int_{G/H=S_2} dn\, |j, n\rangle\langle j, n|$$

with

$$|j, n\rangle = g_n|j, j\rangle$$

• In 4D we work with $SU(2)\times SU(2)$ coherent states $|j_+, n_+\rangle \otimes |j_-, n_-\rangle$

$$1_{j_+} \otimes 1_{j_-} = \mathsf{d}_{j_+}\mathsf{d}_{j_-} \int dn_+ dn_- |j_+, n_+\rangle \otimes |j_-, n_-\rangle\langle j_+, n_+| \otimes \langle j_-, n_-|$$

- 3D BF propagator

  Since $\mathbb{P}^2 = \mathbb{P}$ we introduce two $SU(2)$ gauge-averaging variables, u and v at the ends of the propagator, (u on the side where $\epsilon_{v\ell} = -1$ and v on the side where $\epsilon_{v\ell} = +1$). Between these two variables we insert the partition of unity. This does not modify the propagator.

$$\mathbb{P}(g; g') = \int du dv \prod_{f=1}^{4} \sum_{j_f} \mathsf{d}_{j_f} \mathsf{Tr}_{V_{j_f}} \left(u g_f (g'_f)^{-1} v^{-1} 1_{j_f} \right)$$

  The index f labels the four strands of the propagator, which belong to four different faces

- 3D BF amplitude

  The amplitude is again factorized over faces:

$$A_{\mathcal{G}} = \int \prod_{\ell \in L_{\mathcal{G}}} du_\ell dv_\ell \prod_{f \in \mathcal{F}_{\mathcal{G}}} \mathcal{A}_f .$$

  To write down $\mathcal{A}_f$, let us number the vertices and lines in the (anti)-cyclic order along a face f of length p as $\ell_1, v_1 \cdots \ell_p, v_p$, with $\ell_{p+1} = \ell_1$.

$$\mathcal{A}_f = \sum_j d_j^{p+1} \int \prod_{a=1}^{p} dn_{\ell_a f} < j, n_{\ell_a f} | h_{\ell_a, v_a}^{\eta_{\ell_a f}} h_{\ell_{a+1}, v_a}^{\eta_{\ell_{a+1} f}} | j, n_{\ell a+1, f} >$$

  h_{ℓ_a, v_a} is v_{ℓ_a} if $\epsilon_{\ell_a v_a} = +1$ and u_{ℓ_a} if $\epsilon_{\ell_a v_a} = -1$.

- 4D BF propagator [Ooguri]

$$\mathbb{P}(g;g') \;=\; \int_{SU(2)\times SU(2)} du dv \prod_{f=1}^{4} \sum_{j_{f+},j_{f-}} \mathsf{d}_{j_{f+}} \mathsf{d}_{j_{f-}}$$
$$\mathrm{Tr}_{V_{j_{f+}}\otimes V_{j_{f-}}} \Big(u g_f (g'_f)^{-1} v^{-1} 1_{j_{f+}} \otimes 1_{j_{f-}} \Big)$$

- 4D BF amplitude

Obvious generalization of 3D

$$\mathcal{A}_f \;=\; \sum_{j_+ j_-} (d_{j+} d_{j-})^{p+1} \int \prod_{a=1}^{p} dn^+_{\ell_a f} dn^-_{\ell_a f}$$
$$< j_+, n^+_{\ell_a f} | h^{\eta_{\ell_a f}}_{\ell_a, v_a +} h^{\eta_{\ell_{a+1} f}}_{\ell_{a+1}, v_a +} | j_+, n^+_{\ell a+1, f} > \times \;[-]$$

The EPRL/FK GFT [EPRLS ’08, FK ’08]

The EPRL/FK model introduces a modification of the propagator of the 4D BF model, while the vertex remains the same.

It implements in two steps the Plebanski constraints with a non trivial value of the Immirzi parameter γ.

- $j_+/j_- = (1+\gamma)/(1-\gamma)$ and $n_+ = n_- = n$

$$\gamma > 1 \qquad\qquad j_\pm = \frac{\gamma \pm 1}{2} j,$$

$$\gamma < 1 \qquad\qquad j_\pm = \frac{1 \pm \gamma}{2} j.$$

• In each strand the identity $1_{j_+} \otimes 1_{j_-}$ is replaced by a projector T_j^γ

$$T_j^\gamma = d_{j_+ + j_-} \Big[\delta_{j_-/j_+ = (1-\gamma)/(1+\gamma)}\Big] \int dn |j_+, n\rangle \otimes |j_-, n\rangle \langle j_+, n| \otimes \langle j_-, n|.$$

In the angular momentum basis

$$T_j^\gamma \;=\; \sum_{k, \tilde{k}, m, \tilde{m}} \Big(j_+, k; j_-, \tilde{k} | j_+ + j_-, k + \tilde{k}\Big) \Big(j_+ + j_-, m + \tilde{m} | j_+, m; j_-, \tilde{m}\Big)$$
$$|k\tilde{k} >< m\tilde{m}| \delta_{m+\tilde{m}, k+\tilde{k}},$$

$$\implies \quad (T_j^\gamma)^2 = T_j^\gamma$$

Grouping the four strands of a line defines a $\mathbb{T}^\gamma$ operator that acts separately and independently on each strand of the propagator:

$$\mathbb{T}^\gamma = \oplus_{j_f} \otimes_{f=1}^4 T_{j_f}^\gamma$$

so that the EPRL/FK propagator is $\quad C = \mathbb{P}\mathbb{T}^{\gamma}\mathbb{P}$ [FK]

$$C(g,g') = \int du dv \prod_{f=1}^{4} \sum_{j_f} \Big[\delta_{j_{f-}/j_{f+}=(1-\gamma)/(1+\gamma)}\Big] \alpha_{j_f} \beta_{j_f} \int dn_f$$

$$\mathrm{Tr}_{j_{f+}\otimes j_{f-}} \Big(u g_f \, (g_f^{'})^{-1} v^{-1} | j_{f+}, n_f > \otimes | j_{f-}, n_f >< j_{f+}, n_f | \otimes < j_{f-}, n_f | \Big)$$

where

$$\alpha_j = d_{j_+} d_{j_-}, \quad \beta_j = d_{j_{f+}+j_{f-}}$$

The operator C is symmetric

- Since the propagator is hermitian, Feynman amplitudes are independent of the orientations of faces and propagators.

- Since $\mathbb{T}^\gamma$ and $\mathbb{P}$ do not commute, the propagator C can have non-trivial *spectrum* (with eigenvalues between 0 and 1).

- Slicing the eigenvalues should allow a renormalization group analysis. This is why we call this kind of theories *dynamic* GFT's.

- Since $\mathbb{T}^\gamma$ is a projector, the propagator C of the EPRL/FK theory is bounded in norm by the propagator of the BF theory, as well as Feynman amplitudes

EPRL/FK Amplitudes

We combine the propagator and the vertex expressions and integrate over all g, g' group variables $\Longrightarrow$ we get the amplitude of any graph $\mathcal{G}$.

It is given by an integral of a product over all faces of the graph.

- closed faces (no external edges)

$$\mathcal{A}_f = \sum_{j_f \leq \Lambda} \alpha_{j_f} \mathsf{Tr}_{j_{f+} \otimes j_{f-}} \prod_{a=1}^{p} \left(h^{\eta_{\ell_a f}}_{\ell_a, v_a} h^{\eta_{\ell_a f}}_{\ell_a, v_{a+1}} T^{\gamma}_{j_f} \right).$$

that is

$$\mathcal{A}_f \;=\; \sum_{j_f \leq \Lambda} \alpha_{j_f} \int \prod_{a=1}^{p} \beta_{j_f} dn_{\ell_a,f} < j_{f+} n_{\ell_a,f} | h^{\eta_{\ell_a f}}_{\ell_a, v_a,+} h^{\eta_{\ell_a+1 f}}_{\ell_{a+1}, v_a,+} | j_{f+} n_{\ell_{a+1},f} >$$
$$\times < j_{f-} n_{\ell_a,f} | h^{\eta_{\ell_a f}}_{\ell_a, v_a,-} h^{\eta_{\ell_a+1 f}}_{\ell_{a+1}, v_a,-} | j_{f-} n_{\ell_{a+1},f} > .$$

• open faces (which end on external edges)

slightly modified but irrelevant in the large spin approximation.

We recover the $SU(2)$ BF model in the limit $\gamma = 1$.

It can be shown that the graph amplitude is the same as spin foam amplitudes passing to spin representation.

Stationary phase for BF and EPRL/FK models

$\mathcal{G}$ a graph in a GFT corresponding to the BF or EPRL/FK models, with V vertices, L edges and F faces

$$\mathcal{A}_{\mathcal{G}} = \sum_{j_f \leq \Lambda} \mathcal{N} \int \prod dh \prod dn \, \mathsf{exp} \Big\{ \sum_f j_f S_f[h,n] \Big\},$$

$\mathcal{N} = \mathcal{N}(j)$

Λ ultraspin cutoff

To derive the superficial power counting, we set $j_f = j k_f$ with $k_f \in [0,1]$ and use the stationary phase method to derive the large j behavior of

$$\int \prod dh \prod dn \, \mathsf{exp} \Big\{ j \sum_f k_f S_f[h,n] \Big\}.$$

If the action is complex but has a negative real part, the contribution to this integral are quadratic fluctuations around zeroes of the real part of S which are stationary points of its imaginary part, otherwise the integral is exponentially suppressed as $j \to \infty$.

• Saddle point for 3D BF models

we use

$\langle n, j|g|n', j\rangle = \langle n|g|n'\rangle^{2j}$, and $|n\rangle\langle n| = \frac{1}{2}\Big(1 + \sigma \cdot n\Big)$, then

$$S_f[h, n] = k_f \log \mathrm{Tr}\Big[\Big(\overrightarrow{\prod_{\ell\in\partial f}} h_\ell^{\eta_{\ell,f}}\Big)\Big(1 + \sigma \cdot n\Big)\Big].$$

Since the action is the logarithm of the trace of the product of a unitary element and a projector, its real part is negative and maximal at $h_\ell = 1$

To perform the saddle point expansion, we expand the group element to second order as

$$h_\ell = 1 - \frac{A_\ell^2}{2} + \mathrm{i}\, A_\ell \cdot \sigma + O(A_\ell^3)$$

with $A \in su(2)$. Also

$$n_f = n_f^{(0)} + \xi_f - \frac{\xi_f^2}{2} n_f^{(0)} + O(\xi_f^3), \qquad \text{with} \quad n_f^{(0)} \cdot \xi_f = 0.$$

(because $n_f^2 = 1$ up to third order terms).

Let us consider a face with edges $\ell_1, \ldots, \ell_p$, then to second order

$$\overrightarrow{\prod_{\ell \in \partial f}} h_\ell^{\eta_{\ell,f}} = 1 - \frac{A_f^2}{2} + \mathrm{i}\,\sigma \cdot A_f - \mathrm{i}\,\sigma \cdot \Phi_f$$

with

$$A_f = \sum_{1 \leq a \leq p} \eta_{\ell_a,f} A_{\ell_a} \qquad \text{and} \qquad \Phi_f = \sum_{1 \leq a < b \leq p} \eta_{\ell_a,f} \eta_{\ell_b,f}\, A_{\ell_a} \wedge A_{\ell_b}.$$

$\Longrightarrow$

$$S_f[A_\ell, \xi_f] = 2k_f \Big\{ \mathrm{i}\, n_f \cdot A_f - \frac{A_f^2}{2} + \frac{(n_f \cdot A_f)^2}{2} + \mathrm{i}\, \xi_f \cdot A_f + \mathrm{i}\, n_f \cdot \Phi_f \Big\}$$

and we have to estimate

$$\int \prod_{\ell} dA_\ell \prod_f d\xi_f \; \exp 2j \sum_f k_f \Big\{ \mathrm{i}\, n_f \cdot A_f - \frac{A_f^2}{2} + \frac{(n_f \cdot A_f)^2}{2} + \mathrm{i}\, \xi_f \cdot A_f + \mathrm{i}\, n_f \cdot \Phi_f \Big\}$$

as $j \to \infty$.

• We do not integrate over the vectors n_f. They have to be chosen so that they are extrema of the imaginary part of S.

• Because all terms except the first one $\sum_f k_f n_f \cdot A_f$ are of second order, the imaginary part is stationary if and only if

$$\sum_f \mathrm{i}\, k_f n_f \cdot A_f = \sum_{\ell,f} \mathrm{i}\, \eta_{\ell,f} k_f n_f \cdot A_\ell = 0 \qquad \forall A_\ell \in \mathbb{R}^3,$$

which amounts to the closure condition

$$\sum_f \eta_{\ell,f} k_f n_f = 0, \quad \forall \ell$$

This is the requirement that, in the semi-classical limit, the vectors $j_f n_f$ are the sides of a triangle (resp. the area bivectors of a tetrahedron) that propagates along ℓ in dimension 3 (resp. dimension 4).

The solutions of the closure conditions range from non degenerate to maximally degenerate.

In three dimensional (resp. four dimensional) BF theory, a solution is said to be **non degenerate** if all the tetrahedra (resp. 4-simplices) corresponding to the vertices of the graph have maximal dimension.

At the opposite end, we say that a solution is **maximally degenerate** if all the vectors n_f are proportional to a single one n_0,

$$n_f = \sigma_f n_0 \qquad \text{with} \qquad \sigma_f \in \{-1, +1\}\,.$$

In both cases we find

$$\mathcal{A}_{\mathcal{G}} \sim \Lambda^{3F-3r}$$

with r the rank of the $L \times F$ incidence matrix $\eta_{\ell,f}$.

In particular for the self-energy graph we find Λ^9.

The self-energy in the EPRL/FK model (2P-fig)

The self-energy graph $\mathcal{G}_2$ (figure) has 4 open faces. It has 6 closed faces with two edges.

$$\mathcal{A}_{\mathcal{G}_2} = \sum_{j_f} \int \prod_a du_a^{\pm} \prod_a dv_a^{\pm} \prod_i dn_i \prod_f \left\{ (d_{j_f})^2 d_{j_f^+} d_{j_f^-} \exp \left\{ j S_f^+ + j S_f^- \right\} \right\}$$

with $j_f^{\pm} = j\gamma^{\pm} k_f$, $k_f \in [0,1]$ and j large. There is one coherent state per strand $i = (f, l)$ such that $\eta_{l,f} \neq 0$.

$$S_f^{\pm} = 2\gamma^{\pm} k_f \log \left\{ \langle n_{f,a} | u_a^{\pm} (u_b^{\pm})^{-1} | n_{f,b} \rangle \langle n_{f,b} | v_b^{\pm} (v_a^{\pm})^{-1} | n_{f,a} \rangle \right\}$$

- We employ the saddle point technique around the identity

$$u_a^{\pm} = 1 - \frac{(A_a^{\pm})^2}{2} + \mathrm{i}\,\sigma \cdot A_a^{\pm} + O(A_a^{\pm})^3, \quad v_a^{\pm} = 1 - \frac{(B_a^{\pm})^2}{2} + \mathrm{i}\,\sigma \cdot B_a^{\pm} + O(B_a^{\pm})^3$$

- we introduce the projector $\quad |n_i\rangle\langle n_i| = \frac{1+\mathrm{i}\,\sigma\cdot n_i}{2}$

- the action at the identity for the face $f = ab$ reads

$$S_f^{\pm}[1, 1, n_i] = \gamma^{\pm} k_{ab} \log \left\{ \frac{1 + n_{f,a} \cdot n_{f,b}}{2} \right\}$$

which is negative except for $n_{f,a} = n_{f,b} = n_f$.

$\Longrightarrow$ we perform the expansion of the coherent state around a unit vector common to all the strands of the face

$$n_i = n_f + \xi_i - \frac{(\xi_i)^2}{2} n_f + O(\xi_i)^3, \qquad \text{with} \qquad n_f \cdot \xi_i = 0,$$

otherwise the integral is exponentially damped.

• It is convenient to perform the following change of variables

$$A_a^{\pm} = A_a \pm \gamma^{\mp} X_a \qquad \text{and} \qquad B_a^{\pm} = B_a \pm \gamma^{\mp} Y_a$$

• Terms linear in $A^{\pm}$ and $B^{\pm}$ only involve A and B, while in the quadratic terms, the pair of variables A and B on one side and the pair X and Y on the other side decouple.

$\Longrightarrow$ For arbitrary graph, we can separate the action, at the level of the quadratic approximation, into a SU(2) BF action (variables A and B) and an ultralocal potential that only involves uncoupled variables attached to the vertices (variables X and Y).

Back to the self-energy, performing the Gaussian integration over the two dimensional vector $\chi_f = \xi_{f,a} - \xi_{f,b}$,

$$\mathcal{A}_{\mathcal{G}_2} = \sum_{j_f} j^{18} \Big\{ \int \prod_a dA_a \prod_a dB_a \prod_f d\xi_f \exp j S_{BF}(A, B, \xi) \times \int \prod_a dX_a \exp jQ(X) \times \int \prod_a dY_a \exp jQ(Y) \Big\}$$

with $\xi_f = \xi_{f,a} + \xi_{f,b}$. The BF-like action is

$$S_{BF}[A, B, \xi] = \sum_{a<b} k_{ab} \Big\{ -\frac{1}{2} \Big[n_f \wedge \Big(A_a - A_b + B_b - B_a \Big) \Big]^2 + \mathrm{i}\, n_{ab} \cdot \Big(A_a - A_b + B_b - B_a \Big) + \mathrm{i}\, n_{ab} \cdot \Big(A_a \wedge A_b + B_b \wedge B_a \Big) + \mathrm{i}\, \xi_{ab} \cdot \Big(A_a - A_b + B_b - B_a \Big) \Big\}$$

while the ultra local terms are

$$Q[X] = \gamma^+\gamma^- \sum_{a<b} k_{ab}\Big\{\Big[n_{ab}\wedge(X_a - X_b)\Big]^2 + \mathrm{i}\, n_{ab}\cdot\Big(X_a\wedge X_b\Big)\Big\}.$$

The Gaussian integral over the variables A and B can be evaluated using the same techniques as previously.

$$\sum_{\text{6 independent spins } \sim j < \Lambda} j^{12}\times j^{-9}\sim \Lambda^9,$$

which is the known result for SU(2) BF theory.

while the the Gaussian integral over the independent variables X_a and Y_a,

$$\int \prod_a dX_a \exp jQ(X) \quad \sim \quad j^{-\frac{\text{rank}(Q)}{2}}$$

yield a power of $j^{-9/2}$ each.

Therefore, we obtain the power counting for the self-energy with non degenerate configurations as follows

$$\sum_{\text{6 independent spins } \sim j < \Lambda} j^{24} \times j^{-6} \times j^{-9} \times \left(j^{-9/2}\right)^2 \sim \Lambda^6,$$

- j^{24} arises from a $d_{j^+} d_{j^-} \sim j^2$ for each of the 6 faces

- $d_j \sim j$ for each of the two strands in each face

- j^{-6} results from the Gaussian integration over the 6 variables $\chi_f = (\xi_{f,a} - \xi_{f,b})$

- j^{-9} from the integration over A and B

This reproduces the result of [PRS], with non degenerate configurations.

However, degenerate configurations are more divergent. Maximally degenerate have a behavior in Λ^9.

Conclusions

• Hint for a phase transition: we can expect the amplitude of $\mathcal{G}_2$ to provide the dominant correction to the effective propagator of the model.

Since it is positive, the whole self-energy correction Σ should be also positive. The corresponding geometric power series for the dressed or effective propagator

$$C_{dressed} = C + C\Sigma C + C\Sigma C\Sigma C + ... = C(\frac{1}{1 - \Sigma C}).$$

should be singular when the spectrum of ΣC has eigenvalue 1. This should occur for λ large enough, depending on the ultraviolet cutoff Λ. This is usually the signal of a phase transition.

• Renormalization analysis