On the SU(N) gauge beta functions in the conformal window

ERG 2010 @ Corfu

Sep .16 2010

Yuki Kusafuka and Haruhiko Terao

(Nara Women's Univ. Japan)

Introduction

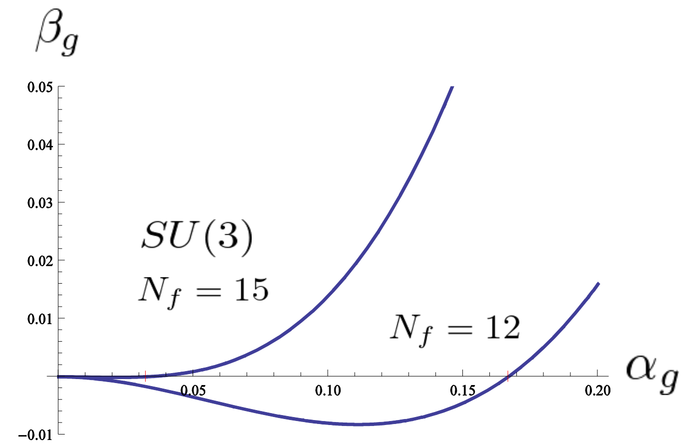
Conformal window

- **SU(Nc) gauge theory with Nf massless flavors**
- **2-loop gauge beta function** $\alpha_g = g^2/(4\pi)^2$

$$\beta_g^{[2]} \equiv \mu \frac{d\alpha_g}{d\mu} = -2b_0\alpha_g^2 - 2b_1\alpha_g^3$$

$$b_0 = \frac{11}{3}N_c - \frac{2}{3}N_f$$

$$b_1 = \frac{34}{3}N_c^2 - N_f\left(\frac{N_c^2-1}{N_c} + \frac{10}{3}N_c\right)$$



- **IR (Banks-Zaks) fixed point**
  - goes towards strong coupling region as Nf decreases.
- **Spontaneous breaking of the chiral symmetry for Nf < Ncr**
  - Scale invariance is lost there. ⇒ Fixed point cannot exist !
  - Conformal window: $N_{\rm cr} < N_f < \frac{11}{2}N_c$

Introduction

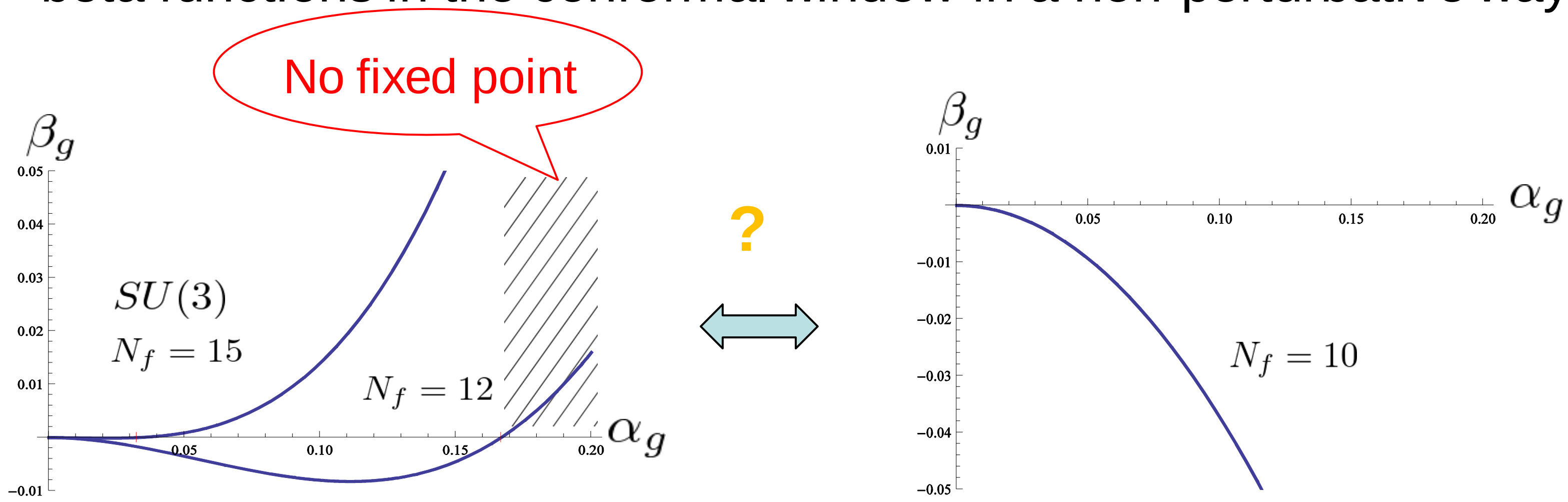
Lower bound of the conformal window Ncr

- Dyson-Schwinger analyses
- Lattice MC simulations

$$\Rightarrow \quad 8 < N_{\rm cr} \leq 12$$

How does the beta function change vs. Nf ?

- Perturbative beta functions cannot transform into the asymptotically free one smoothly, then what happens?
- In order to look into this problem, it is necessary to examine the beta functions in the conformal window in a non-perturbative way.



Introduction

Use of ERG

- Scale invariance:

  Dyson-Schwinger equations treating the chiral order parameters are useless in the conformal window. Also it would be difficult to study almost scale invariant theories by the Lattice MC simulation.

- Non-perturbative calculation:

  In the Wilson RG, renormalized theories can be defined by the renormalized trajectories (RTs), which are given as the continuum limit of the Wilson RG flows. J.Polchinski, N.P. B231 (1984)

- Beta function:

  Then **non-perturbative beta functions** can be given by scale transformation on the RTs.

⇒ So the ERG is a quite suitable framework.

Non-perturbative beta function

Scalar field theory as a toy model

- **RG flows in (λ_4, λ_6) space**

J.Polchinski, N.P. B231 (1984)

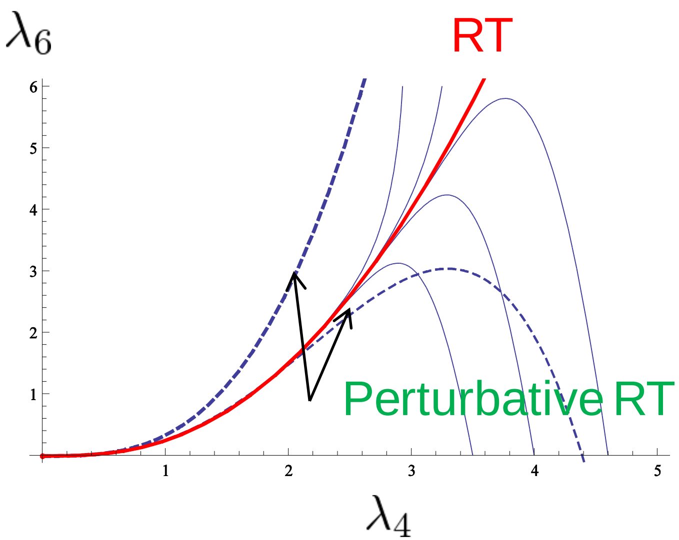
  - Operator truncation

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda_4}{4!}\phi^4 - \frac{\lambda_6}{6!\Lambda^2}\phi^6$$

  - Wetterich eqn (sharp cutoff limit)

$$\Lambda \frac{d\lambda_4}{d\Lambda} = a\lambda_4^2 - b\lambda_6$$

$$\Lambda \frac{d\lambda_6}{d\Lambda} = 2\lambda_6 - c\lambda_4^3 + 2d\lambda_4\lambda_6$$



$$a = 9A, \quad b = 10A, \quad c = 27A, \quad d = (45/2)A \qquad (A = 1/(4\pi)^2)$$

- **Renormalized trajectory**
  - Perturbative analysis

$$\lambda_6 = \frac{c}{2}\lambda_4^3 + \frac{c}{4}(3a - 2d)\lambda_4^4 + O(A^2)$$

Non-perturbative beta function

- **Non-perturbative beta function**
  - Renormalized trajectory

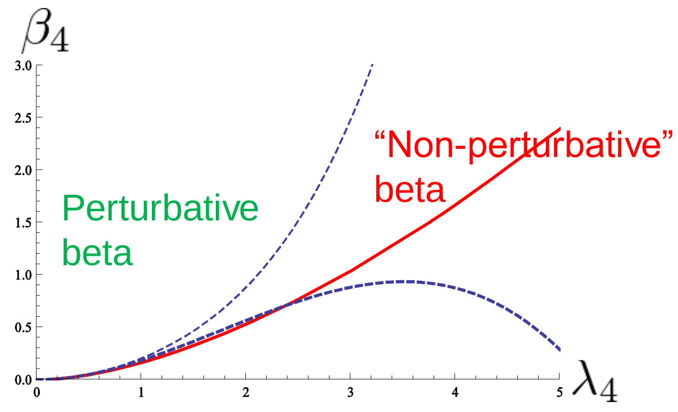
    We can find the RT numerically without perturbative expansion.

$$\lambda_6 = \lambda_6^*(\lambda_4)$$

  - "Non-perturbative" beta function

    The beta function of a renormalized parameter is given by the scale transformation on the RT.

$$\begin{aligned}\beta_4(\lambda_4) &= a\lambda_4^2 - b\lambda_6^*(\lambda_4) \\ &= a\lambda_4^2 - \frac{bc}{2}\lambda_4^3 - \frac{bc}{4}(3a-2d)\lambda_4^4 + \cdots\end{aligned}$$



RG flow equations for SU(N) gauge theories

Wilsonian effective action

- **Four-fermi operators**
  - Important to describe the chiral symmetry breaking
  - Symmetries
    - Gauge symmetry : $SU(N_c)$
    - Chiral flavor symmetry : $SU(N_f)_L \times SU(N_f)_R$
    - Parity

$$\psi_L^{ai} = L^{ai}, \psi_R^{ai} = R^{ai}$$
$$(a = 1, \cdots, N_c)$$
$$(i = 1, \cdots, N_f)$$

  - 4 invariant four-fermi operators

$$\mathcal{L}_{4\mathrm{f}} = \frac{G_S}{\Lambda^2}\mathcal{O}_S + \frac{G_V}{\Lambda^2}\mathcal{O}_V + \frac{G_{V1}}{\Lambda^2}\mathcal{O}_{V1} + \frac{G_{V2}}{\Lambda^2}\mathcal{O}_{V2}$$

$$\mathcal{O}_S = 2\bar{L}_i R^j \bar{R}_j L^i \qquad (\bar{L}_i R^j = \bar{L}_{ai} R^{aj} \text{ etc.})$$
$$\mathcal{O}_V = \bar{L}_i \gamma^\mu L^j \bar{L}_j \gamma_\mu L^i + (L \leftrightarrow R)$$
$$\mathcal{O}_{V1} = 2\bar{L}_i \gamma^\mu L^i \bar{R}_j \gamma_\mu R^j$$
$$\mathcal{O}_{V2} = (\bar{L}_i \gamma^\mu L^i)^2 + (L \leftrightarrow R)$$

H.Gies, J.Jackel, C.Wetterich, PRD69 (2004)
H.Gies, J.Jackel, EPJC46 (2006)

RG flow equations for SU(N) gauge theories

- **Spontaneous breaking of the chiral symmetry**

K.-I.Aoki, K.Morikawa, W.Souma, J.-I.Sumi, H.T.,M.Tomoyose,

PTP97 (1997), PTP102 (1999), PRD61 (2000)

  - $G_S \to \infty$: Chiral symmetry breaking

$$\langle \bar{\psi}_i \psi^j \rangle = M^3 \delta_i^j \quad \Rightarrow \quad SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$$

- **Approximation scheme**
  - Operator truncation

$$\mathcal{L}_{\mathrm{eff}} = -\frac{1}{4g^2} F_{\mu\nu}^A F^{A\mu\nu} + \bar{\psi}_f i \not{D} \psi^f + \mathcal{L}_{4\mathrm{f}}$$

  - We discard gauge non-invariant corrections.

Note: Cutoff breaks gauge invariance. Gauge non-invariant corrections may be controlled by the modified WT identities.

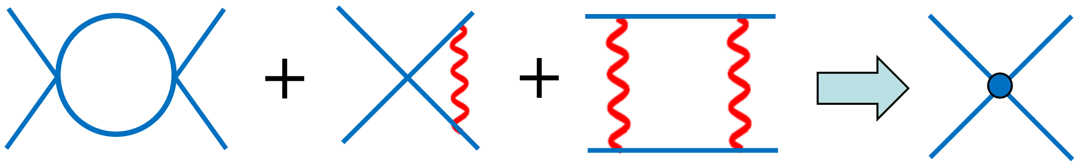
RG flow equations (sharp cutoff limit)

- **Four-fermi couplings** $(g_i = G_i/4\pi^2,\ \alpha_g = g^2/(4\pi)^2)$

$$\begin{aligned}
\Lambda\frac{dg_S}{d\Lambda} &= 2g_S - 2N_c g_S^2 + 2N_f g_S g_V + 6g_S g_{V1} + 2g_S g_{V2} \\
&\quad -12C_2(F)g_S\alpha_g + 12g_{V1}\alpha_g - \frac{3}{2}\left(3N_c - \frac{4}{N_c} - \frac{1}{N_c^2}\right)\alpha_g^2 \\
\Lambda\frac{dg_V}{d\Lambda} &= 2g_V + (N_f/4)g_S^2 + (N_c+N_f)g_V^2 - 6g_V g_{V2} \\
&\quad -\frac{6}{N_c}(g_V + g_{V2})\alpha_g - \frac{3}{4}\left(N_c - \frac{8}{N_c} + \frac{3}{N_c^2}\right)\alpha_g^2 \\
\Lambda\frac{dg_{V1}}{d\Lambda} &= 2g_{V1} - (1/4)g_S^2 - g_S g_V - 3g_{V1}^2 - N_f g_S g_{V2} + 2(N_c+N_f)g_V g_{V1} \\
&\quad +2(N_c N_f + 1)g_{V1}g_{V2} + \frac{6}{N_c}g_{V1}\alpha_g + \frac{3}{4}\left(1 + \frac{3}{N_c^2}\right)\alpha_g^2 \\
\Lambda\frac{dg_{V2}}{d\Lambda} &= 2g_{V2} - 3g_V^2 - N_c N_f g_{V1}^2 + (N_c N_f - 2)g_{V2}^2 - N_f g_S g_{V1} \\
&\quad +2(N_c N_f + 1)g_V g_{V2} + 6(g_V + g_{V2})\alpha_g - \frac{3}{4}\left(3 + \frac{1}{N_c^2}\right)\alpha_g^2
\end{aligned}$$

RG flow equations for SU(N) gauge theories

- **Loop corrections for the four-fermi operators**



- **Large Nc, Nf limit** $(r = N_f/N_c \;:\; \text{fixed})$
  - rescale as $N_c g_{S(V)} \;\to\; g_{S(V)},\; N_c^2 g_{V1(V2)} \;\to\; g_{V1(V2)},$
    $N_c \alpha_g \;\to\; \alpha_g$

$$\boxed{\begin{aligned} \Lambda\frac{dg_S}{d\Lambda} &= 2g_S - 2g_S^2 + 2rg_Sg_V - 6g_S\alpha_g - \frac{9}{2}\alpha_g^2 \\ \Lambda\frac{dg_V}{d\Lambda} &= 2g_V + \frac{r}{4}g_S^2 + (1+r)g_V^2 - \frac{3}{4}\alpha_g^2 \end{aligned}}$$

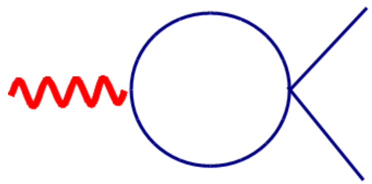
  - Note: Four-fermi couplings g_{V1}, g_{V2} do not involve in the large Nc and Nf limit.
  - Note: The large Nc corrections contain only the ladder diagrams. But the non-ladder ones come through the large Nf part.

RG flow equations for SU(N) gauge theories

- **Gauge coupling**
  - We use the perturbative 2-loop beta function in the large N_c, N_f limit and add a part of higher order corrections via the four-fermi effective couplings.

1. Vertex correction : H.Gies, J.Jackel, C.Wetterich, PRD69 (2004)

   We discard all vertex corrections with the four-fermi couplings, since the gauge symmetry should forbid them.

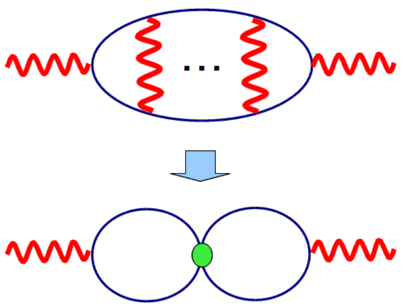


2. Vacuum polarization :

   The higher order corrections via four-fermi effective operators should be incorporated into the vacuum polarization.

$$\beta_g^{[2]} = -\frac{2}{3}(11-2r)\alpha_g^2 - \frac{2}{3}(34-13r)\alpha_g^3$$

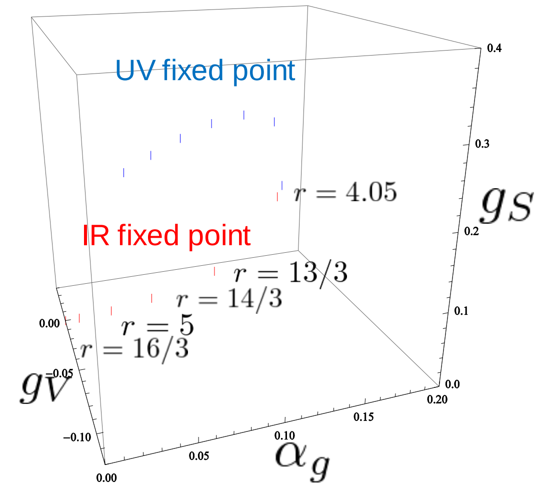
$$\boxed{\Lambda \frac{d\alpha_g}{d\Lambda} = \beta_g^{[2]} + 2rg_V\alpha_g^2}$$

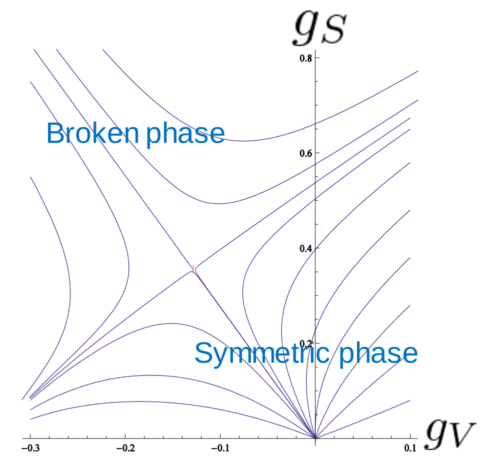


Aspect of RG flows

Numerical analysis of the flow equations

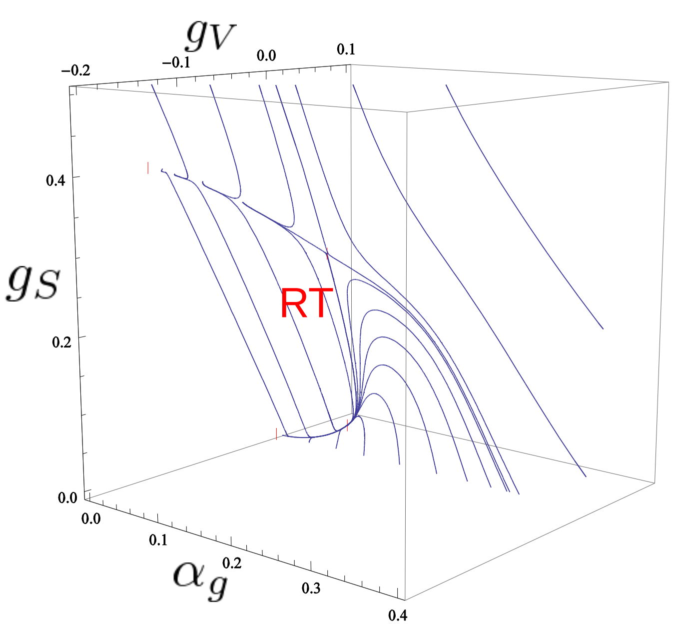
- **RG flows in large Nc and Nf**
  - RG flows are given in 3 dimensional coupling space of (α_g, g_V, g_S).
- **Fixed points in the conformal window**
  - A UV fixed point exists as well as the IR fixed point.
  - The UV fixed point and the IR fixed point merge with each other at r = 4.05.
- **RG flows in (gs, gv) space**
  - One linear combination of gs and gv gives the relevant operator, which induces the chiral phase transition.



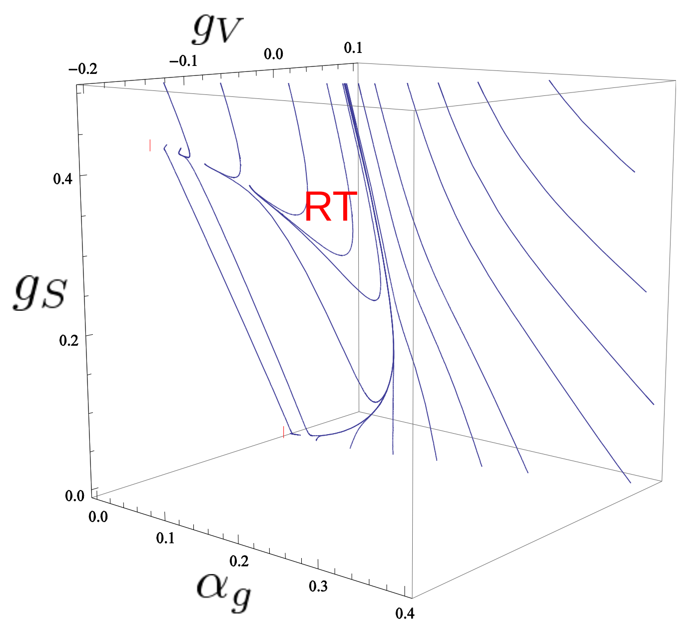


Aspect of RG flows

- **RG flows in the 3D space**
  - There is the phase boundary of chiral symmetry and the UV fixed point lies on the boundary.
  - Flows in the unbroken phase approach towards the IR fixed point.
  - The phase boundary disappears for r < 4.05 and the entire region becomes the broken phase.



$N_f = 13$



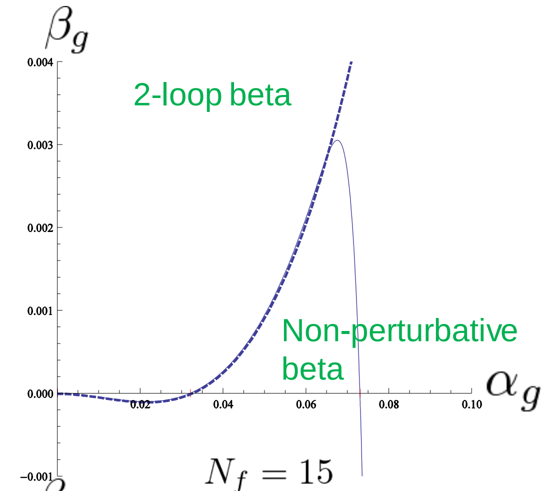
$N_f = 12$

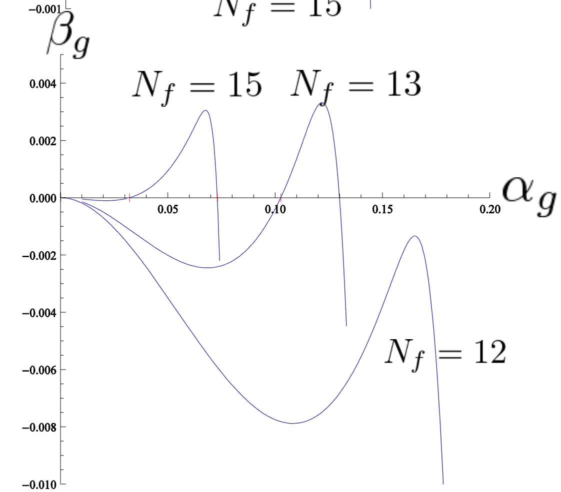
“Non-perturbative” gauge beta functions

- **Gauge beta functions**
  - We define the non-perturbative gauge beta function by scale transformation of the gauge coupling on the RTs.
  - A UV fixed point appears in the gauge beta function due to higher order corrections via four-fermi operators.
- **“Conformality lost”**
  - The IR fixed point merges with the UV fixed point at the edge of conformal window.
  - Fermion mass generation

    Miransky scaling

$$m_f \sim M e^{-\frac{C}{\sqrt{N_{\rm cr}-N_f}}} \qquad N_f \leq N_{\rm cr}$$





D.B. Kaplan, J-W. Lee, D.T. Son, M.A. Stephanov, PRD80 (2009)

J. Braun, H. Gies, JHEP 1005 (2010)

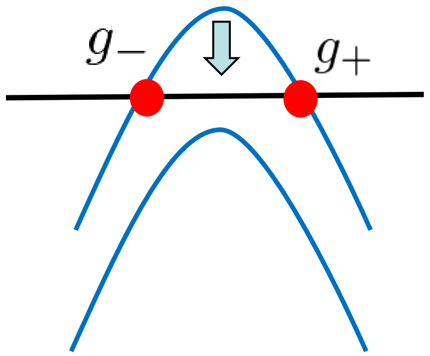
"Non-perturbative" gauge beta functions

"Conformality lost"

D.B. Kaplan, J-W. Lee, D.T. Son, M.A. Stephanov, PRD80 (2009)

- **BKT type phase transition**
  - Suppose that a UV fixed point and an IR fixed point merge.
    The beta function with an external parameter α is given as
    $$\beta(g;\alpha) = \mu\frac{dg}{d\mu} = (\alpha - \alpha_*) - (g - g_*)^2$$
    Then the fixed point couplings are $g_\pm = g_* \pm \sqrt{\alpha - \alpha_*}$
  - For $\alpha = \alpha_* - \epsilon$

$$\begin{aligned}\frac{\Lambda_{\mathrm{IR}}}{\Lambda_{\mathrm{UV}}} &= \exp\left(\int_{g_{\mathrm{UV}}}^{g_{\mathrm{IR}}} \frac{dg}{\beta(g;\alpha)}\right) \\ &\simeq \exp\left(-\frac{\pi}{\sqrt{\alpha_* - \alpha}}\right)\end{aligned}$$

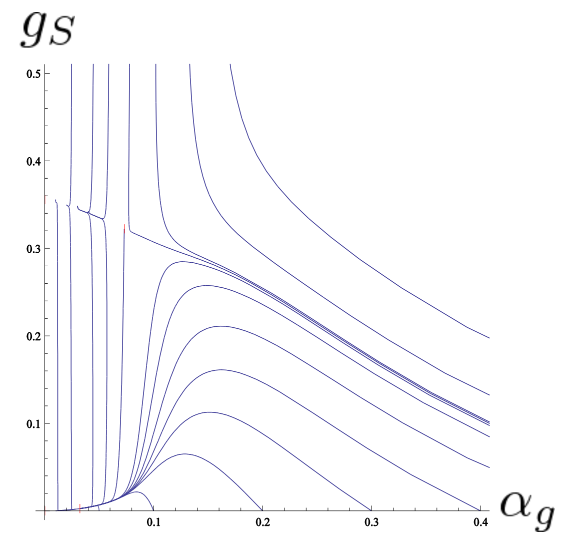


More on the RT

Near upper bound of the conformal window

- **Perturbative RT**
  - We may extract the RT by solving the truncated RG flow equations in perturbative expansion.
  - This survives for $N_f > (11/2)N_c$.
  - This “RT” does not seem to give a continuum limit.
- **Non-perturbative RT = QCD***
  - The flow steming from the UV fixed point gives a continuum limit. This RT defines a different theory from the ordinary QCD !. (non-perturbative renormalizable or asymptotically safe theory)

⇒ Two “renormalized trajectories”?



$N_f = 15$

Summary and discussions

- **We extended the RG flow equations for the gauge couplings so as to include "non-perturbative" corrections via the effective four-fermi operators.**
- **We analyzed the RG flows in large N_c, N_f limit and found the non-perturbative RT transforms smoothly to the continuum limit of the asymptotically free gauge theories.**
- **We gave the non-perturbative gauge beta functions by scale transformation on the RT, which shows merge of the UV and the IR fixed points. ⇒ manifestation of ``Conformality Lost''.**

Future issues

- Derivation of the RG flow equation for the gauge coupling including the four-fermi couplings from the Wetterich equation.
- Evaluation of the chiral order parameters near conformal regime.
- Relation to the supersymmetric conformal gauge theories.

Many Thanks

Appendix

Invariant four-fermi operators

- **Apparent invariants**

$$\psi_L^{ai} = L^{ai}, \psi_R^{ai} = R^{ai} \quad (a = 1, \cdots, N_c) \quad (i = 1, \cdots, N_f)$$

$$2\bar{L}_{ai}\gamma^\mu L^{ai}\bar{R}_{bj}\gamma_\mu R^{bj} = \frac{1}{2}\left[(\bar{\psi}_{ai}\gamma^\mu\psi^{ai})^2 - (\bar{\psi}_{ai}\gamma^\mu\gamma_5\psi^{ai})^2\right]$$

$$2\bar{L}_{ai}\gamma^\mu L^{bi}\bar{R}_{bj}\gamma_\mu R^{aj} = \frac{1}{2}\left[\bar{\psi}_{ai}\gamma^\mu\psi^{bi}\bar{\psi}_{bj}\gamma_\mu\psi^{aj} - \bar{\psi}_{ai}\gamma^\mu\gamma_5\psi^{bi}\bar{\psi}_{bj}\gamma_\mu\gamma_5\psi^{aj}\right]$$

$$\bar{L}_{ai}\gamma^\mu L^{ai}\bar{L}_{bj}\gamma_\mu L^{bj} + (L \leftrightarrow R) = \frac{1}{2}\left[(\bar{\psi}_{ai}\gamma^\mu\psi^{ai})^2 + (\bar{\psi}_{ai}\gamma^\mu\gamma_5\psi^{ai})^2\right]$$

$$\begin{aligned}\bar{L}_{ai}\gamma^\mu L^{bi}\bar{L}_{bj}\gamma_\mu L^{aj} &+ (L \leftrightarrow R) \\ &= \frac{1}{2}\left[\bar{\psi}_{ai}\gamma^\mu\psi^{bi}\bar{\psi}_{bj}\gamma_\mu\psi^{aj} + \bar{\psi}_{ai}\gamma^\mu\gamma_5\psi^{bi}\bar{\psi}_{bj}\gamma_\mu\gamma_5\psi^{aj}\right]\end{aligned}$$

$$\begin{aligned}\bar{L}_{ai}\gamma^\mu L^{aj}\bar{L}_{bj}\gamma_\mu L^{bi} &+ (L \leftrightarrow R) \\ &= \frac{1}{2}\left[\bar{\psi}_{ai}\gamma^\mu\psi^{aj}\bar{\psi}_{bj}\gamma_\mu\psi^{ai} + \bar{\psi}_{ai}\gamma^\mu\gamma_5\psi^{aj}\bar{\psi}_{bj}\gamma_\mu\gamma_5\psi^{bi}\right]\end{aligned}$$

$$\begin{aligned}\bar{L}_{ai}\gamma^\mu L^{bj}\bar{L}_{bj}\gamma_\mu L^{ai} &+ (L \leftrightarrow R) \\ &= \frac{1}{2}\left[\bar{\psi}_{ai}\gamma^\mu\psi^{bj}\bar{\psi}_{bj}\gamma_\mu\psi^{ai} + \bar{\psi}_{ai}\gamma^\mu\gamma_5\psi^{bj}\bar{\psi}_{bj}\gamma_\mu\gamma_5\psi^{ai}\right]\end{aligned}$$

Appendix

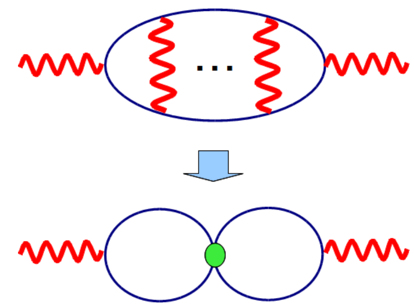
- **Fiertz identities**
  - $$\bar\psi_1\gamma^\mu\psi_2\ \bar\psi_3\gamma_\mu\psi_4+\bar\psi_1\gamma^\mu\gamma_5\psi_2\ \bar\psi_3\gamma_\mu\gamma_5\psi_4$$ $$=\ \bar\psi_1\gamma^\mu\psi_4\ \bar\psi_3\gamma_\mu\psi_2+\bar\psi_1\gamma^\mu\gamma_5\psi_4\ \bar\psi_3\gamma_\mu\gamma_5\psi_2$$
  - $$\bar\psi_1\psi_2\ \bar\psi_3\psi_4-\bar\psi_1\gamma_5\psi_2\ \bar\psi_3\gamma_5\psi_4$$ $$=\ -\frac{1}{2}\left[\bar\psi_1\gamma^\mu\psi_4\ \bar\psi_3\gamma_\mu\psi_2-\bar\psi_1\gamma^\mu\gamma_5\psi_4\ \bar\psi_3\gamma_\mu\gamma_5\psi_2\right]$$
- **Current-current interactions**

$$2\sum_{A=1}^{\dim G}\left(T^A\right)^a_{\ d}\ \left(T^A\right)^c_{\ b}=\delta^a_b\ \delta^c_d-\frac{1}{N_c}\delta^a_d\ \delta^c_b$$

  - $$2\sum_A \bar L_i T^A\gamma^\mu L^i\ \bar R_j T^A\gamma_\mu R^j=-\mathcal{O}_S-\frac{1}{2N_c}\mathcal{O}_{V1}$$
  - $$\sum_A \bar L_i T^A\gamma^\mu L^i\ \bar L_j T^A\gamma_\mu L^j+(L\leftrightarrow R)=\frac{1}{2}\mathcal{O}_V-\frac{1}{2N_c}\mathcal{O}_{V2}$$
  - $$\sum_A \bar L_i T^A\gamma^\mu L^j\ \bar L_j T^A\gamma_\mu L^i+(L\leftrightarrow R)=\frac{1}{2}\mathcal{O}_{V2}-\frac{1}{2N_c}\mathcal{O}_V$$

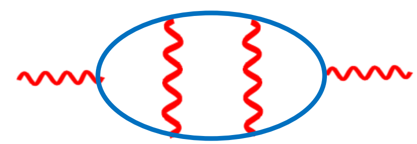
Appendix

- **Higher order correction to the vacuum polarization**
  - Here we consider to take in the abelian gauge type corrections, which are partly given as corrections via effective four-fermi operators.



  - In the large Nc limit, the four-fermi operator Ov is generated.
  - For the 3-loop correction, the induced coupling is given by

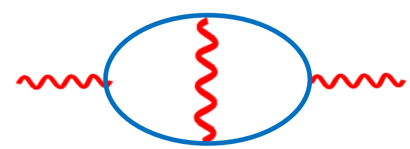
$$G_V = -\frac{3N_c}{16}\alpha_g^2$$



In large Nc, the induced effective operator is represented as

$$\mathcal{L}_{\rm eff} \simeq \frac{2G_V}{\Lambda^2}\left[\bar{L}_i\gamma^\mu T^A L^i \bar{L}_j\gamma_\mu T^A L^j + (L \leftrightarrow R)\right]$$

Therefore it may be regarded as a 2-loop correction with the effective coupling.



Appendix

- **Modification of the 2-loop beta function**
  - We may evaluate divergence in the effectively 2-loop vacuum polarization as

$$\Pi_{\mu\nu}(q) = -N_f \left[\frac{4}{3}\alpha_g + 2\alpha_g^2 + \alpha_g g_V \right] \log\Lambda \left(q^2 g_{\mu\nu} - q_\mu q_\nu \right)$$

  - Eventually we may incorporate the higher order corrections via the four-fermi operator in the 2-loop gauge beta function as

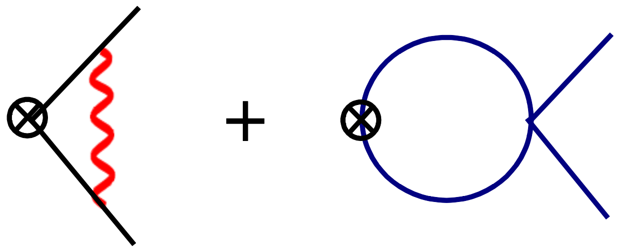
$$\begin{aligned} \beta_g &= \Lambda \frac{d\alpha_g}{d\Lambda} = \eta_A \alpha_g \\ \eta_A &= -2b_0 - 2b_1\alpha_g^2 + 2N_f \alpha_g g_V \end{aligned}$$

Appendix

Anomalous dimensions

- **Dimension of $\bar{\psi}\psi$ at the fixed points**

$$\gamma_{\bar{\psi}\psi} =$$



$$= -6C_2(F)\alpha_g - 2N_c g_S + 4g_{V1}$$

$$\simeq -3\alpha_g - 2g_S$$

$$\Delta_{\bar{\psi}\psi} = 3 + \gamma_{\bar{\psi}\psi}$$

- Non-ladder corrections to the 4-fermi operators are significant for a large Nf.