# Nonperturbative functional renormalization group for the random field Ising model:

## Supersymmetry and its spontaneous breaking

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# Random field Ising model (RFIM)

- prototypical model in theory of "disordered systems"

$$S_h[\phi] = S_B[\phi] - \int_x h(x)\phi(x);\; S_B = \int d^dx \left\{ \frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{\tau}{2}\phi(x)^2 + \frac{u}{4!}\phi(x)^4 \right\}$$

with <u>quenched</u> random field (e.g., Gaussian): $\overline{h(x)} = 0,\ \overline{h(x)h(y)} = \Delta_B\, \delta^{(d)}(x-y)$

=>Partition function: $Z_h[J] = e^{W_h[J]} = \int \mathcal{D}\phi\, e^{-S_h[\phi] + \int_x J(x)\phi(x)}$

- Generic difficulties:

  ∗ Due to quenched disorder (*h*), loose translational invariance
  Way out: <u>average over disorder</u>, but what ?, how ?

  ∗ Presence of many <u>"metastable states"</u>

# Average over the disorder

["self-averaging", "replica trick", etc]

- $W_h[J]$ is a random functional of the source =>
  - ✳ in principle, needs its whole probability distribution
  - ✳ or equivalently, the infinite set of its cumulants:

$$W_1[J] = \overline{W_h[J]},\ W_2[J_1, J_2] = \overline{W_h[J_1]W_h[J_2]}|_c,\ \cdots$$

- Crucial to get the full functional dependence (physics of disordered systems at T=0 may involve nonanalytic dependence in the arguments). However, usually, only the cumulants at equal arguments are considered.

# Metastable states

- Many minima of the bare action in the region of interest:

  At small $\Delta_B$, low T, the stochastic field equation (SFE)

  $$\frac{\delta S_B[\phi]}{\delta\phi(x)} = h(x) + J(x)$$

  has many solutions.

- What is their effect on the long-distance properties ?

  Also known to go with slow relaxation, hysteresis and "glassiness"

## Focus on the critical behavior of the RFIM

Known:

- ✳ Existence of a $Z_2$ symmetry breaking transition for $d>2$
- ✳ Critical behavior associated with a zero-temperature fixed point (can directly work at $T=0$)
- ✳ For a given $h$ realization, the ground state is unique

Yet, many puzzles, e.g. problem of "dimensional reduction"

# Parisi-Sourlas supersymmetric approach

- At T=0, generating functional of the correlation functions:

$$\mathcal{Z}_h[J,\hat{J}] = \int \mathcal{D}\phi\, \delta\Big[\frac{\delta S_B[\phi]}{\delta\phi} - h - J\Big] \left| \frac{\delta^2 S_B[\phi]}{\delta\phi\delta\phi} \right| e^{\int_x \hat{J}(x)\phi(x)}$$

If unique solution of SFE, usual manipulations:

Introduce auxiliary fields $\hat{\phi}(x)$, $\psi(x)$, $\overline{\psi}(x)$, then average over disorder;
Introduce a superspace with 2 Grassmann coordinates $\underline{x} = (x, \overline{\theta}, \theta)$,
a superLaplacian $\Delta_{SS} = \partial_\mu^2 + \Delta_B \partial_\theta \partial_{\overline{\theta}}$,
a superfield $\Phi(\underline{x}) = \phi(x) + \overline{\theta}\psi(x) + \overline{\psi}(x)\theta + \overline{\theta}\theta\hat{\phi}(x)$, super-etc...

- Generating functional obtained from a superfield theory

$$S_{SUSY}[\Phi] = \int_{\underline{x}} \Big\{ -\frac{1}{2}\Phi(\underline{x})\Delta_{SS}\Phi(\underline{x}) + \frac{\tau}{2}\Phi(\underline{x})^2 + \frac{u}{4!}\Phi(\underline{x})^4 \Big\}$$

- Invariant under SUSY (super-rotations in superspace)
=> leads to "dimensional reduction": RFIM in *d* dim. is equivalent to pure theory in *d-2*. Beautiful, but wrong!!

# Program for RG study of RFIM

## [Search for proper T=0 (critical) fixed point]

- Start the RG flow with a "regularized" stochastic field equation having a unique solution.
- Select with high probability the ground state at the running IR scale $k$ among the solutions if several of them.
- Describe full functional dependence of cumulants of renormalized disorder and allow for nonanalytical dependence on their arguments.
- Use a nonperturbative truncation and be able to recover dimensional reduction if it has a range of validity.

=> NP-FRG in a superfield setting

# Superfield formalism for the RFIM

- Several copies+weighting factor => Generating functional:

$$\mathcal{Z}_h[\{J_a,\hat{J}_a\}] = \prod_a \int \mathcal{D}\phi_a\, \delta\Big[\frac{\delta S_B[\phi_a]}{\delta \phi_a} - h - J_a\Big] \left| \frac{\delta^2 S_B[\phi_a]}{\delta\phi_a \delta\phi_a} \right| \times e^{-\beta\left(S_B[\phi_a] - \int_x [h(x)+J_a(x)]\phi_a(x)\right)} e^{\int_x \hat{J}_a(x)\phi_a(x)}$$

Average over disorder generates cumulants with full functional dependence:

$$\overline{\mathcal{Z}_h[\{J_a,\hat{J}_a\}]} = \overline{\prod_a e^{\mathcal{W}_h[J_a,\hat{J}_a]}} = e^{\sum_a \overline{\mathcal{W}_h[J_a,\hat{J}_a]} + \frac{1}{2}\sum_{ab} \overline{\mathcal{W}_h[J_a,\hat{J}_a]\mathcal{W}_h[J_b,\hat{J}_b]}|_c + \cdots}$$

- Introduce superfields and ("curved") Grassmannian space

$$\Phi(\underline{\theta}) = \phi + \overline{\theta}\psi + \overline{\psi}\theta + \overline{\theta}\theta\hat{\phi}\,; \quad \int_{\underline{\theta}} = \int\int d\theta d\overline{\theta}(1+\beta\overline{\theta}\theta)$$

=> $$S[\{\Phi_a\}] = \sum_a \int_{\underline{\theta}} S_1[\Phi_a(\underline{\theta})] - \frac{1}{2}\sum_{ab}\int\int_{\underline{\theta}_1\underline{\theta}_2} S_2[\Phi_a(\underline{\theta}_1), \Phi_a(\underline{\theta}_2)]$$

$$S_1 = \int_x \left[\frac{1}{2}\big(\partial_\mu \Phi_a(\underline{\theta},x)\big)^2 + U_B(\Phi_a(\underline{\theta},x))\right]; \; S_2 = \int_x \Delta_B\, \Phi_a(\underline{\theta}_1,x)\Phi_b(\underline{\theta}_2,x)$$

# Superfield formalism (contnd.)

- Add coupling to supersources $\sum_a \int_{\underline{\theta},x} \mathcal{I}_a(\underline{\theta},x)\Phi_a(\underline{\theta},x)$ -> $\mathcal{W}[\{\mathcal{I}_a\}]$

  + Legendre transform -> Effective action $\Gamma[\{\Phi_a\}]$

- Action is invariant under a large group of symmetries and supersymmetries ($S_n$ between copies, global $Z_2$ and Euclidean translations + rotations, isometries of the curved Grassmann subspace copy by copy)

- Additional properties:

  ∗ If a unique solution is included in the partition function => joint expansion (ultra-locality in Grassmann subspace):

$$\mathcal{W}[\{\mathcal{I}_a\}] = \sum_a \int_{\underline{\theta}} W_1[\mathcal{I}_a(\underline{\theta})] + \frac{1}{2}\sum_{ab}\int\int_{\underline{\theta}_1\underline{\theta}_2} W_2[\mathcal{I}_a(\underline{\theta}_1),\mathcal{I}_b(\underline{\theta}_2)] + \cdots$$

  ∗ In addition, for supersources that reduce the theory to a <u>1-copy problem</u> AND for <u>β=0</u>: invariance under superrotations

  => Ward-Takahashi identities

# NP-FRG in superfield formalism

- Add an IR regulator to the action:

$$\Delta S_k[\{\Phi_a\}] = \frac{1}{2}\sum_{ab}\int_{\underline{x}_1}\int_{\underline{x}_2}\Phi_a(\underline{x}_1)\mathcal{R}_{k,ab}(\underline{x}_1,\underline{x}_2)\Phi_b(\underline{x}_2)$$

$\mathcal{R}_{k,ab}(\underline{x}_1,\underline{x}_2) = \delta_{\underline{\theta}_1,\underline{\theta}_2}\widehat{R}_k(q^2) + \widetilde{R}_k(q^2)$ : suppresses fluctuations of ϕ field and random field

- ERGE for the effective average action at scale *k*:

$$\partial_k\Gamma_k[\{\Phi_a\}] = \frac{1}{2}\sum_{ab}\int_{\underline{x}_1}\int_{\underline{x}_2}\partial_k\mathcal{R}_{k,ab}(\underline{x}_1,\underline{x}_2)(\Gamma_k^{(2)}[\{\Phi_a\}] + \mathcal{R}_k)^{-1}_{(b,\underline{x}_2)(a,\underline{x}_1)}$$

- If at scale *k*, the stoch. field eq. has a unique solution or if for large enough β, one only selects the ground state (occasional errors will lead to subdominant terms... another story!): then, joint expansion

$$\Gamma_k[\{\Phi_a\}] = \sum_a\int_{\underline{\theta}}\Gamma_{k,1}[\Phi_a(\underline{\theta})] - \frac{1}{2}\sum_{ab}\int_{\underline{\theta}_1}\int_{\underline{\theta}_2}\Gamma_{k,2}[\Phi_a(\underline{\theta}_1),\Phi_b(\underline{\theta}_2)] + \ldots$$

# NP-FRG and SUSY breaking

- ERGE+joint expansion => hierarchy of ERGE's for the cumulants

$$\begin{cases} \partial_t \Gamma_{k1}[\phi] = \frac{1}{2}\widetilde{\partial}_t\, Tr\{[\Gamma_{k1}^{(2)}[\phi] + \hat{R}_k]^{-1}[\Gamma_{k2}^{(11)}[\phi,\phi] - \widetilde{R}_k]\} \\ \partial_t \Gamma_{k2}[\phi_1,\phi_2] = \cdots \end{cases} \qquad [t = ln(k/\Lambda)]$$

!!!!!! The auxiliary parameter β drops out of the flow equations !!!!!!

- As a result, superrotation invariance for 1 copy is *a priori* preserved along the RG flow: Can show that it leads (nonperturbatively) to dimensional reduction.

- What can go wrong ?
  - ✳ Spontaneous breaking of superrotation invariance: some 1PI vertex blows up when copy fields become equal.
  - ✳ Dimension reduction is broken when a cusp

$$\Gamma_{k,2}^{(11)}(\varphi_1,\varphi_2) - \Gamma_{k,2}^{(11)}(\varphi_1,\varphi_1) \sim |\varphi_2 - \varphi_1| \quad \text{as} \quad \varphi_2 \to \varphi_1$$

appears at a <u>finite</u> scale $k_L$

# SUSY-compatible approximation and RG flow

- Ansatz for effective average action:

$$\Gamma_{k1}[\phi] = \int_x \left[U_k(\phi(x)) + \frac{1}{2} Z_k(\phi(x))(\partial_\mu \phi(x))^2\right]$$

$$\Gamma_{k2}[\phi_1, \phi_2] = \int_x V_k(\phi_1(x), \phi_2(x)), \qquad \Gamma_{k,p>2} = 0$$

+ Regulator: $\hat{R}_k = Z_k k^2\, r(q^2/k^2),\ \tilde{R}_k = -(\Delta_k/Z_k)\partial_{q^2}\hat{R}(q^2)$

[ SUSY Ward identity: $\Delta_k = \Delta_B Z_k$ ]

- Introduce scaling dimensions for T=0 fixed point (critical):

$$\partial_t u'_k(\varphi) = \cdots$$
$$\partial_t z_k(\varphi) = \cdots$$
$$\partial_t \delta_k(\varphi_1, \varphi_2) = \partial_t v_k^{(11)}(\varphi_1, \varphi_2) = \cdots$$

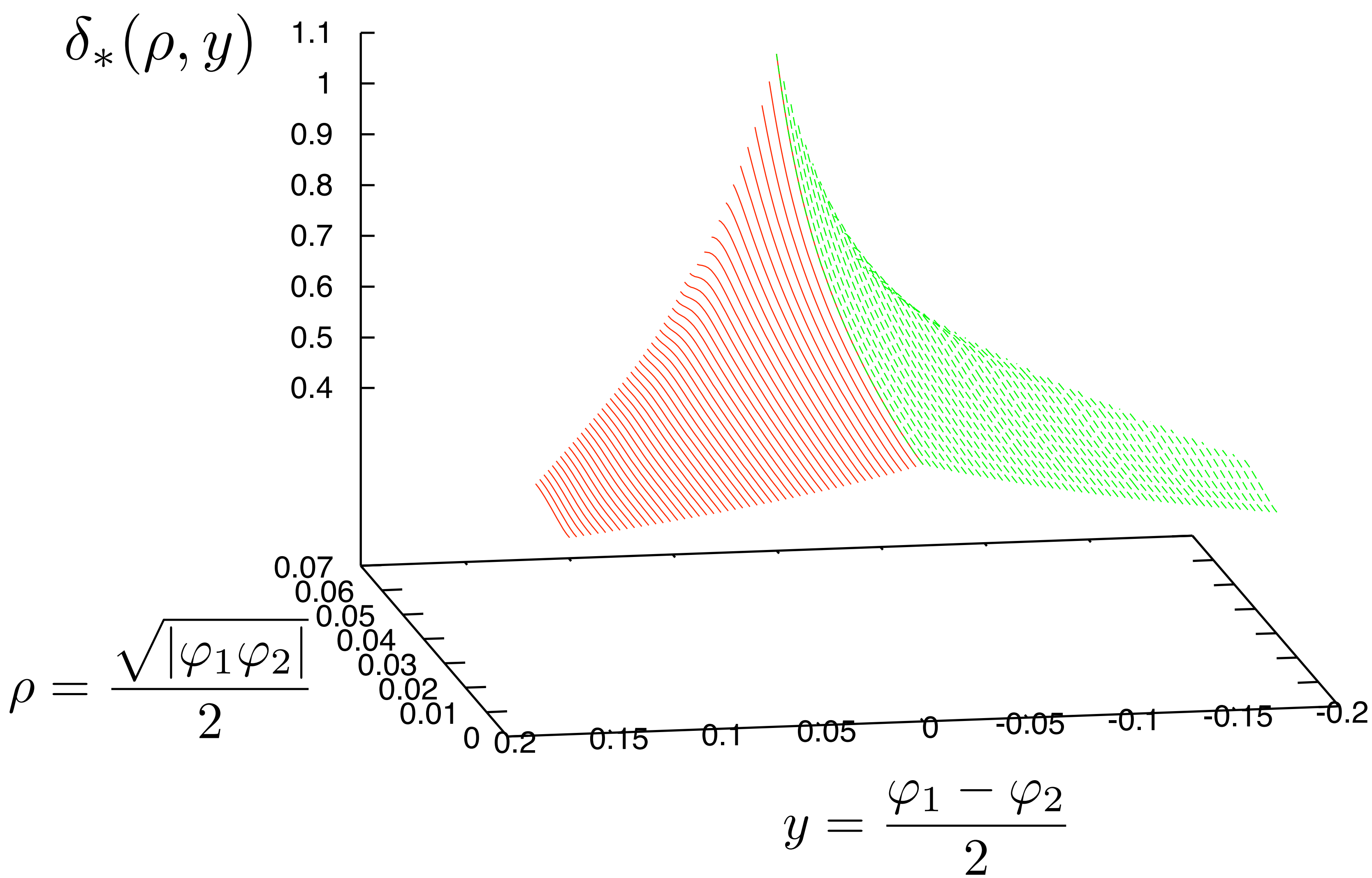
$$\eta_k = -\partial_t Z_k$$
$$\bar{\eta}_k = 2\eta_k + \partial_t \Delta_k$$

- If no linear cusp in $\delta_k(\varphi_1, \varphi_2)$, then $\partial_t \delta_k(\varphi, \varphi) = \partial_t z_k(\varphi)$ (Ward id.) and exact dim. reduction follows: found for $d > d_{DR} \simeq 5.15$

# Results

Above $d_{DR} \simeq 5.15$: no cusp in $\delta_k(\varphi_1, \varphi_2)$ & dim.reduction
Below $d_{DR}$: cusp in $\delta_k(\varphi_1, \varphi_2)$ & breakdown of dim. reduction

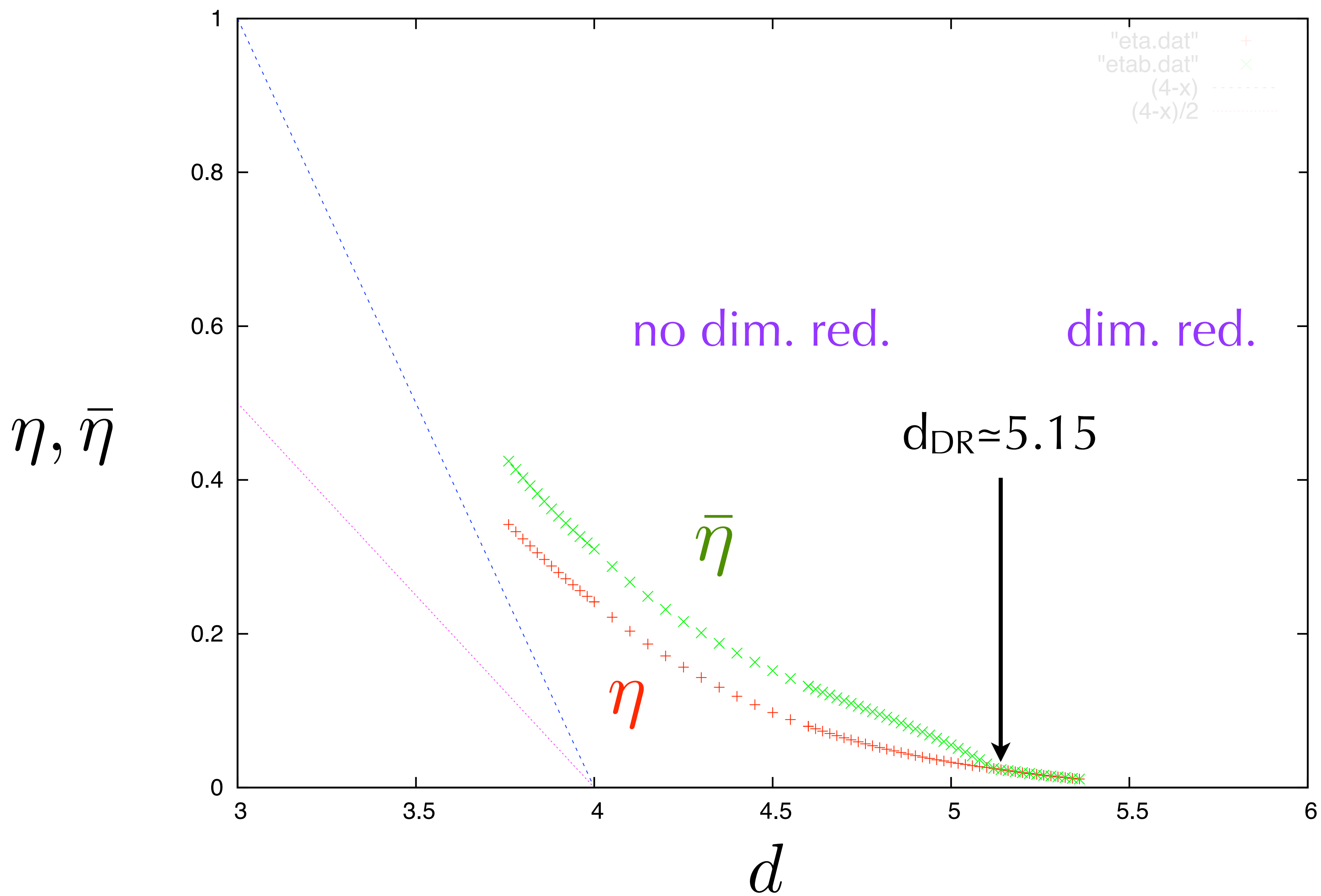
Dimensionless cumulant of disorder at fixed point in d=3



# Results: Critical exponents $\eta$ and $\tilde{\eta}$

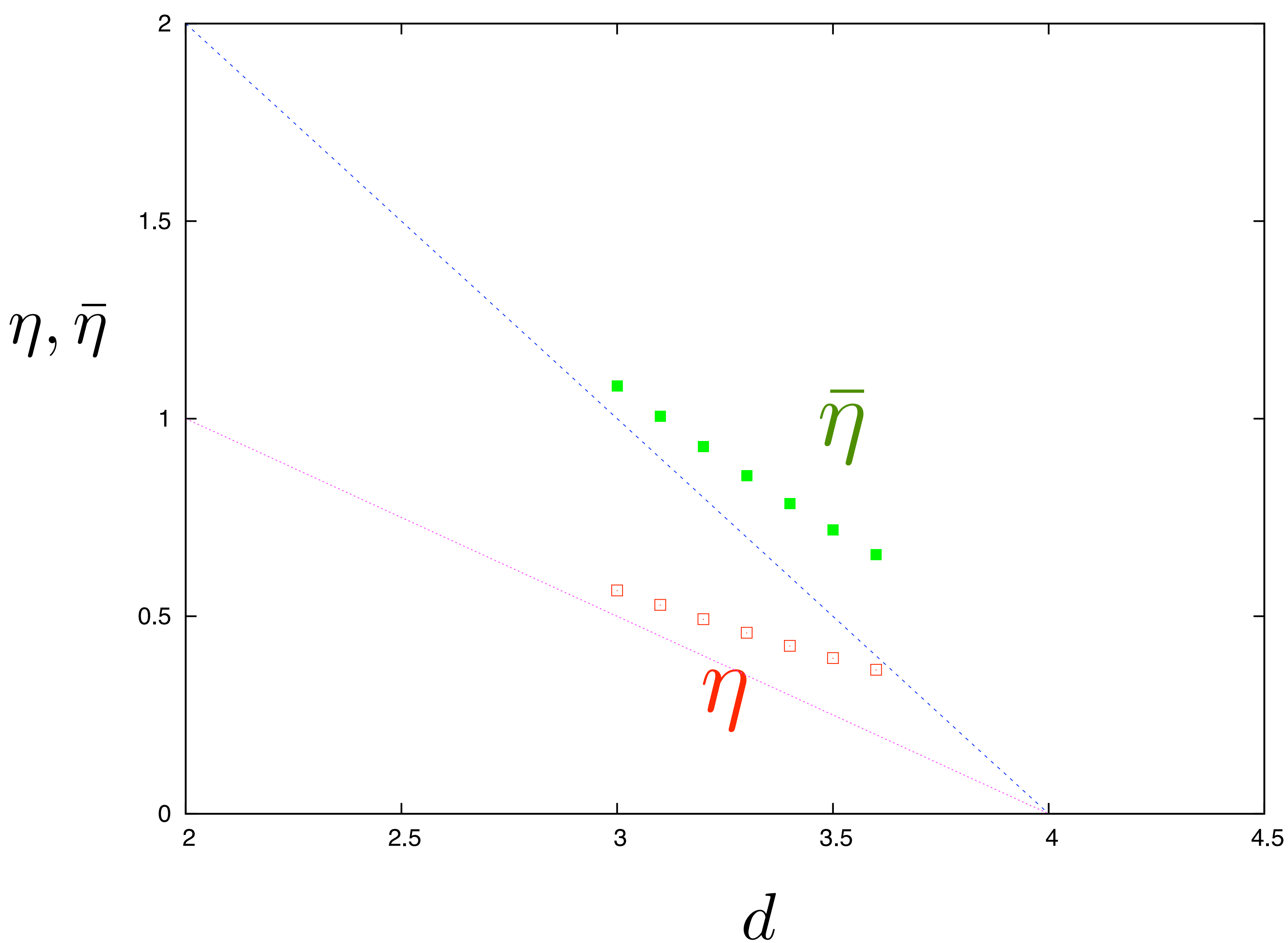
Breakdown from dim. red. appears continuously in dimension *d*

- Dim. reduction: $\bar{\eta} = \eta$
- Pending speculation: $\bar{\eta} = 2\,\eta$ → wrong!



# Results: Critical exponents $\eta$ and $\tilde{\eta}$ (contnd.)

To go to low dimension (d≲4), need optimization of cut-off (versus stability of results)




Very good agreement with "best estimates":

In d=3,

$\eta$ =0.57 ± 0.05 [0.512±0.036]

$\bar{\eta}$ =1.10 ± 0.05 [1.022±0.06]

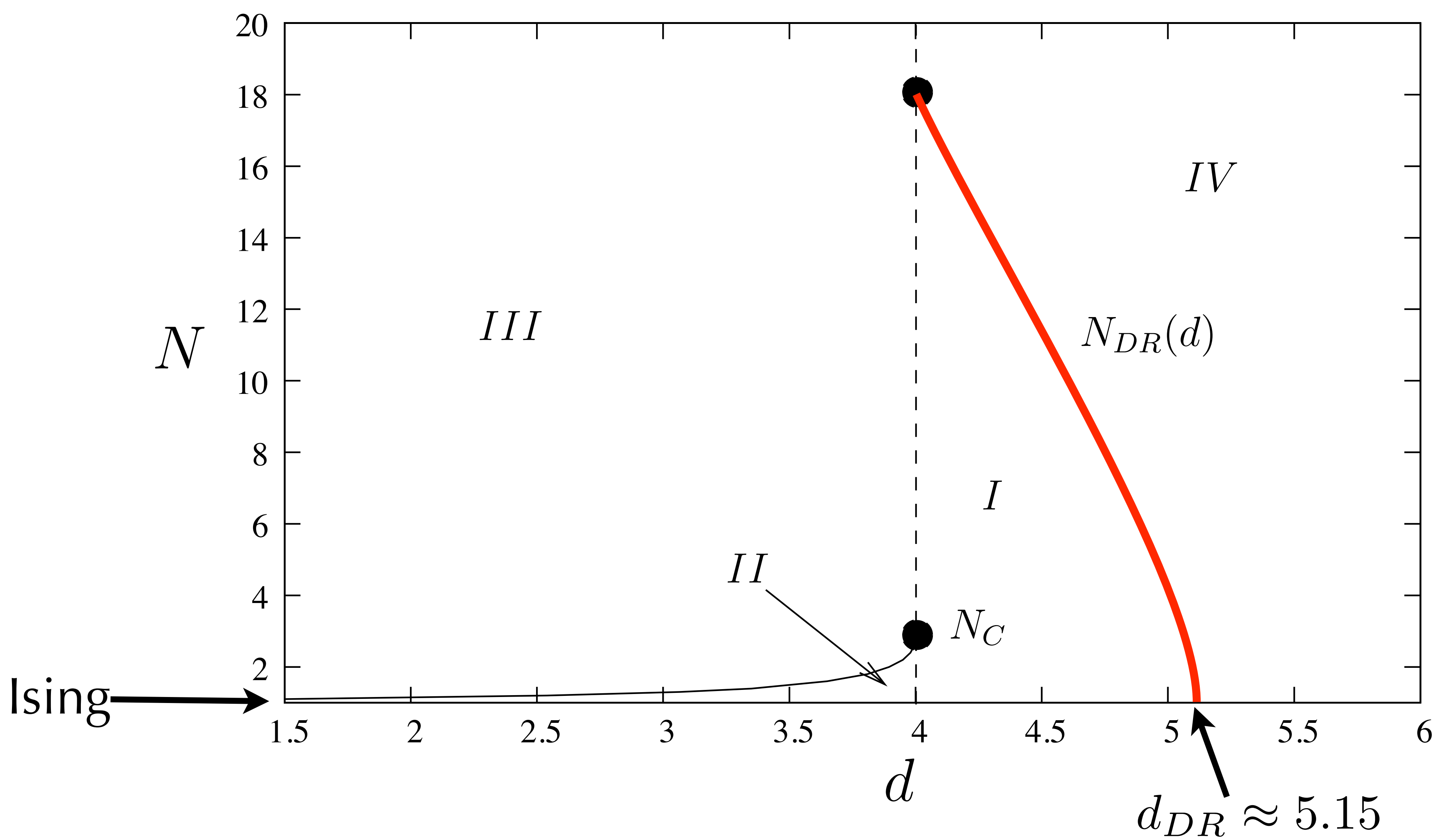
# Conclusion

- NP-FRG in a superfield setting
= useful formalism for dealing with long-distance behavior in (some) disordered systems and solutions of (some ?) stochastic field equations.

- It solves the 30-year-old pending problems concerning the critical behavior in random field systems.

- It can be generalized to treat excitations (droplets) and the effect of temperature, out-of-equibrium criticality in hysteresis behavior, dynamics.

# Results: *N*-*d* phase diagram of the RFO(*N*)M




Region IV: Weak non-analyticity (at fixed pt.); dim. red. predictions O.K.

Regions I and II: Spontaneous SUSY breaking at finite RG scale; cusp in renormalized second cumulant; breakdown of dim. red. (II: QLRO)

Region III: No phase transition