Critical Dynamics and Turbulence

in ultracold gases

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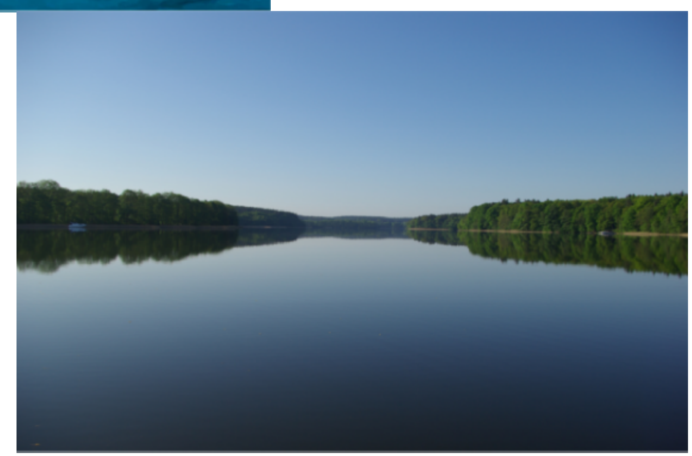
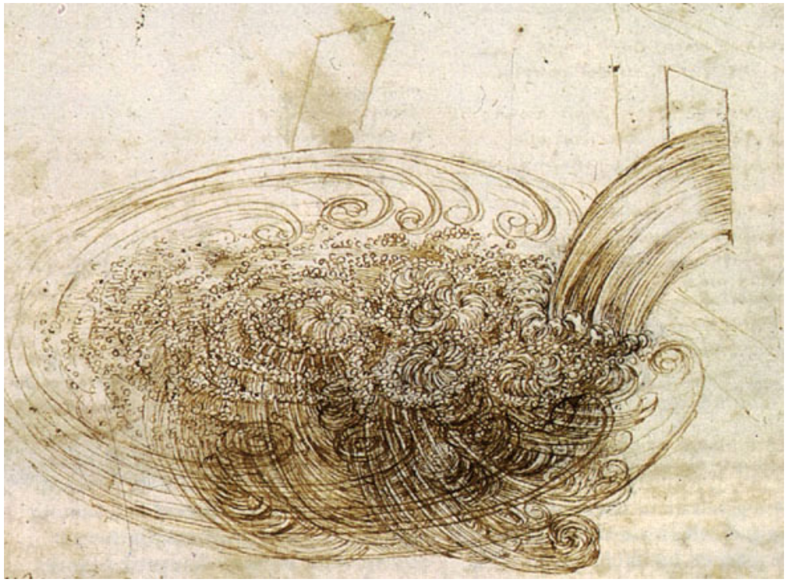
Center for Quantum Dynamics

EMMI

Nonequilibrium Stationary Dynamics



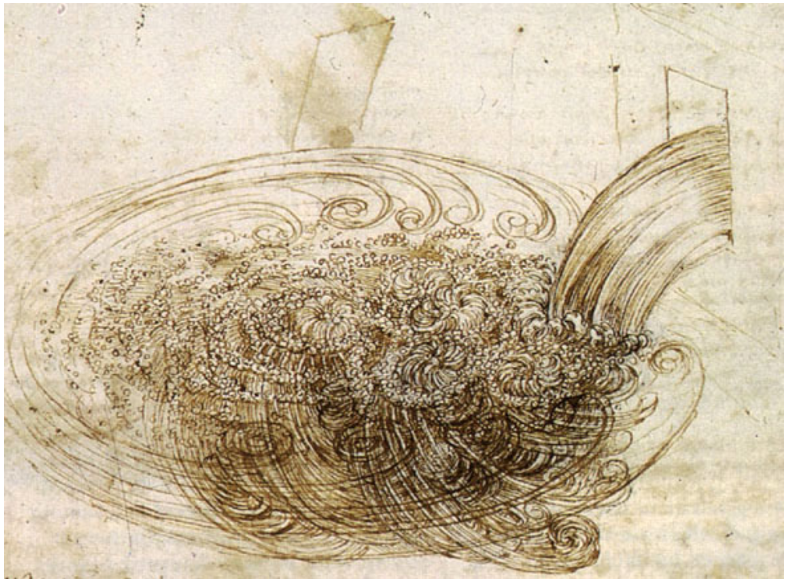
Turbulence

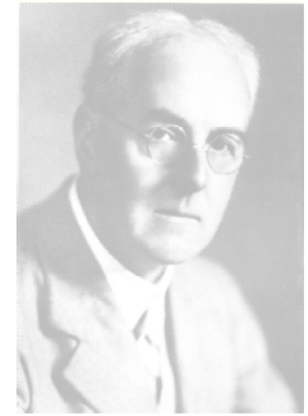


Kinetic energy cascade

large scales (source)
→ small scales
→ dissipation (sink)

Richardson Cascade





Lewis F. Richardson
(1881-1953)

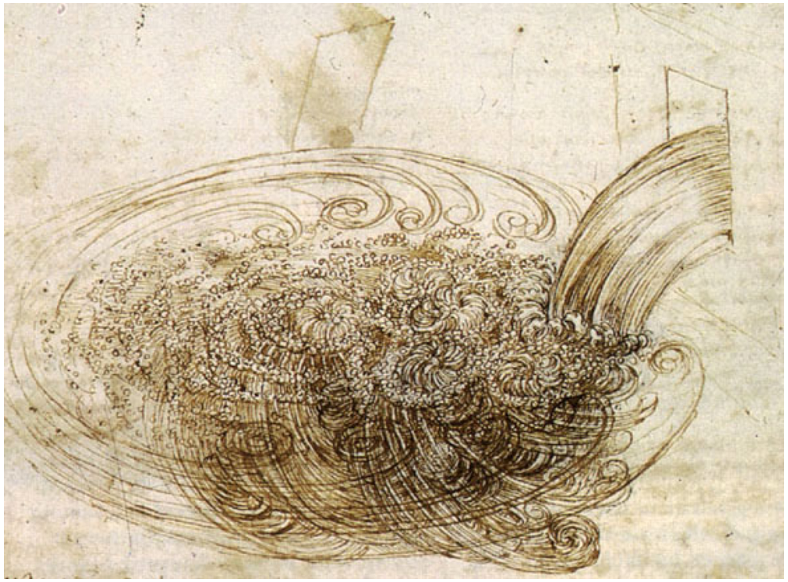
Kinetic energy cascade

large scales (source)
→ small scales
→ dissipation (sink)

"Big whirls have little whirls that feed on their velocity,
and little whirls have lesser whirls and so on to viscosity."

(Richardson, 1920)

Richardson Cascade



Lewis F. Richardson
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Kinetic energy cascade

large scales (source)
$\rightarrow$ small scales
$\rightarrow$ dissipation (sink)

*"Big whirls have little whirls that feed on their velocity,
and little whirls have lesser whirls and so on to viscosity."*

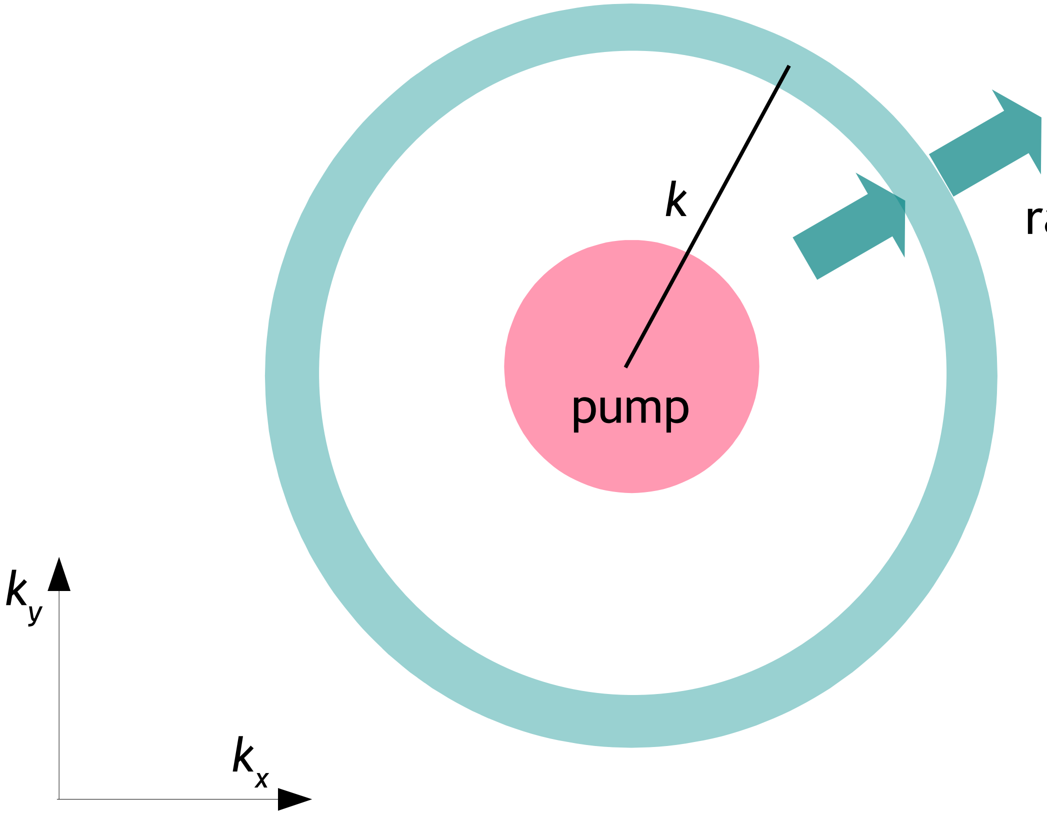
(Richardson, 1920)

Kolmogorov (1941): scale-invariant, incompressible fluid

$$E(k) \sim P^{2/3} \rho^{1/3} k^{-5/3}$$ (Stationary distribution)

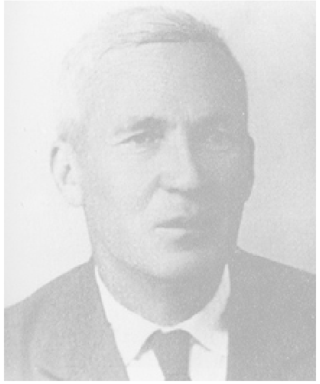
Kolmogorov's theory of turbulence
(1941)

Take isotropic system:
Cascade = Radial momentum transport



$$\partial_t\, \varepsilon(\mathbf{k}) = -\nabla_{\mathbf{k}} \cdot \mathbf{p}(\mathbf{k})$$

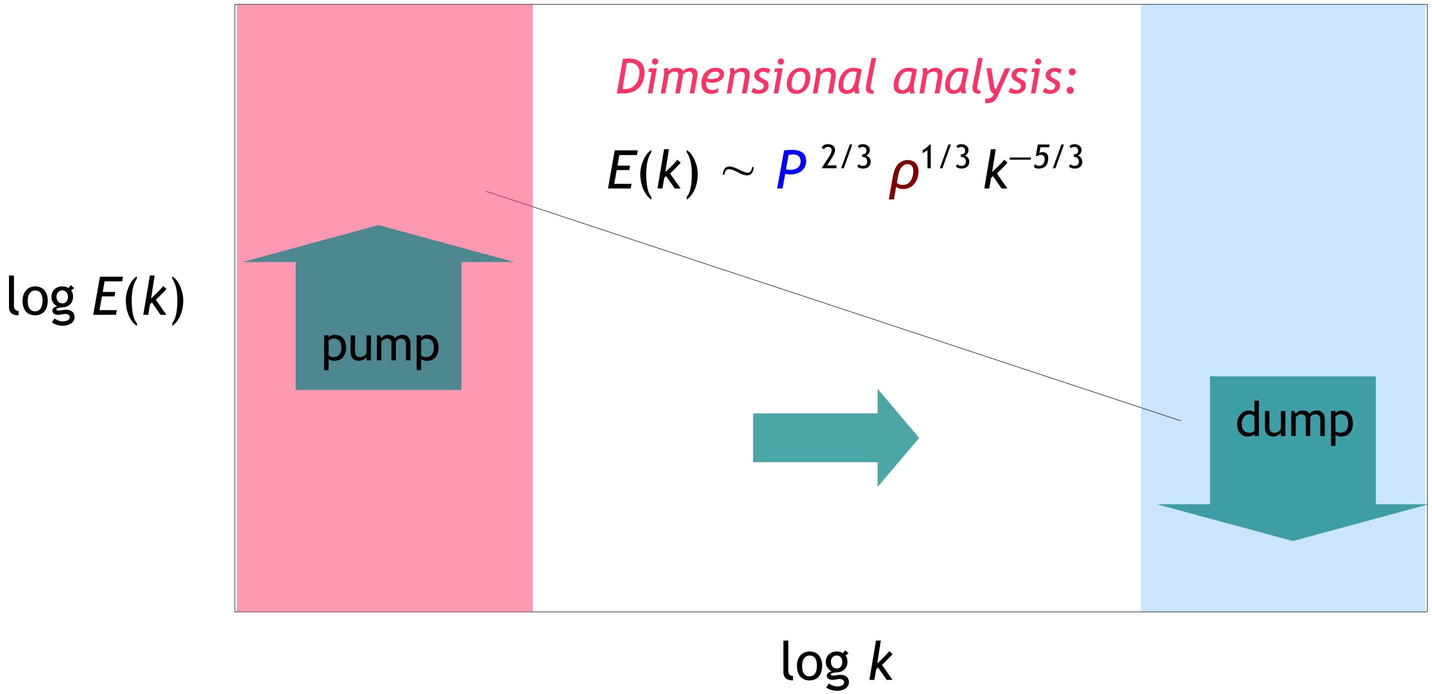
Energy density ε
Energy flux $\mathbf{p}$



Andrey N. Kolmogorov
(1903-1987)

Kolmogorov's theory of turbulence
(1941)

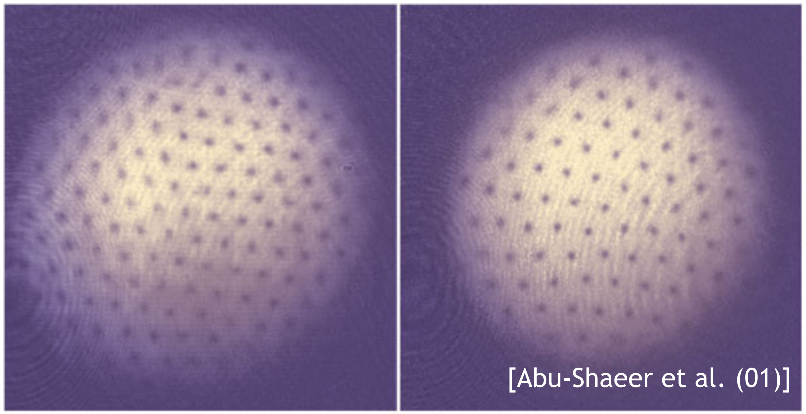
Scale invariant (self-similar) stationary transport:

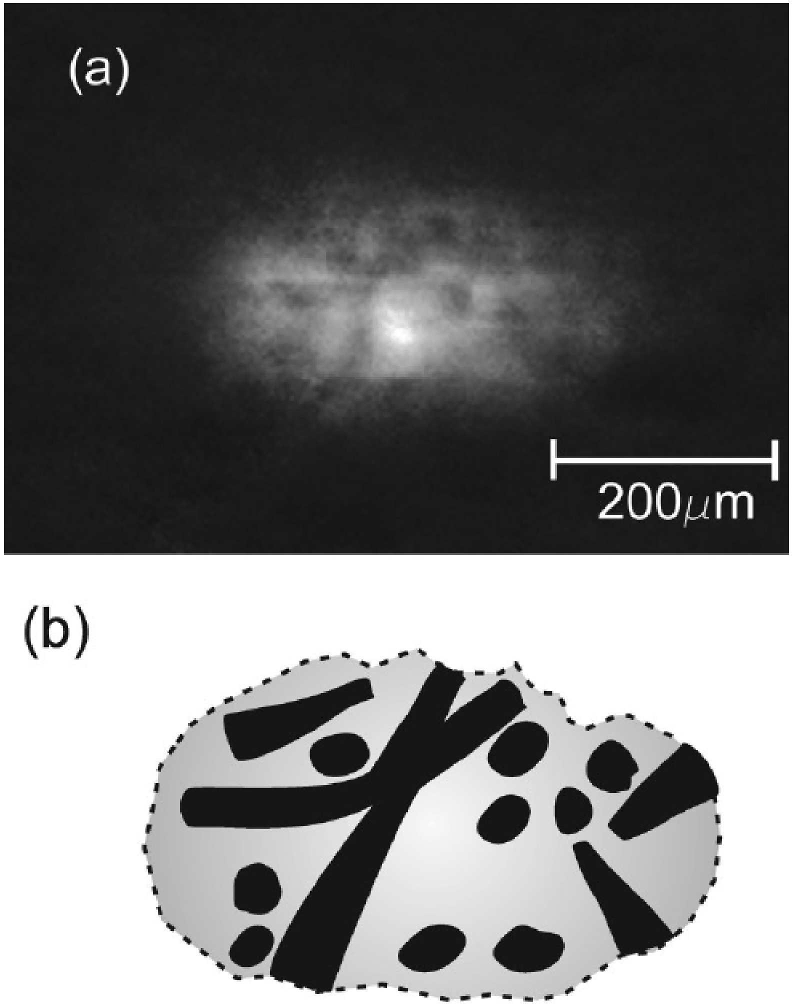


3D:	Radial energy density	E	[kg s^{-2}]
	Radial energy flux	P	[kg m^{-1} s^{-3}]
	Density	ρ	[kg m^{-3}]

Quantum Turbulence

Turbulence in a Bose Einstein Condensate





[E.A.L. Henn et al. PRL (09)]

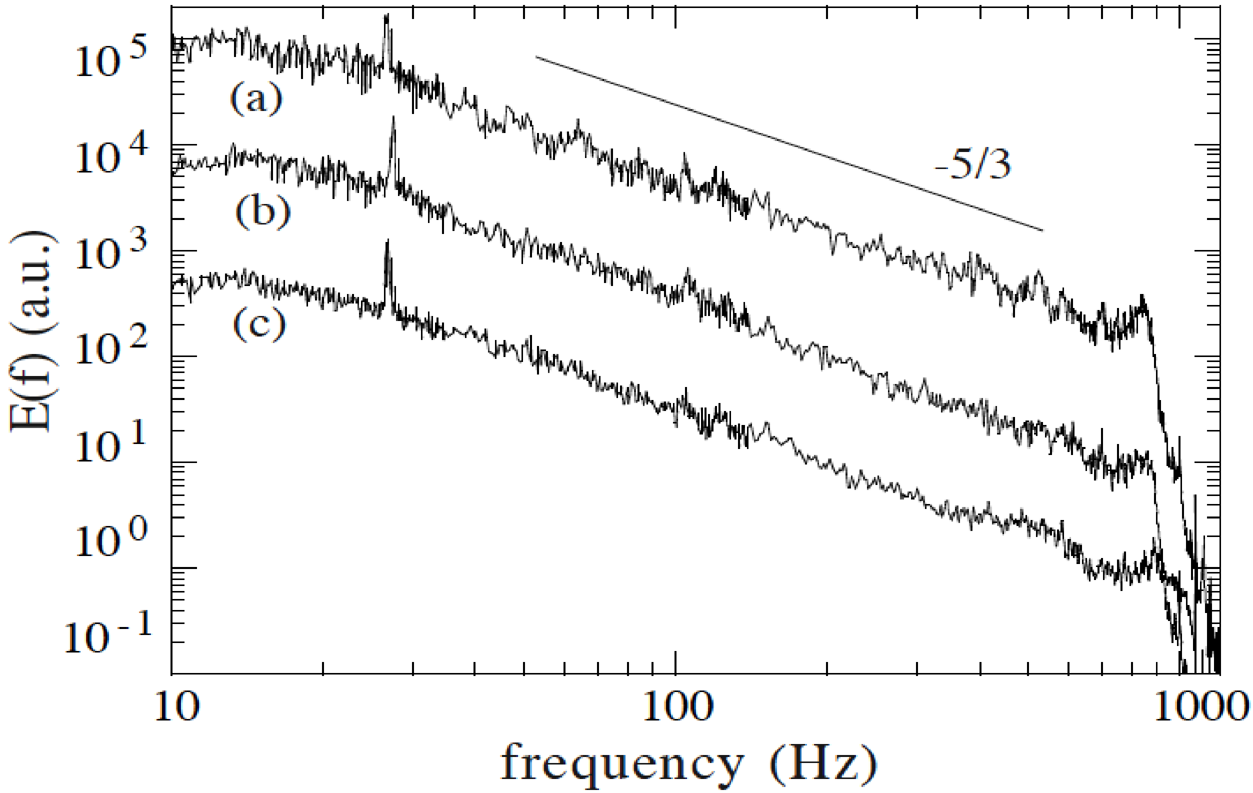
He Experiment

Europhys. Lett., **43** (1), pp. 29-34 (1998)

Local investigation of superfluid turbulence

J. Maurer and P. Tabeling

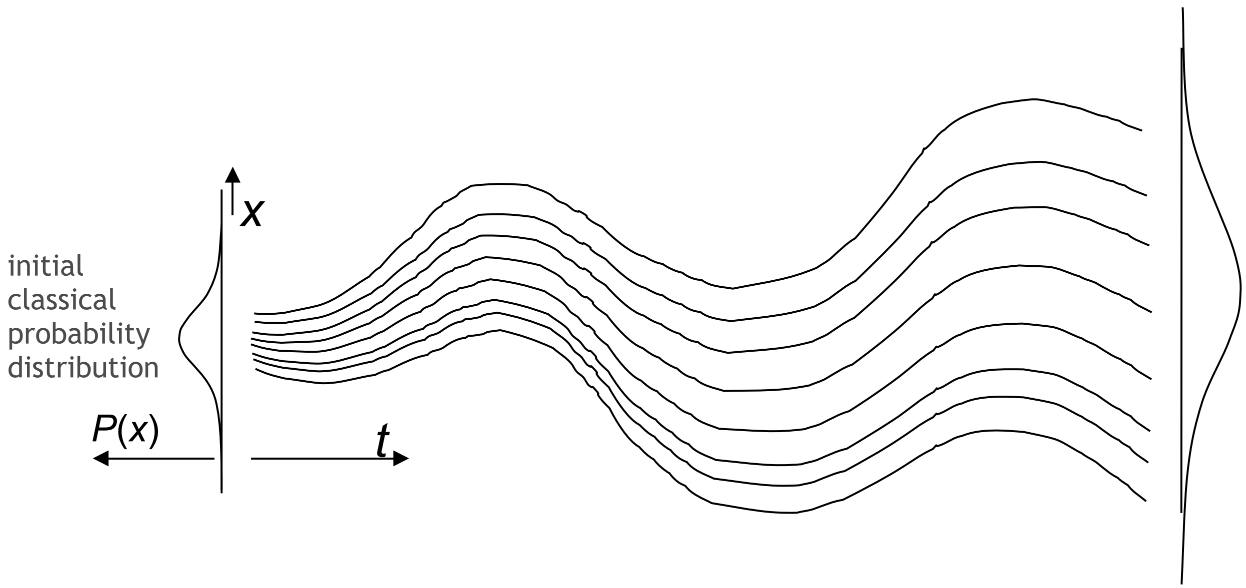
flows of helium IV driven by two counter-rotating disks



Theory: Simulations...

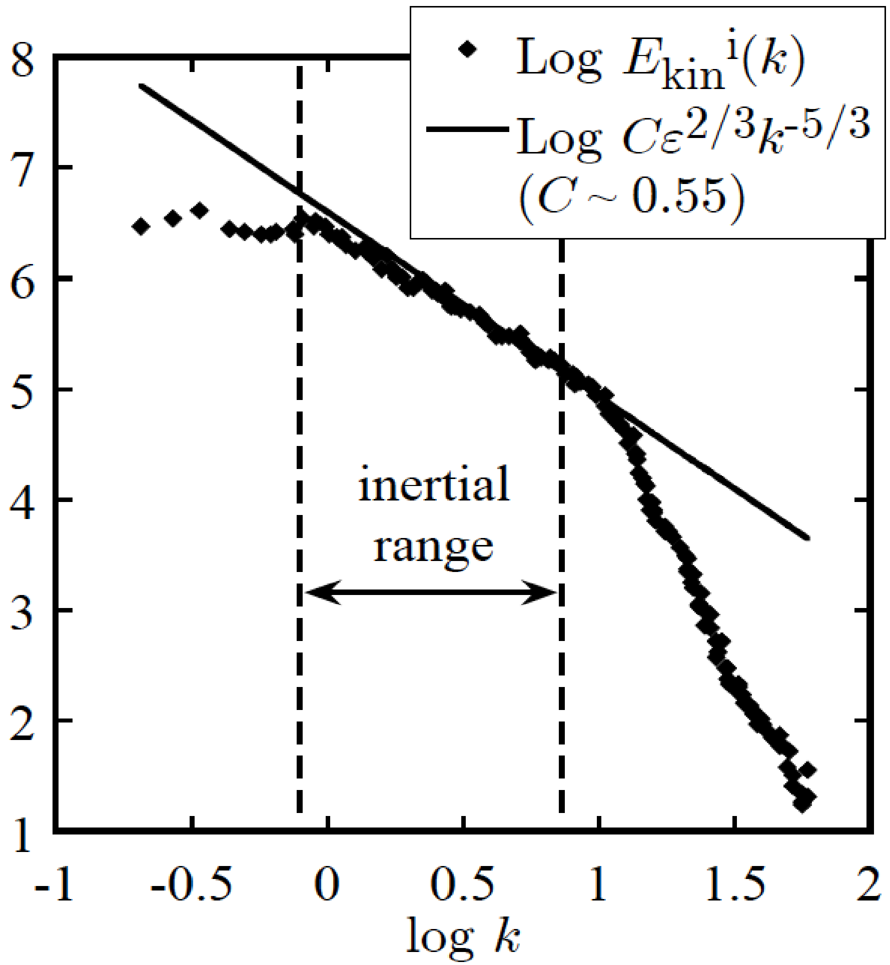
...of the Gross-Pitaevskii Eq. in the classical regime:

$$i\partial_t \psi(\mathbf{x},t) = \left[-\frac{\partial_{\mathbf{x}}^2}{2m} + g|\psi(\mathbf{x},t)|^2 \right] \psi(\mathbf{x},t)$$



Kolmogorov scaling of Qu. Turbulence

(Simulations of Gross-Pitaevskii Equation)



M. Kobayashi & M. Tsubota,
J. Phys. Soc. Jpn (2005).
M. Tsubota, arXiv:1004.5458 [cond-mat].

Extensive work on Qu. Turbulence:

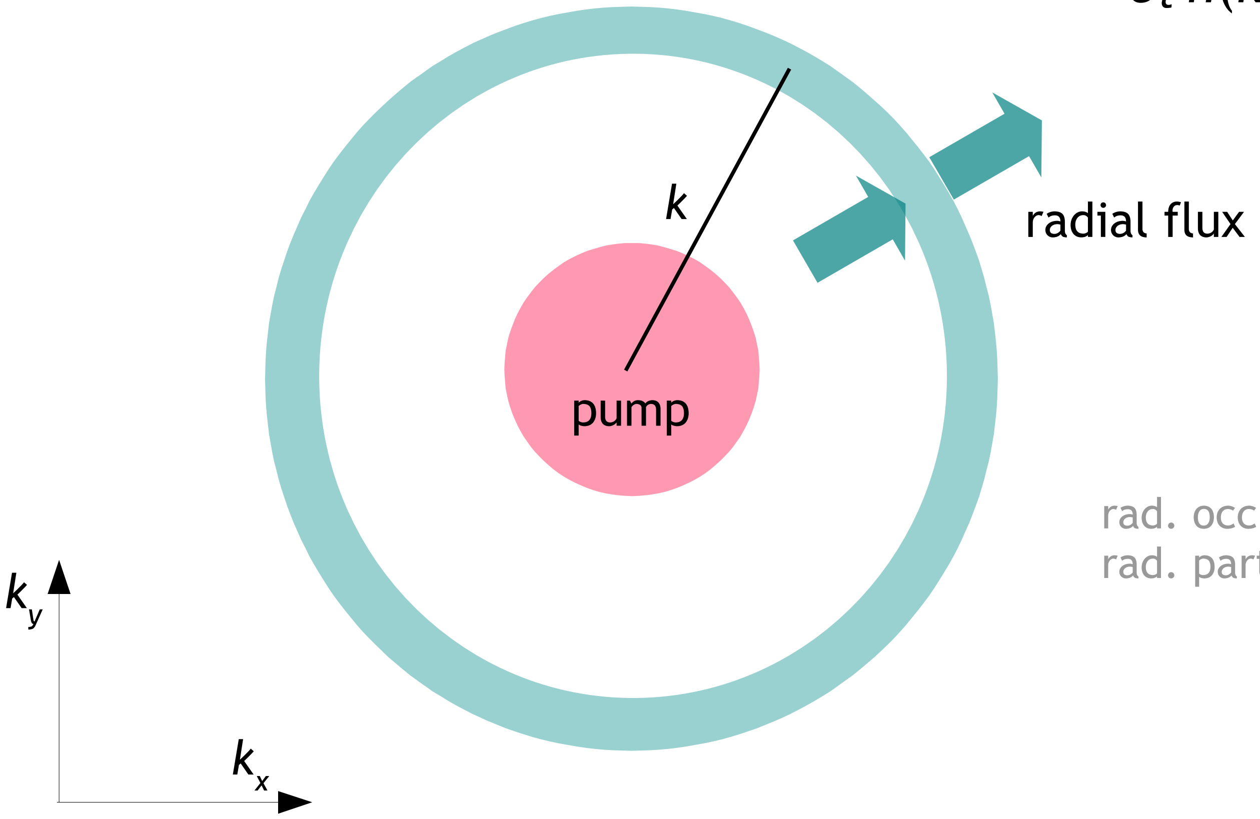
Svistunov, Berloff, Kozik, … 1995-
Nore, Abid, Brachet, 1995-
Vinen 50's-; & Donnelly, 2002, 07.
L'vov, Nazarenko, Budenko, 2007.
Horng, Gou, et al., 2008-
...

What can be expected analytically?

Local radial flux only

Balance equation for radial flux

$$\partial_t n(k) = - \partial_k Q(k)$$



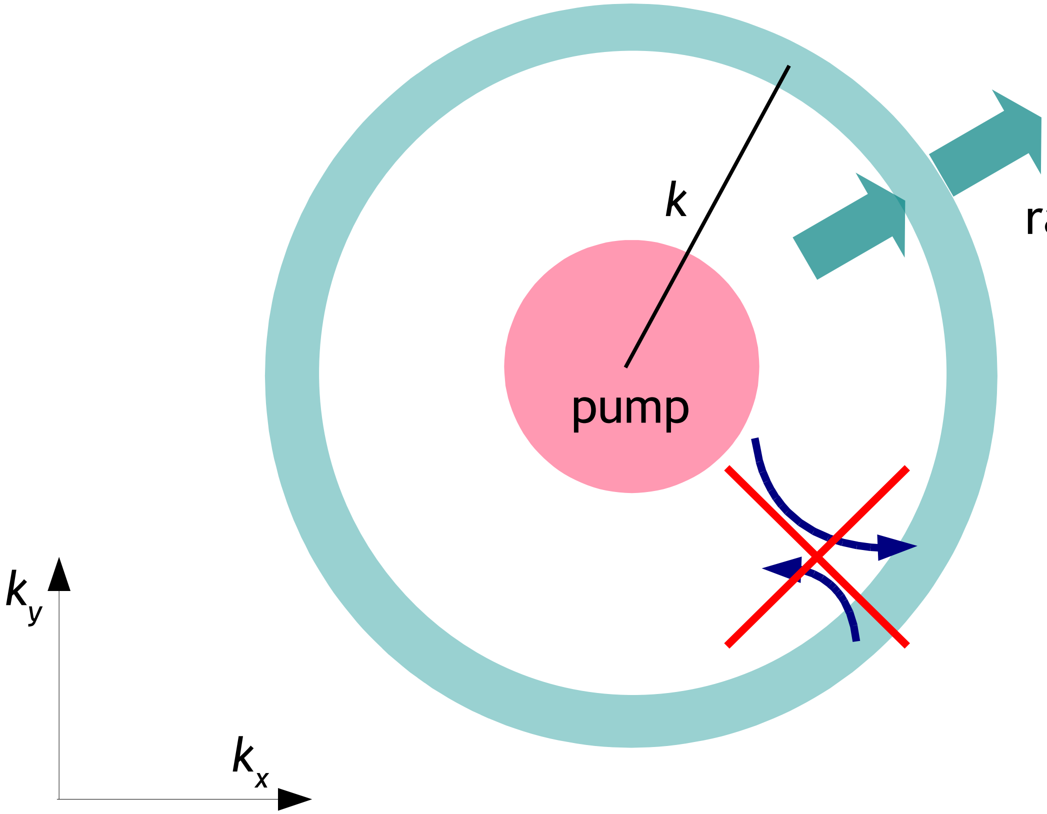
rad. occupation no. n
rad. particle flux Q

Local radial flux only

With kinetic (Boltzmann) eq.

Boltzmann scattering integral

$$\partial_t\, n(k) = -\,\partial_k\, Q(k) \sim k^2\, J(k) \overset{!}{=} 0$$



radial flux

rad. occupation no. n
rad. particle flux Q

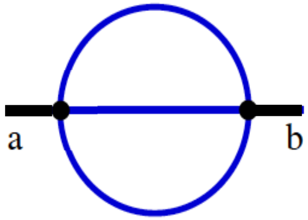
Dynamical field theory

Dynamic (Schwinger-Dyson/Kadanoff-Baym) eq.:

$$G^{-1}_{ab}(x,y) \;=\; G^{-1}_{0,ab}(x,y;\phi) - \Sigma_{ab}(x,y;\phi,G)$$

...from **2PI effective action (Φ - derivable approx.)**

$$\partial_t n(p) \;=\; \Sigma^{\rho}_{ab}(p)F_{ba}(p) - \Sigma^{F}_{ab}(p)\rho_{ba}(p)$$

$\Sigma_{ab}(p) \;=$ 

$p = (p_0, \mathbf{p})$

Scaling solutions

Want to look for scaling solutions fulfilling stationarity condition $J(p) = 0$

Scaling ansatz:

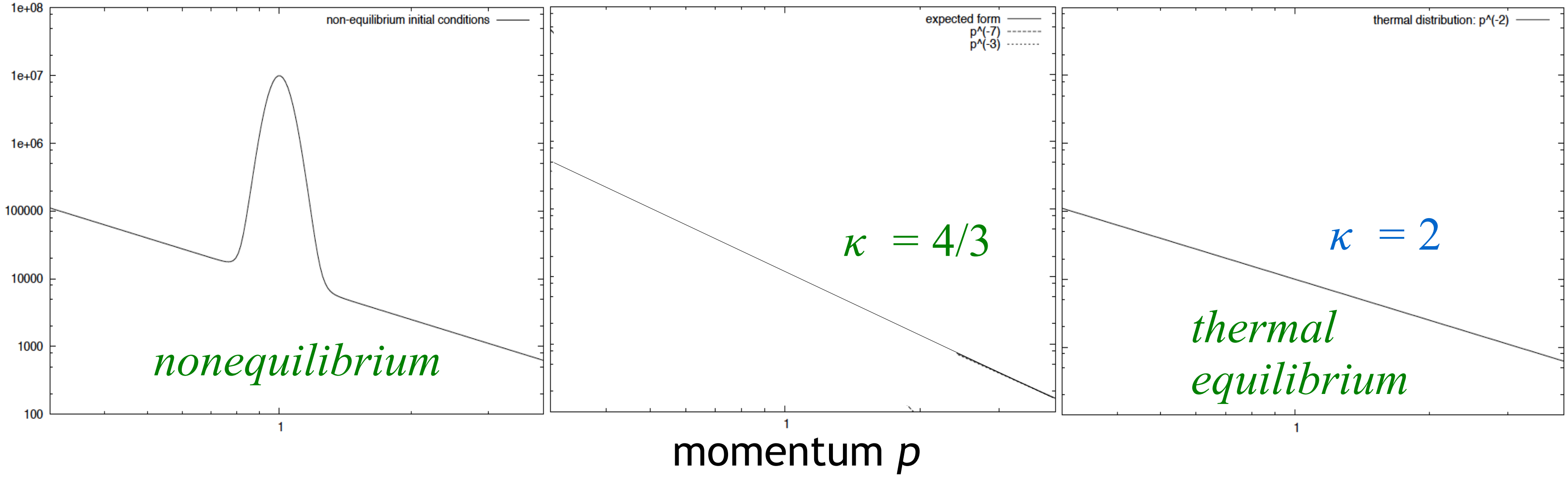
$$\rho_{ab}(s^z p_0, s\mathbf{p}) = s^{-2+\eta}\rho_{ab}(p_0, \mathbf{p})$$
$$F_{ab}(s^z p_0, s\mathbf{p}) = s^{-2+\eta-\kappa}F_{ab}(p_0, \mathbf{p})$$

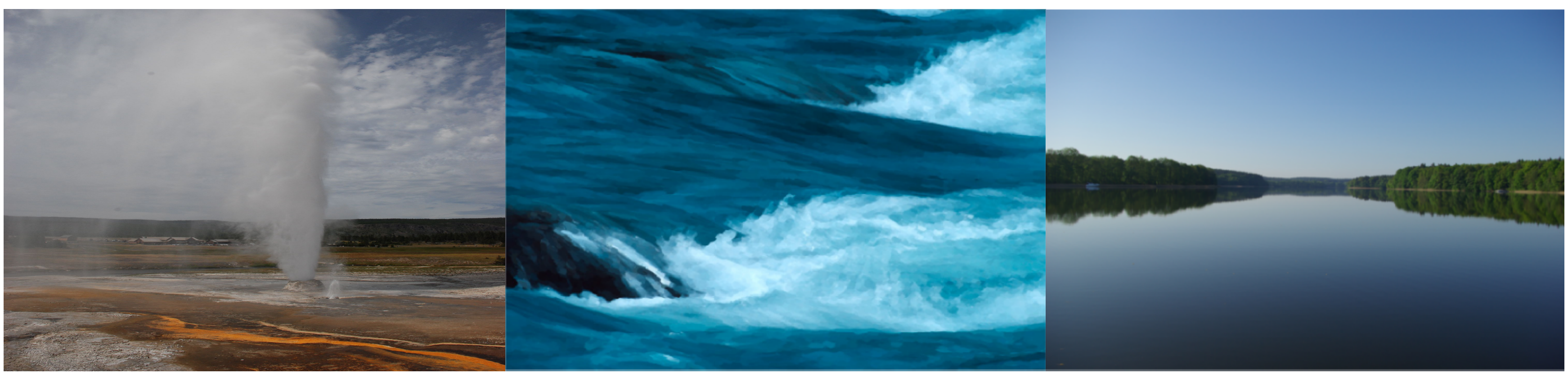
Implies scaling of the single-particle momentum distribution:

$$n(s\mathbf{p}) = s^{z-2-\kappa+\eta}n(\mathbf{p})$$

Turbulent scaling in 2+1 D

$n \sim k^{-\kappa}$



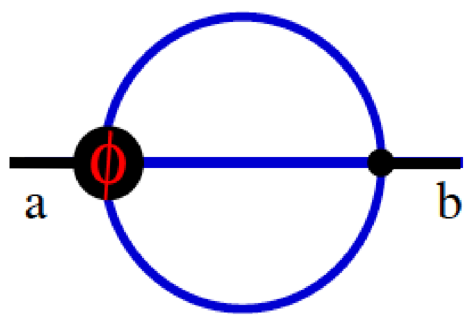


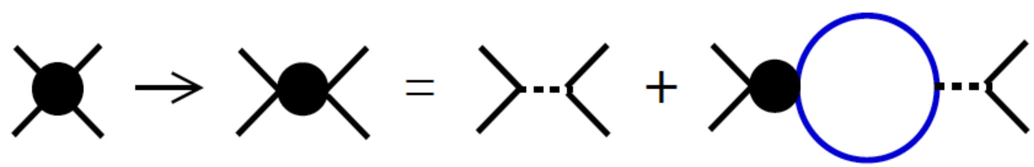
C. Scheppach, J. Berges, TG PRA **81** (10) 033611

Strong turbulence

$p = (p_0, \mathbf{p})$:

$$J(p) := \Sigma^{\rho}_{ab}(p) F_{ba}(p) - \Sigma^{F}_{ab}(p) \rho_{ba}(p) \stackrel{!}{=} 0$$

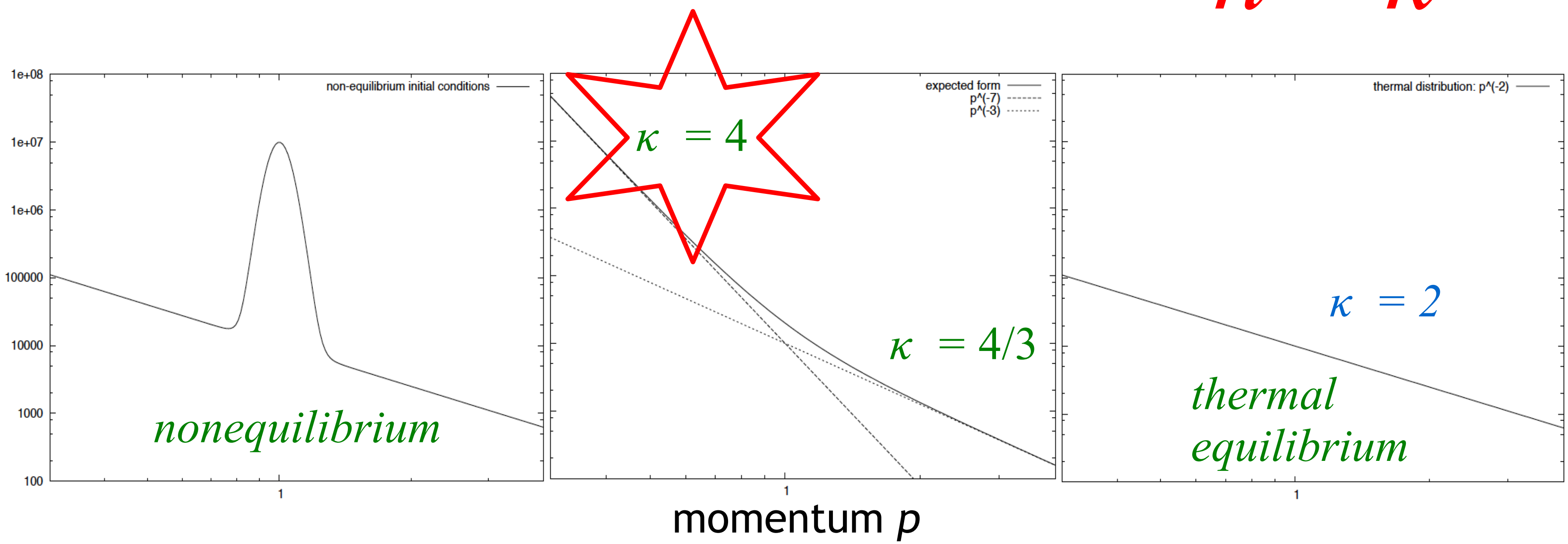
$\Sigma_{ab}(x,y) =$ 

2PI to NLO in $1/N$: Vertex: 

[Dynamics: J. Berges, (02); G. Aarts et al., (02); Kad.Baym: "GW-Approximation", Hedin (65)]

Strong Turbulence in 2D

$$n \sim k^{-\kappa}$$

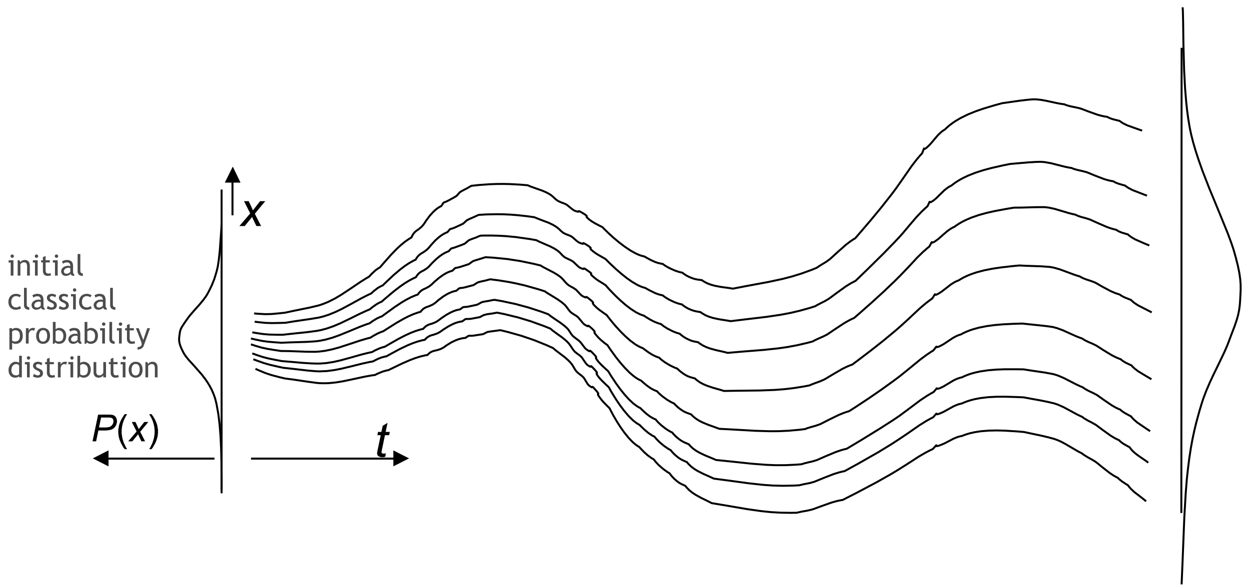


J. Berges, A. Rothkopf, J. Schmidt, PRL **101** (08) 041603
C. Scheppach, J. Berges, TG PRA **81** (10) 033611

Simulations

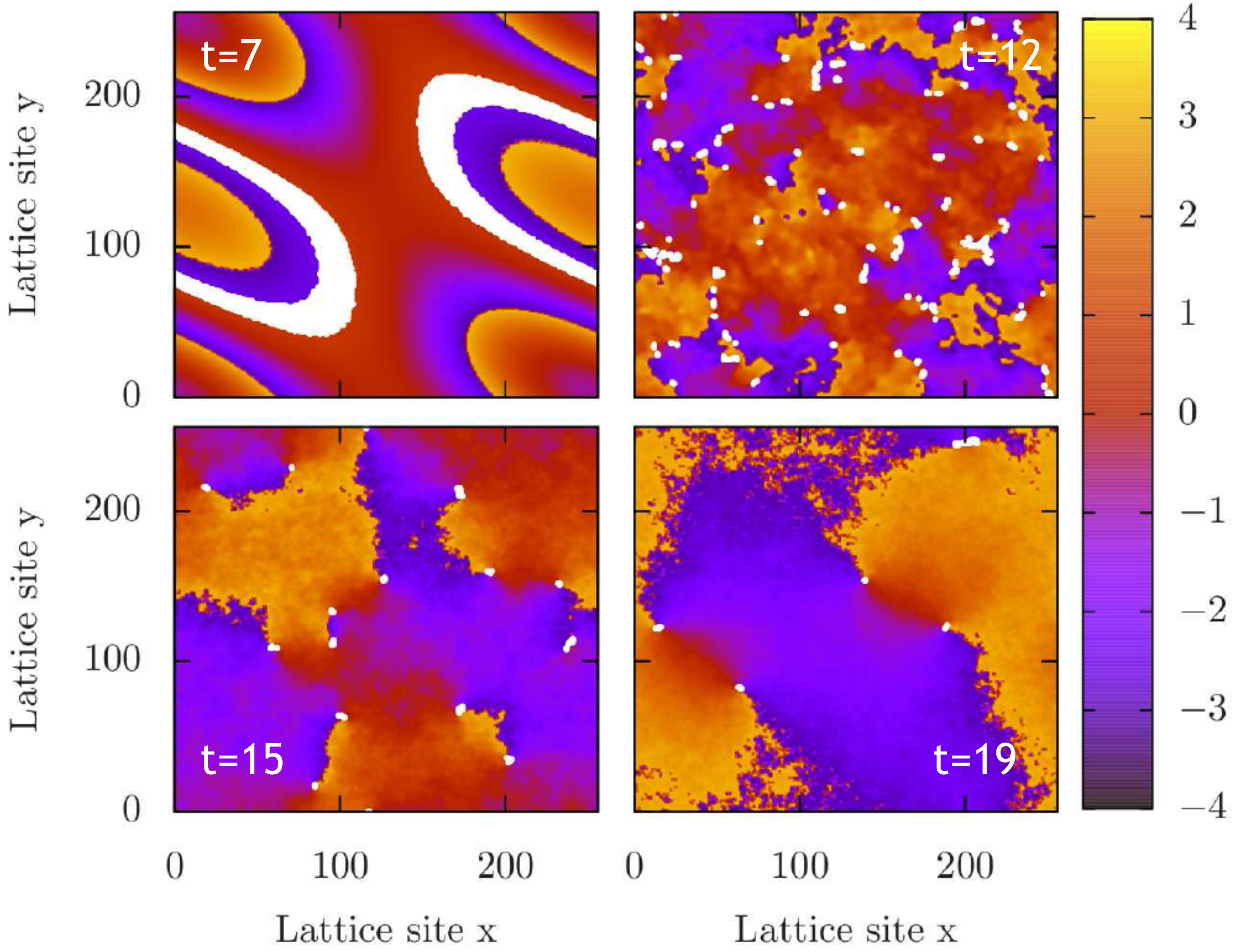
Let's do simulations of the GPE in the classical regime:

$$i\partial_t \psi(\mathbf{x},t) = \left[-\frac{\partial_{\mathbf{x}}^2}{2m} + g|\psi(\mathbf{x},t)|^2\right]\psi(\mathbf{x},t)$$



B. Nowak, D. Sexty, TG (unpublished)

Simulations in 2+1 D (semi-classical)



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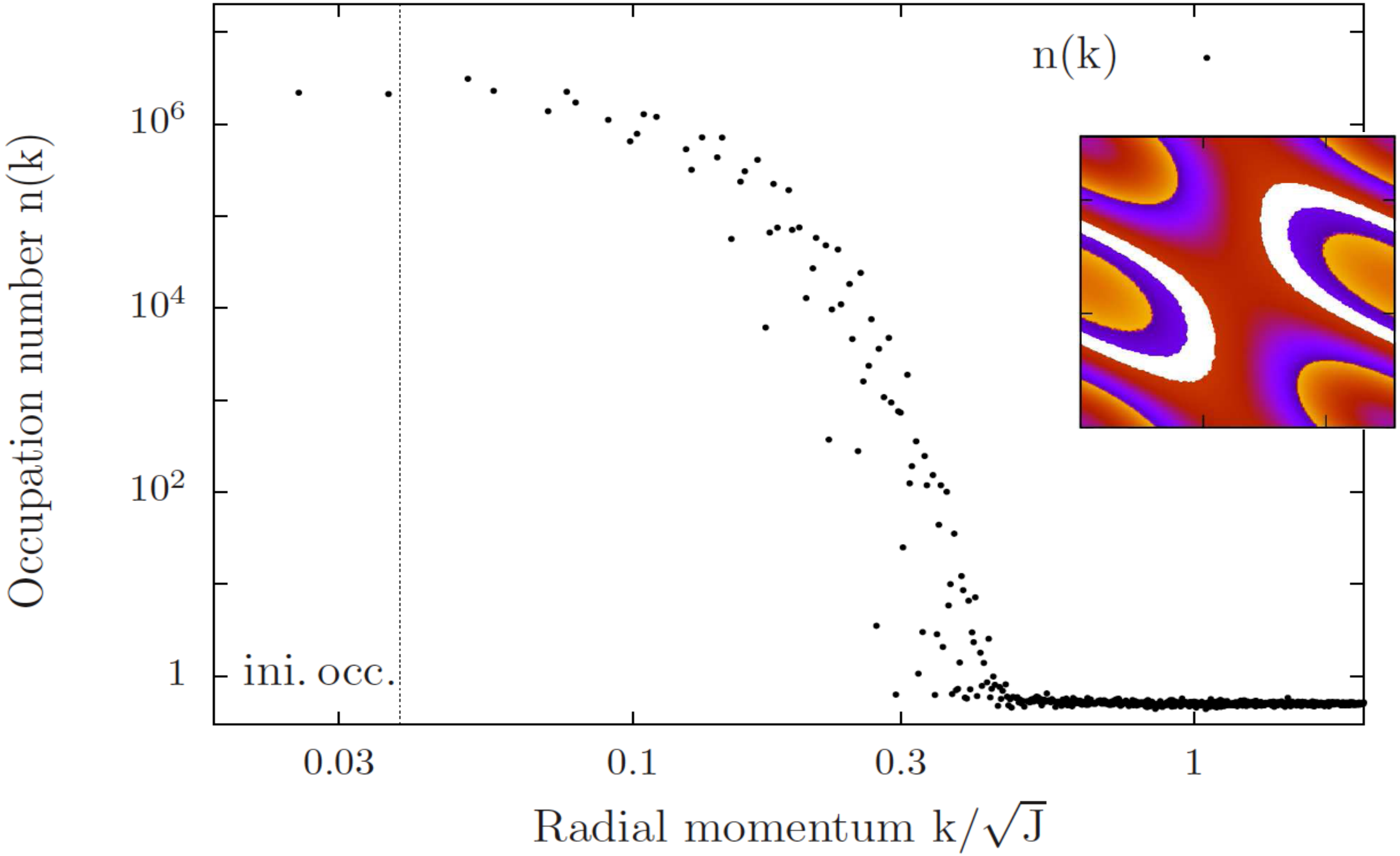

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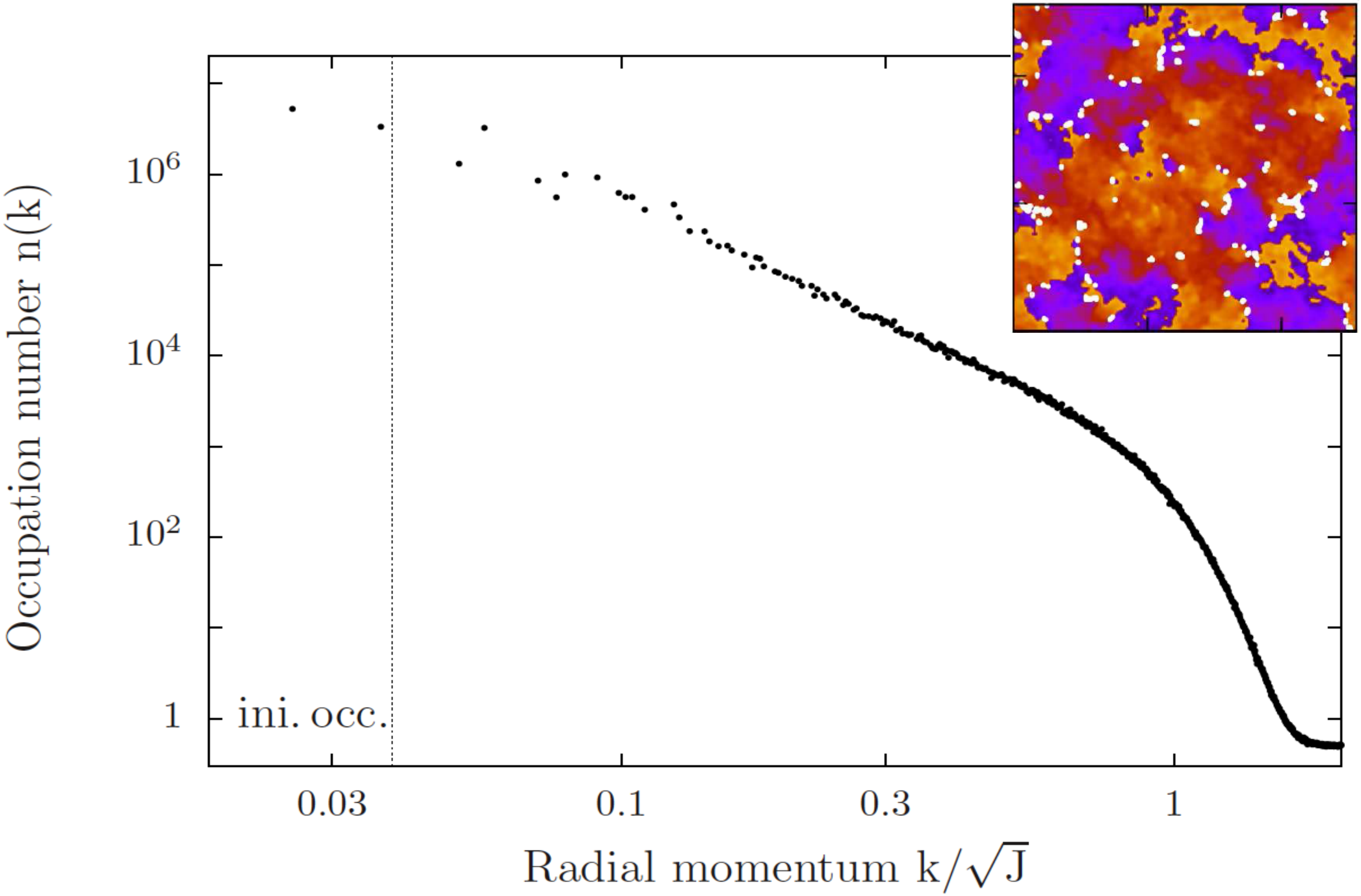

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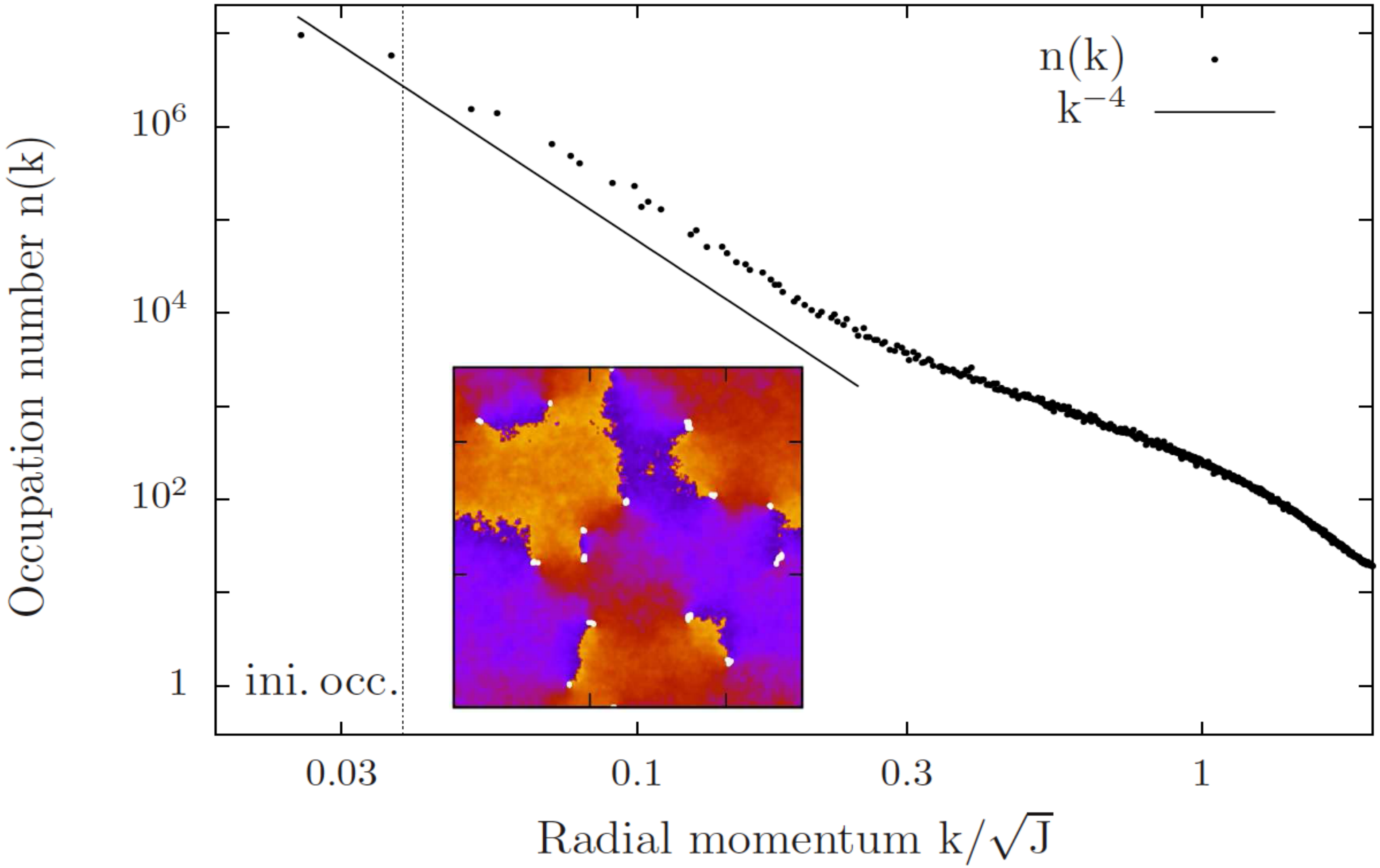

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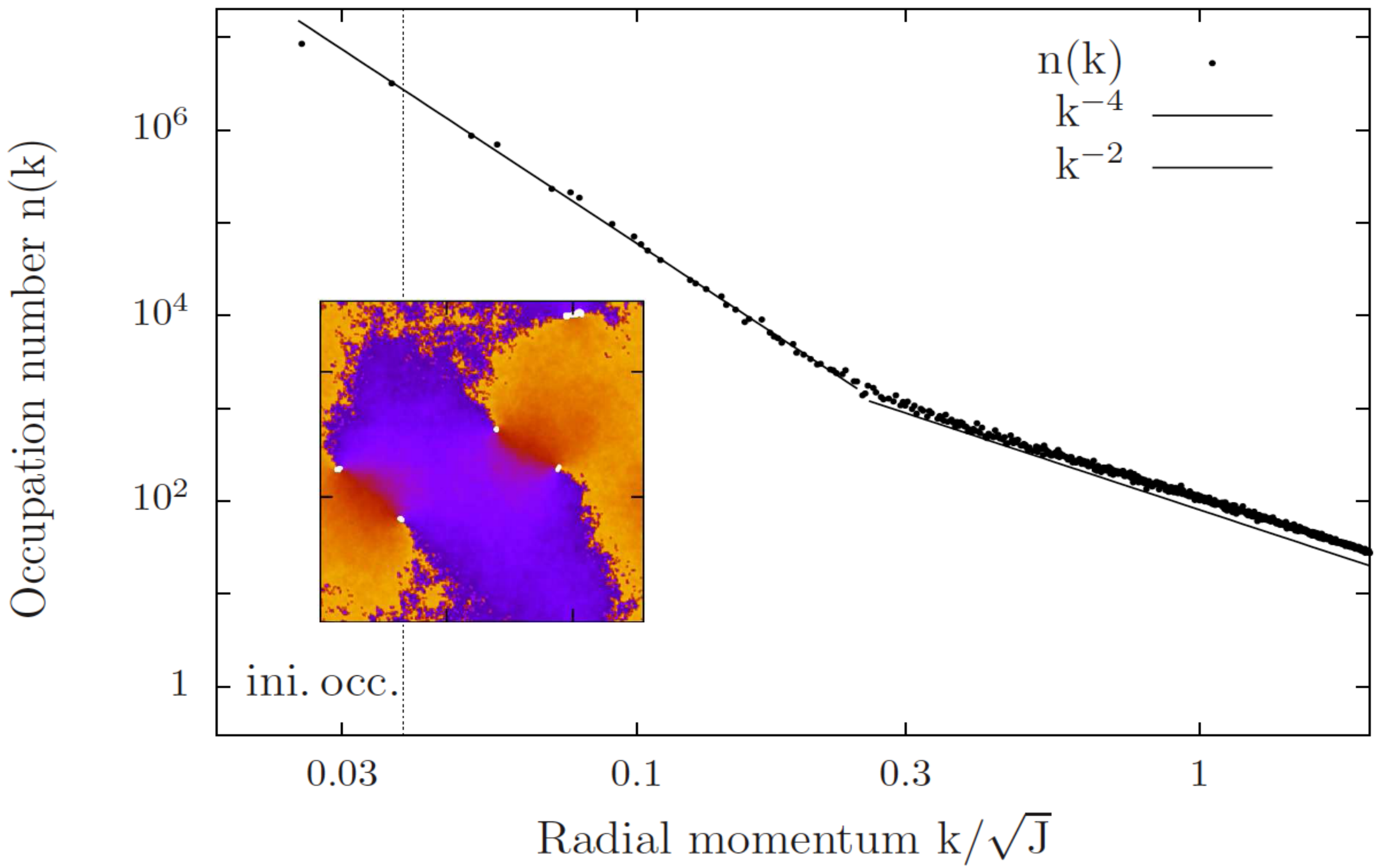
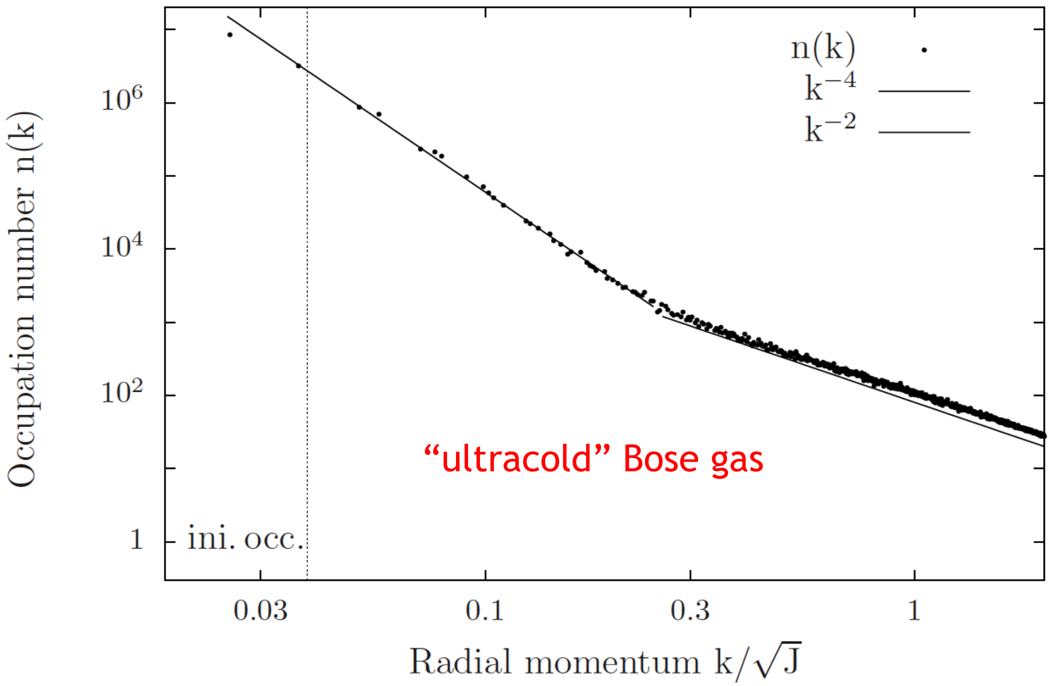
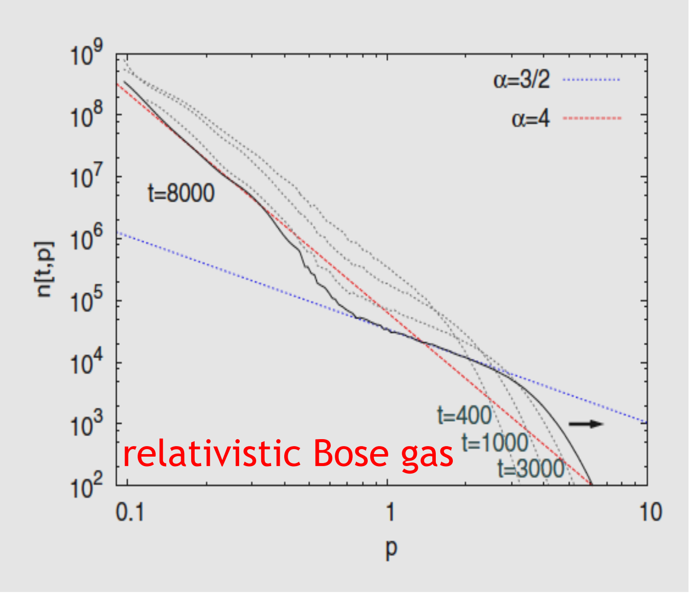


B. Nowak, D. Sexty, TG (unpublished)

Nonrel. vs. rel. Simulation



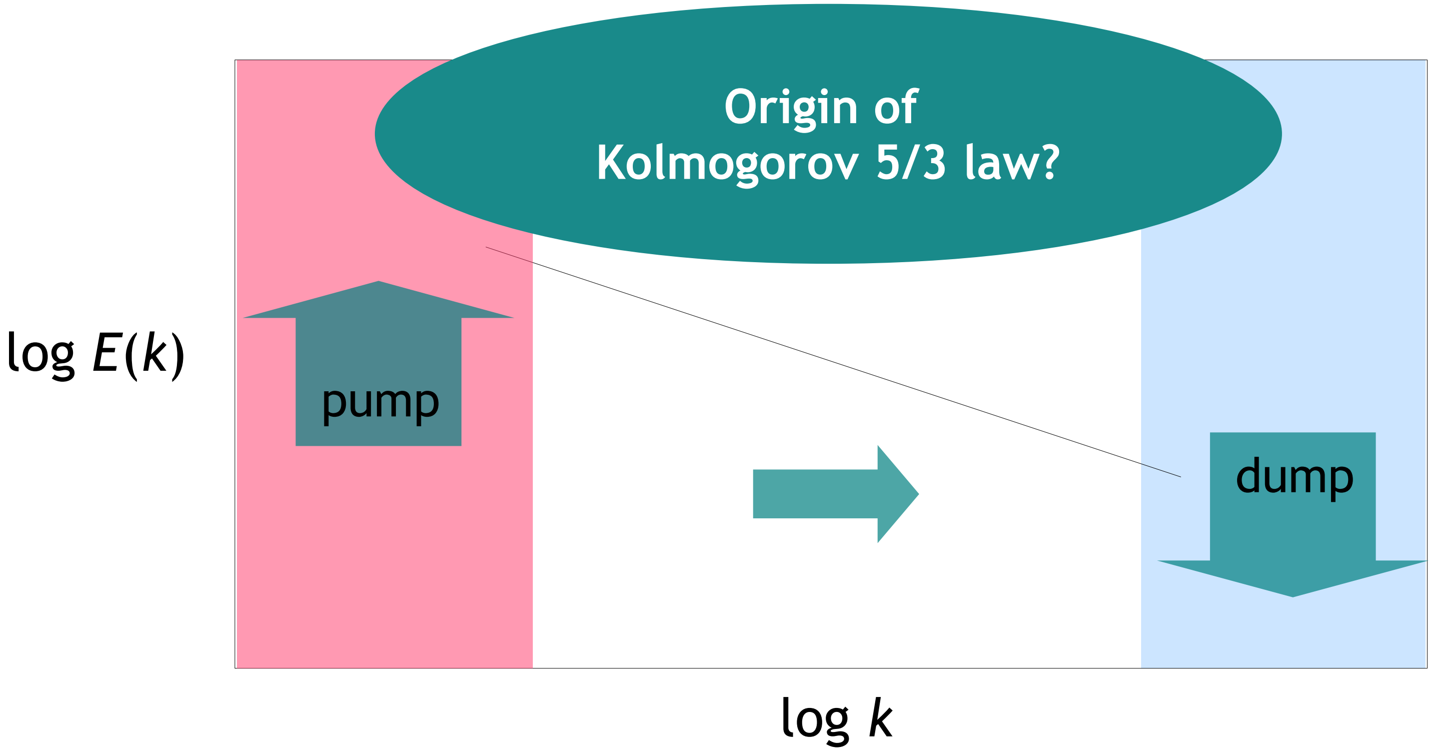
B. Nowak, D. Sexty, TG (unpublished)



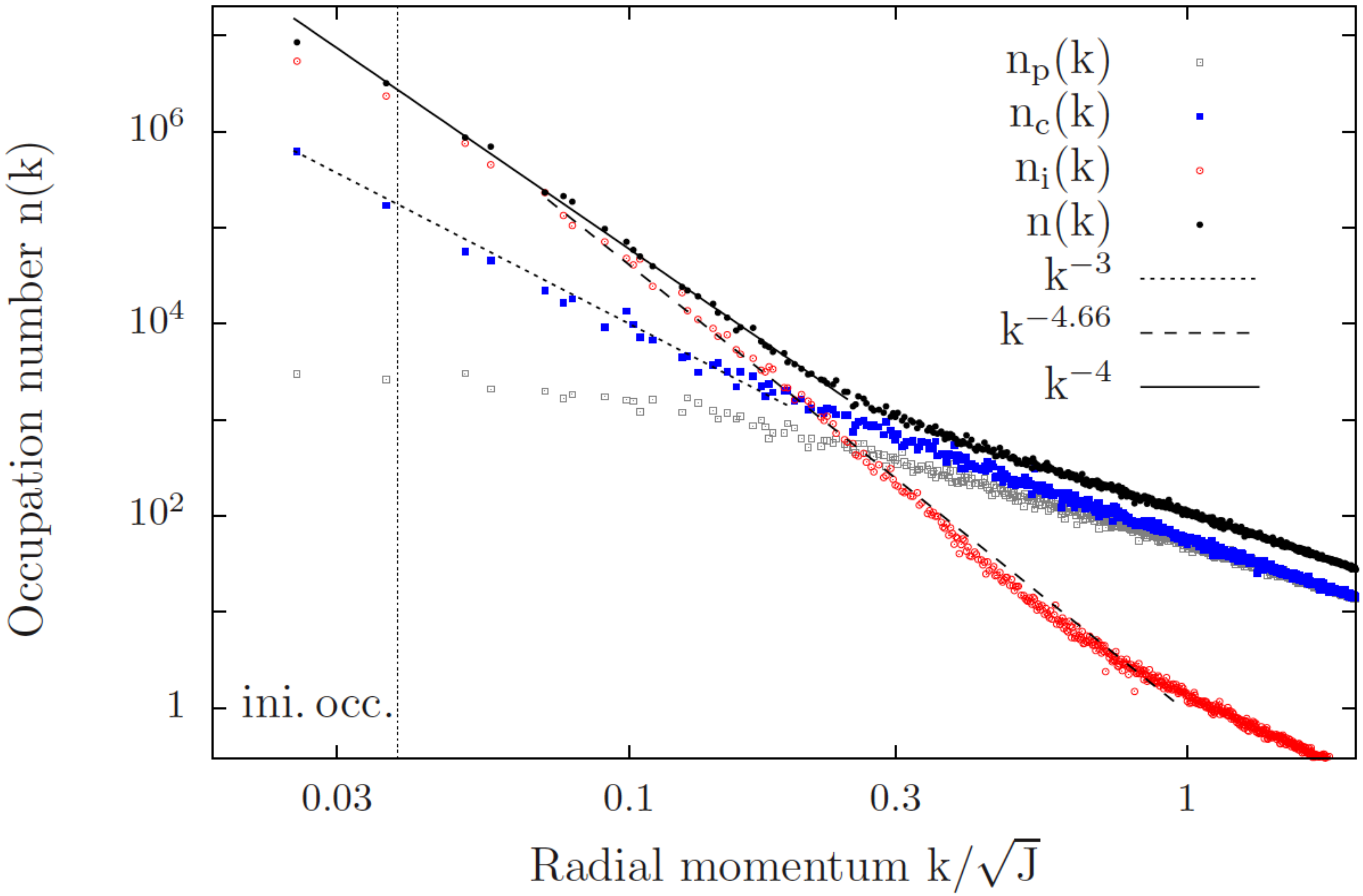
J. Berges, A. Rothkopf, and J. Schmidt, PRL **101** (08) 041603

Recall

Scaling stationary solutions:

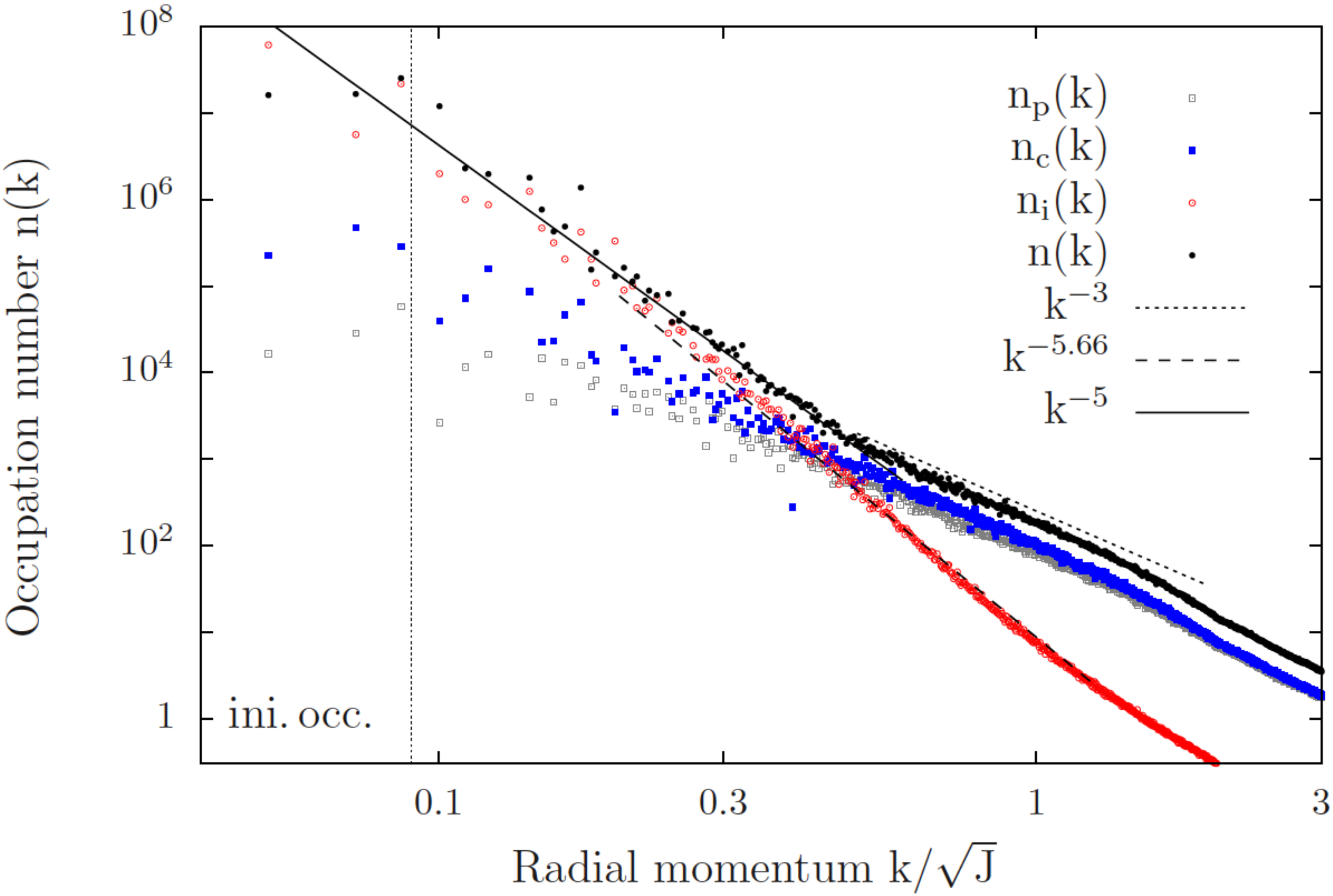


Simulations in 2+1 D (semi-classical)



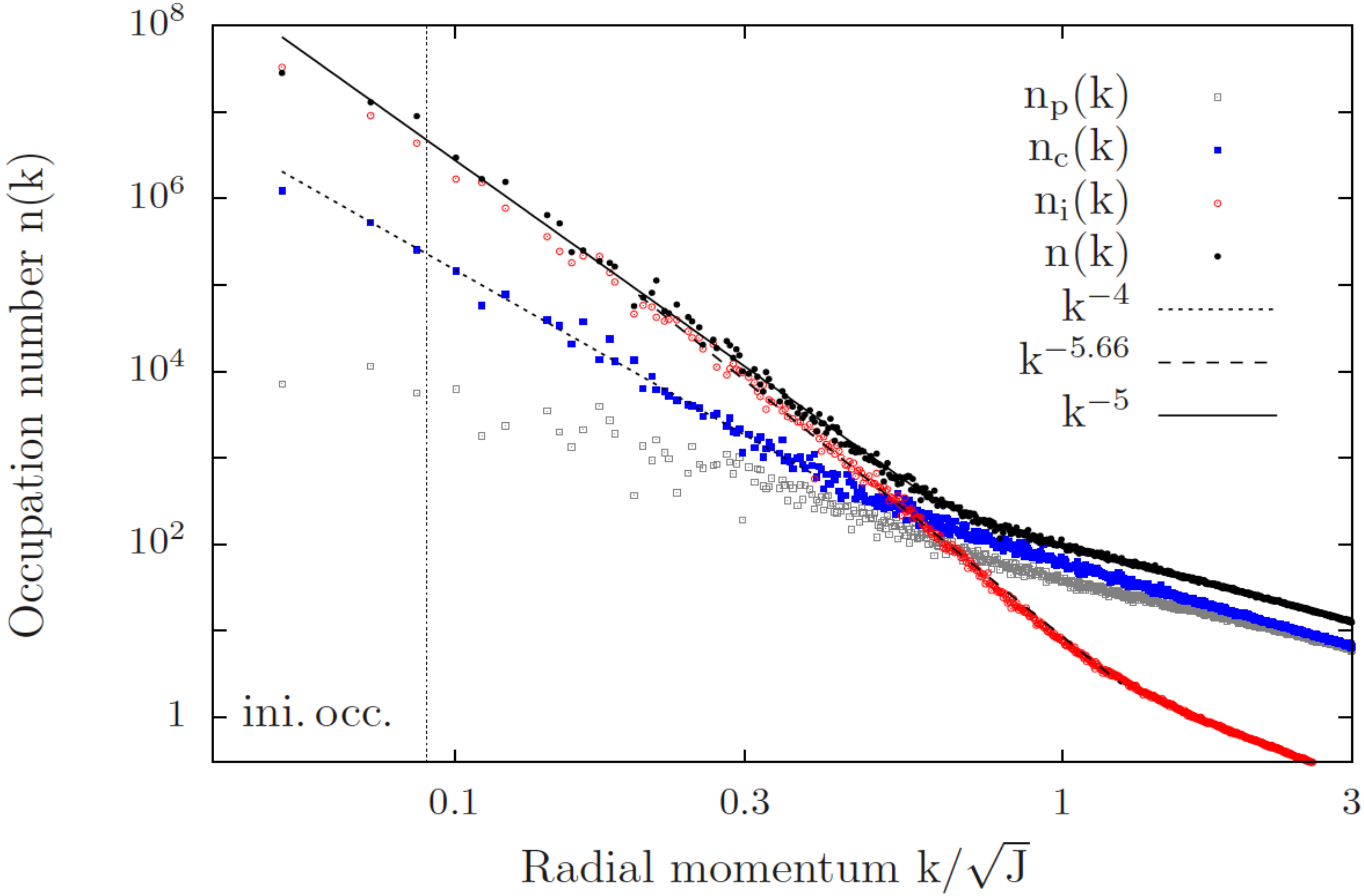
B. Nowak, D. Sexty, TG (unpublished)

Simulations in 3+1 D (semi-classical)



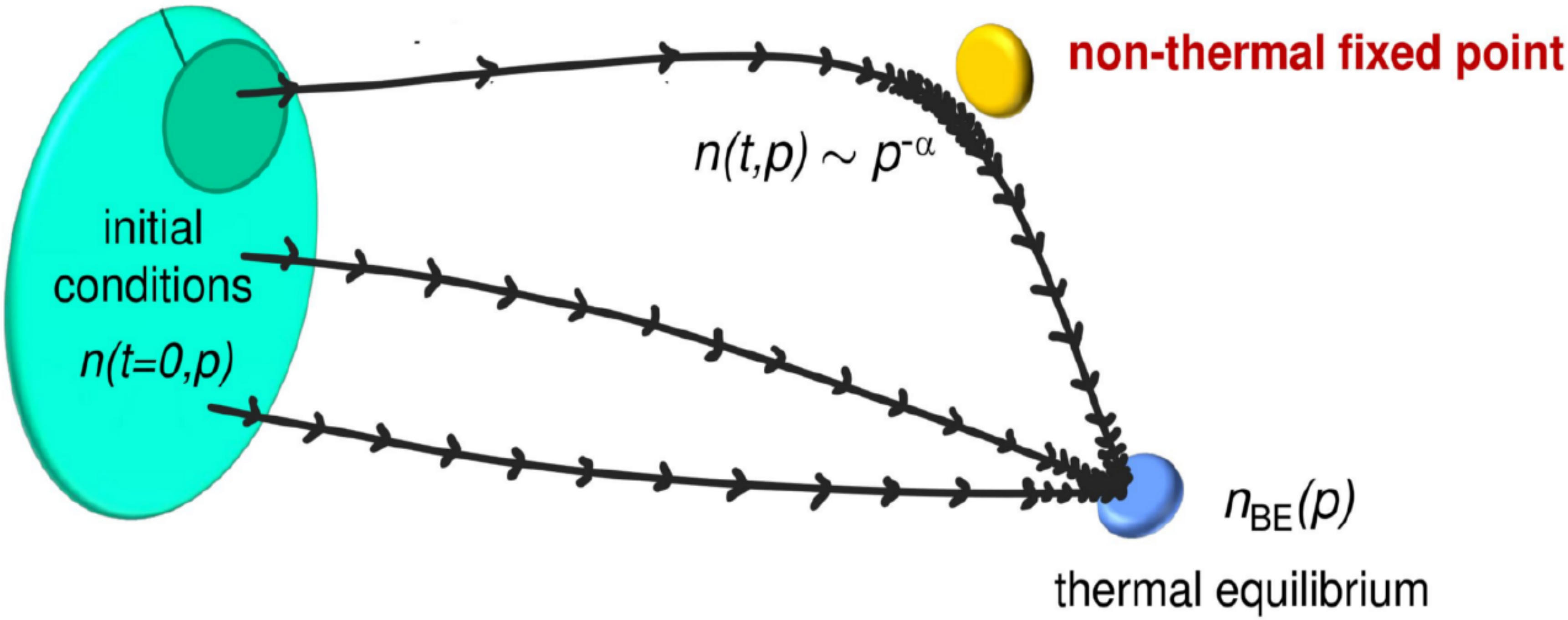
B. Nowak, D. Sexty, TG (unpublished)

Simulations in 3+1 D (semi-classical)



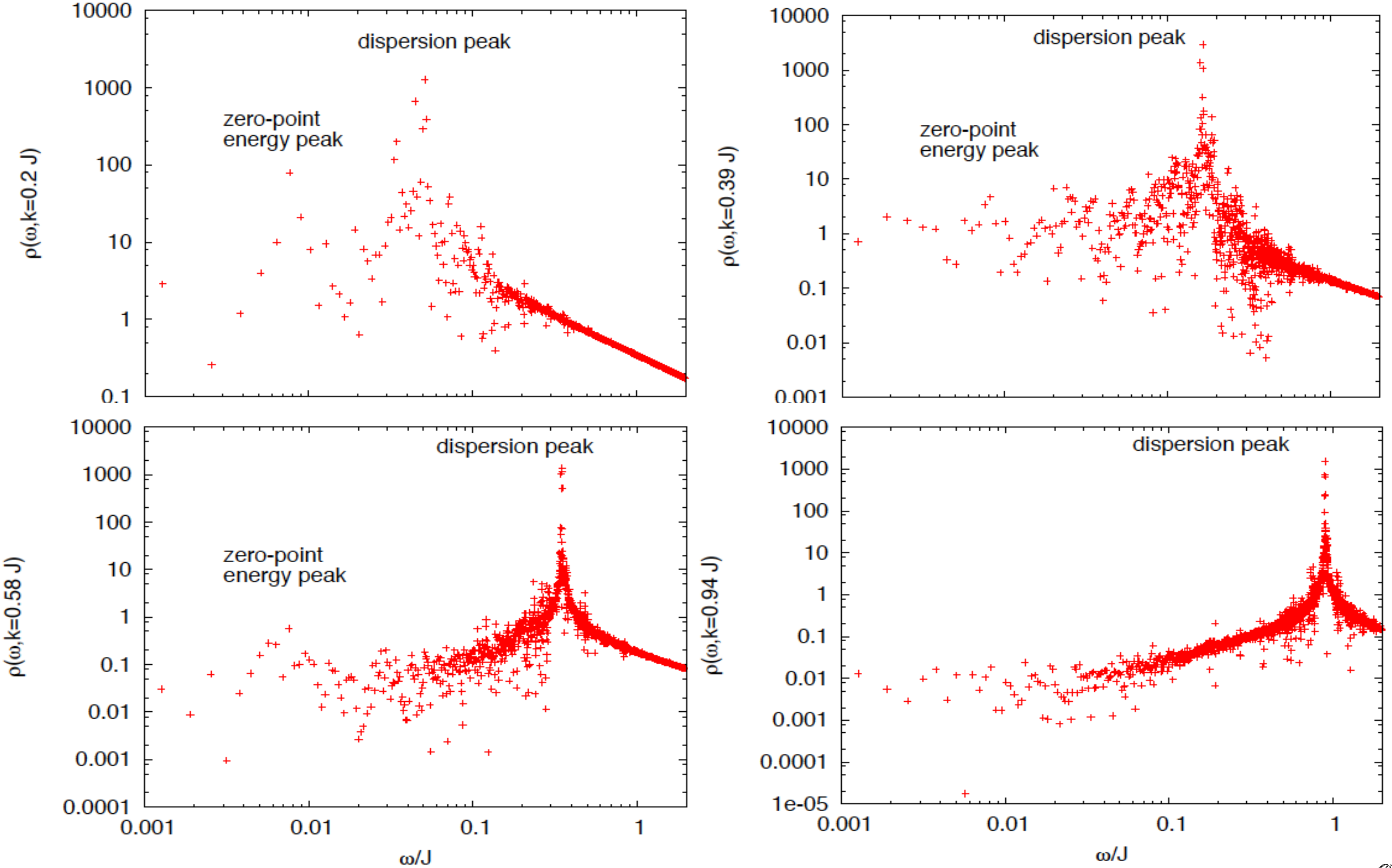
B. Nowak, D. Sexty, TG (unpublished)

Turbulence as a fixed point

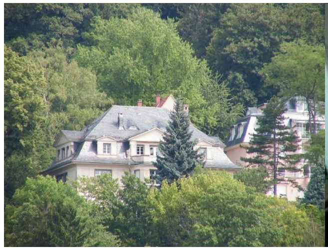


[Fig.: Berges 08]

Spectral functions



Thanks & credits to...



...my work group in Heidelberg:

Cédric Bodet
Martin Gärttner
Matthias Kronenwett
Boris Nowak
Denes Sexty
Martin Trappe
Jan Zill

Alexander Branschädel	(→ Karlsruhe)
Stefan Keßler	(→ LMU München)
Christian Scheppach	(→ Cambridge, UK)
Philipp Struck	(→ Konstanz)
Kristan Temme	(→ Vienna)

€€€...

Heisenberg-Programm
Deutsche Forschungsgemeinschaft

RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

LGFG BaWue

DAAD Deutscher Akademischer Austausch Dienst German Academic Exchange Service

Center for Quantum Dynamics

EMMI

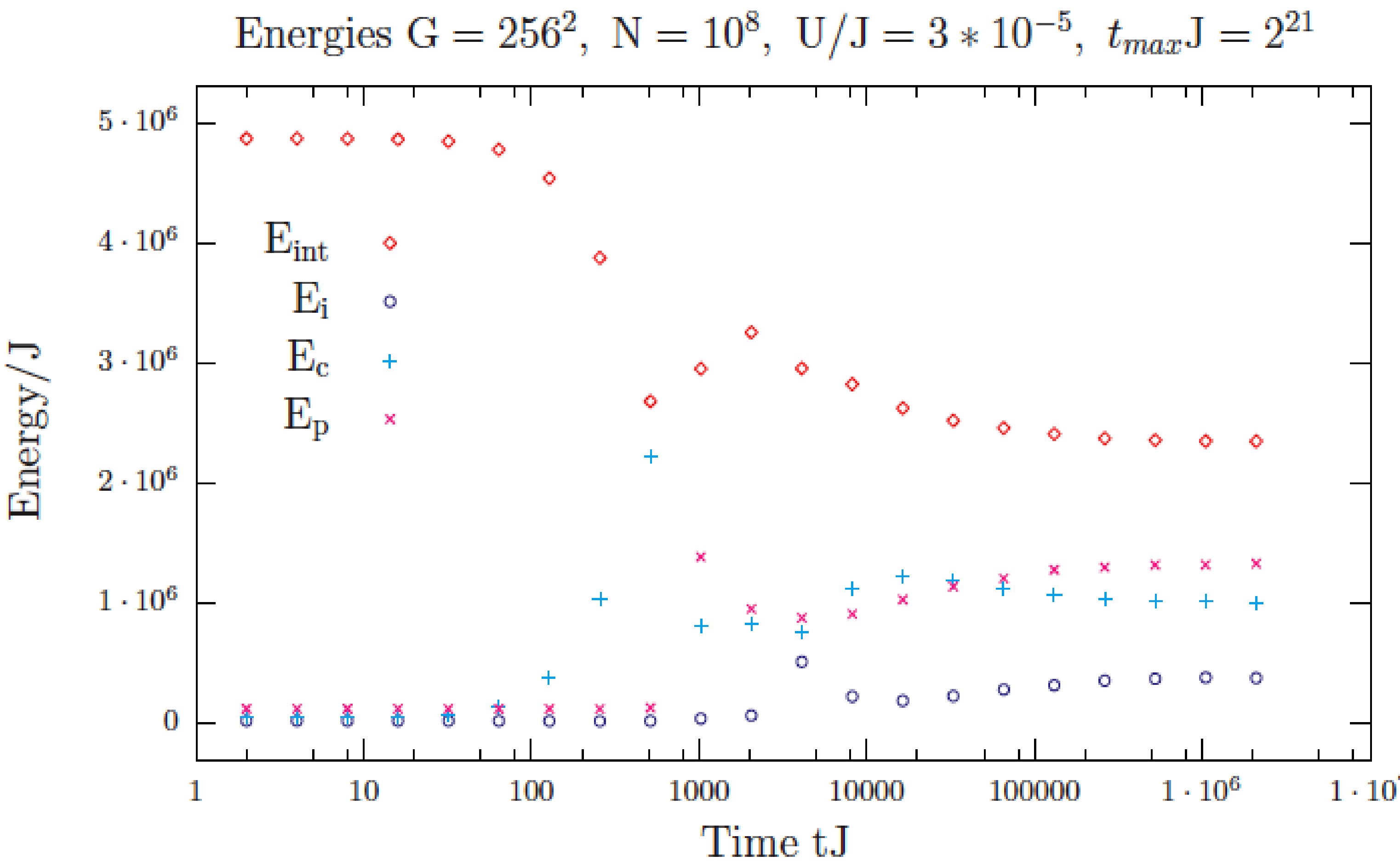
HELMHOLTZ | GEMEINSCHAFT

...my collaborators

Jürgen Berges (Darmstadt) • Hrvoje Buljan (Zagreb) • Keith Burnett (Oxford/Sheffield) • David Hutchinson (Otago) • Paul S. Julienne (NIST Gaithersburg) • Thorsten Köhler (Oxford/UCL) • Otto Nachtmann (Heidelberg) • Markus K. Oberthaler (Heidelberg) • Jan M. Pawlowski (Heidelberg) • Robert Pezer (Sisak) • David Roberts (Oxford/Los Alamos) • Janne Ruostekoski (Southampton) • Michael G. Schmidt (Heidelberg) • Marcos Seco (Santiago de Compostela)

Supplementary slides

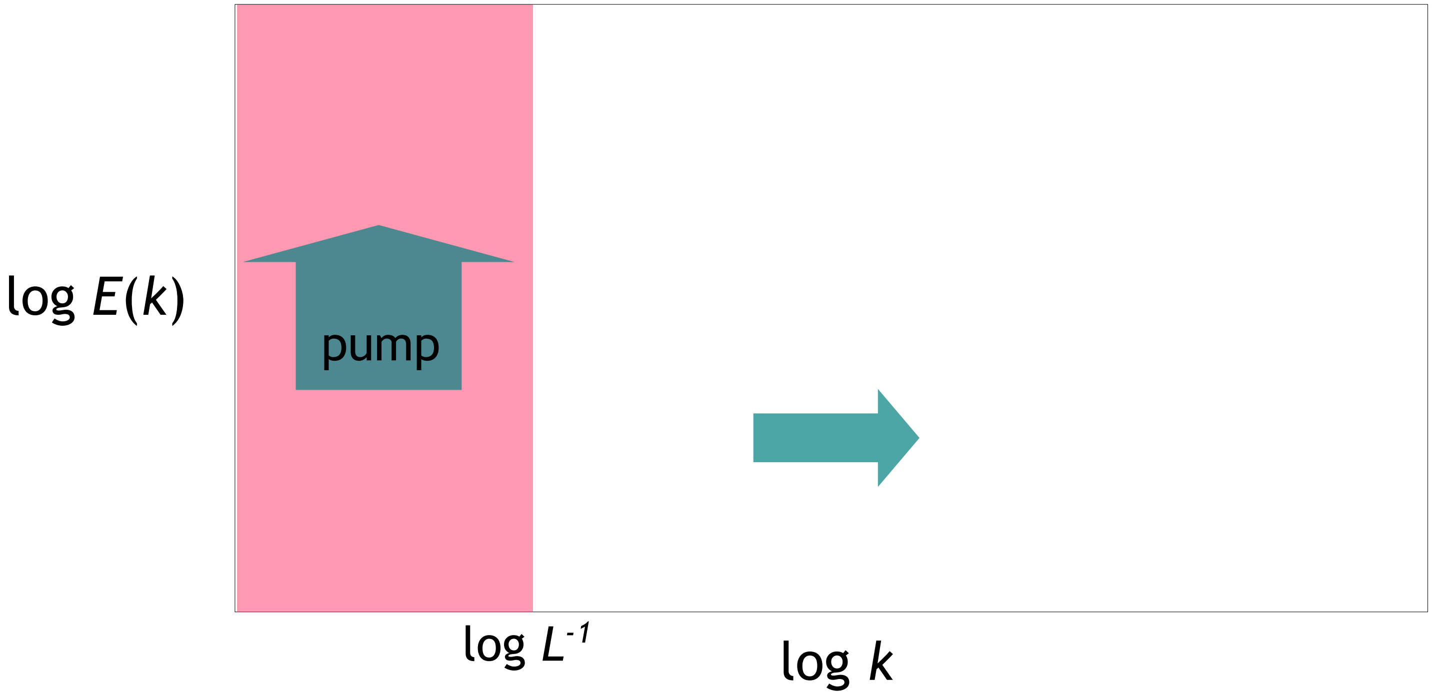
Energies



Energies G = 256², N = 10⁸, U/J = 3 * 10⁻⁵, t_max J = 2²¹
Energy/J
5 · 10⁶
4 · 10⁶
3 · 10⁶
2 · 10⁶
1 · 10⁶
0
1
10
100
1000
10000
100000
1 · 10⁶
1 · 10⁷
Time tJ
E_int
E_i
E_c
E_p

Kolmogorov's theory of turbulence (1941)

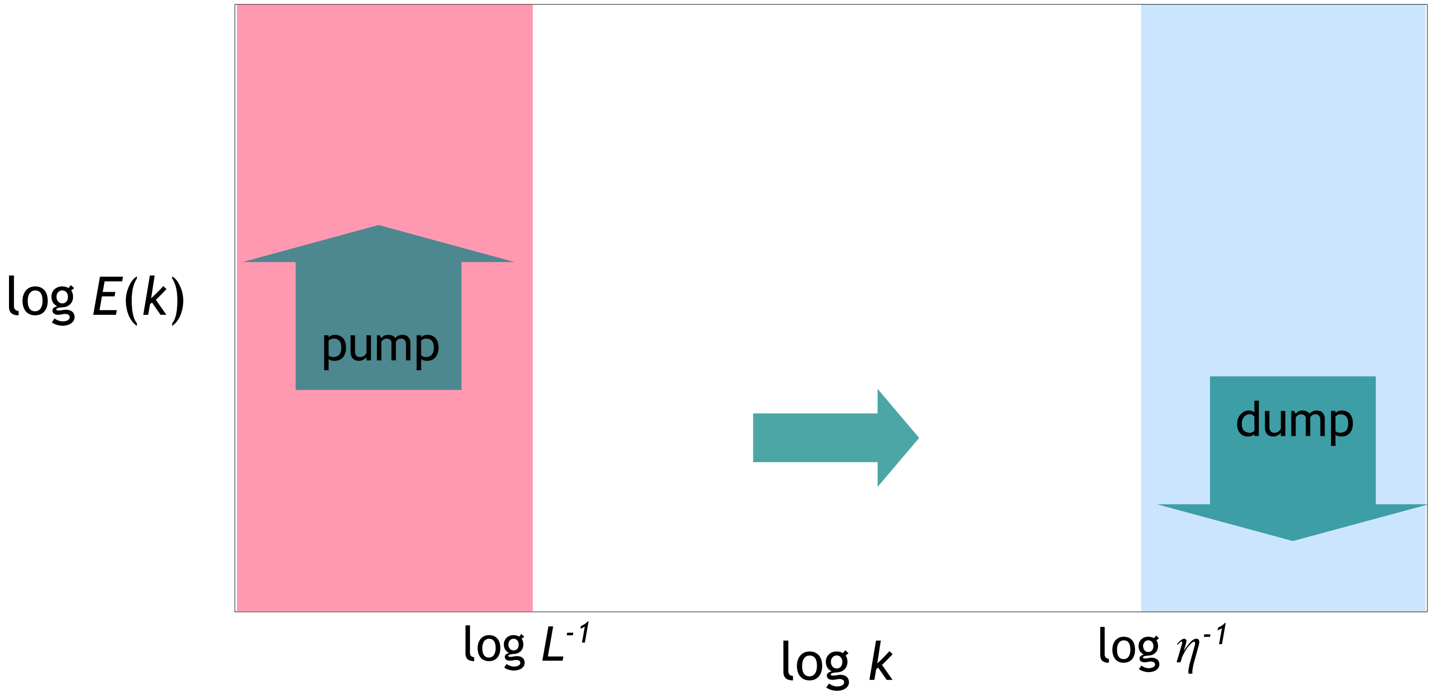
Cascade = Transport in momentum space:



3D: Radial energy density $E = k^2\varepsilon$ [kg s^{-2}]

Kolmogorov's theory of turbulence
(1941)

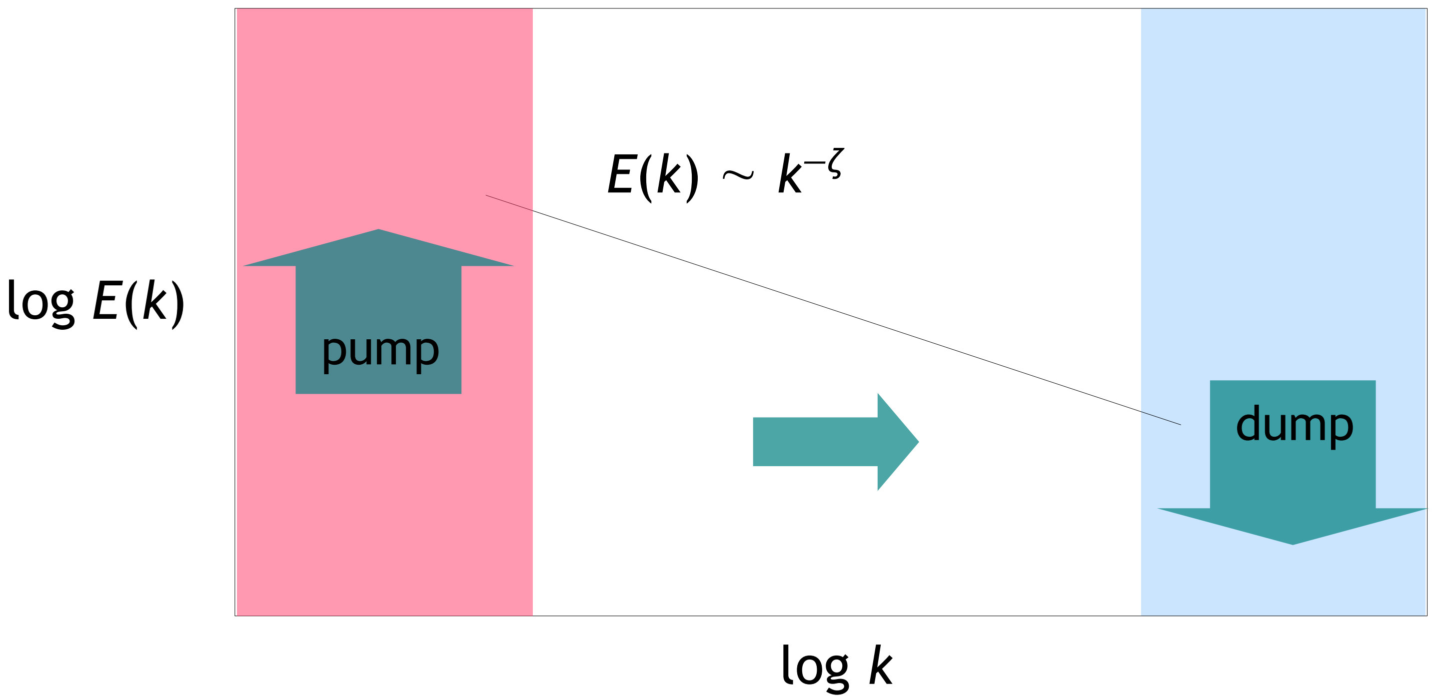
Stationary *E*-distribution in cascade?



3D: Radial energy density $E = k^2\varepsilon$ [kg s^{-2}]

Kolmogorov's theory of turbulence
(1941)

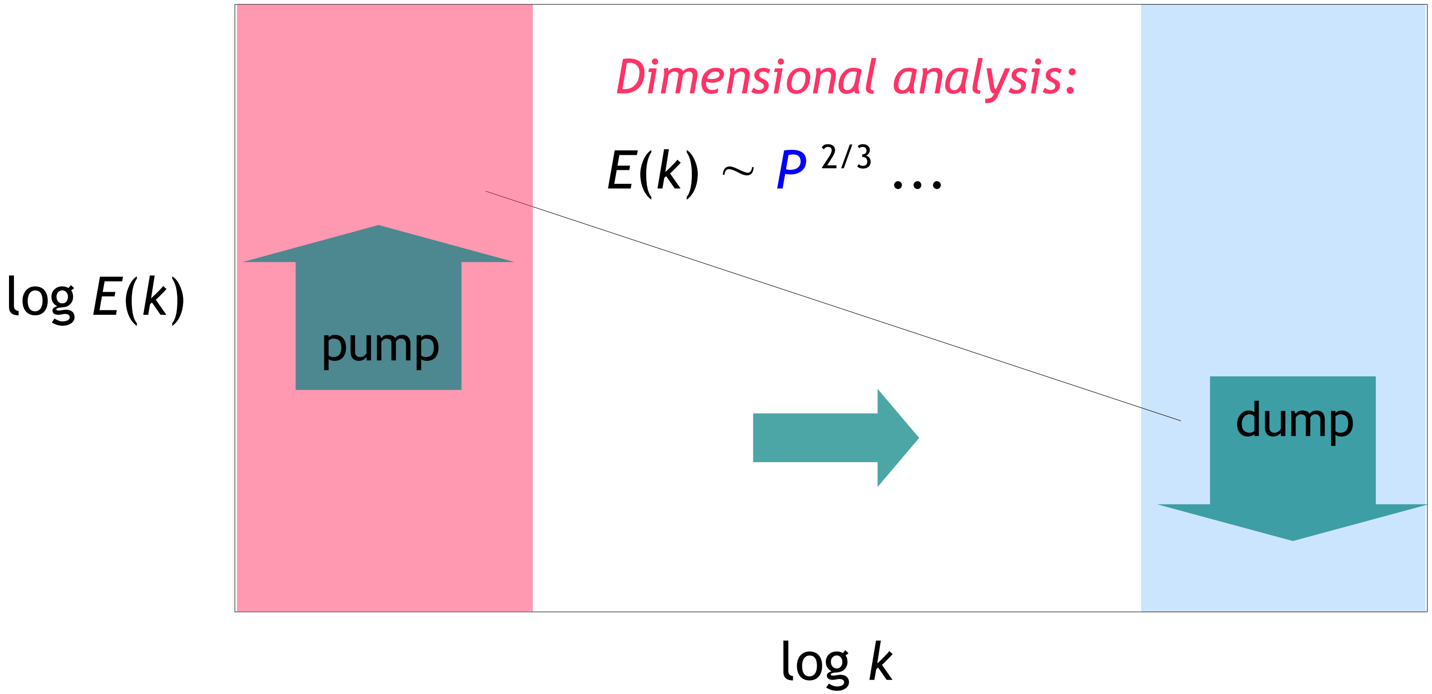
Assume self-similarity:



3D:	Radial energy density	$E = k^2\varepsilon$	[kg s^{-2}]
	Radial energy flux	$P = k^2 \lvert \mathbf{p} \rvert$	[kg m^{-1} s^{-3}]

Kolmogorov's theory of turbulence (1941)

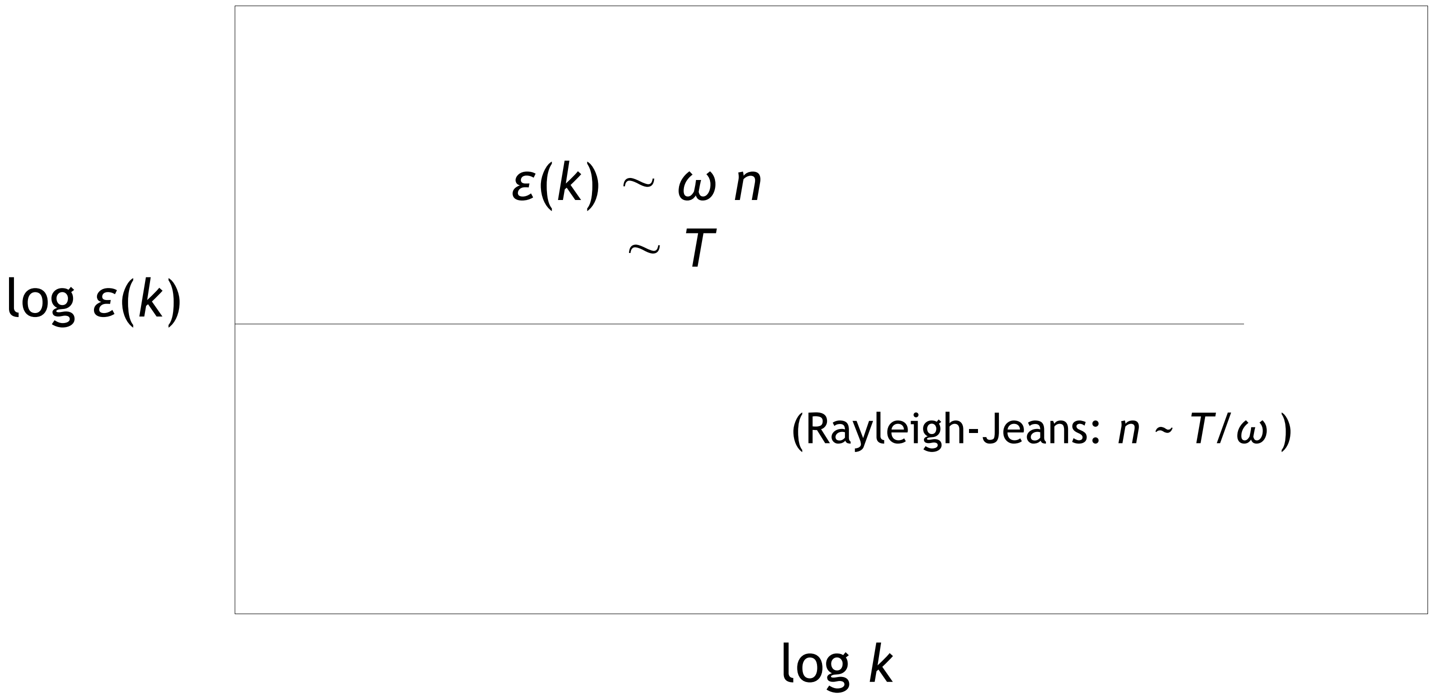
Assume self-similarity:



3D:	Radial energy density	E	[kg s^{-2}]
	Radial energy flux	P	[kg m^{-1} s^{-3}]

Compare: *Thermal* Equilibrium

Constant energy $\varepsilon \sim E/k^2$ (In 3D):



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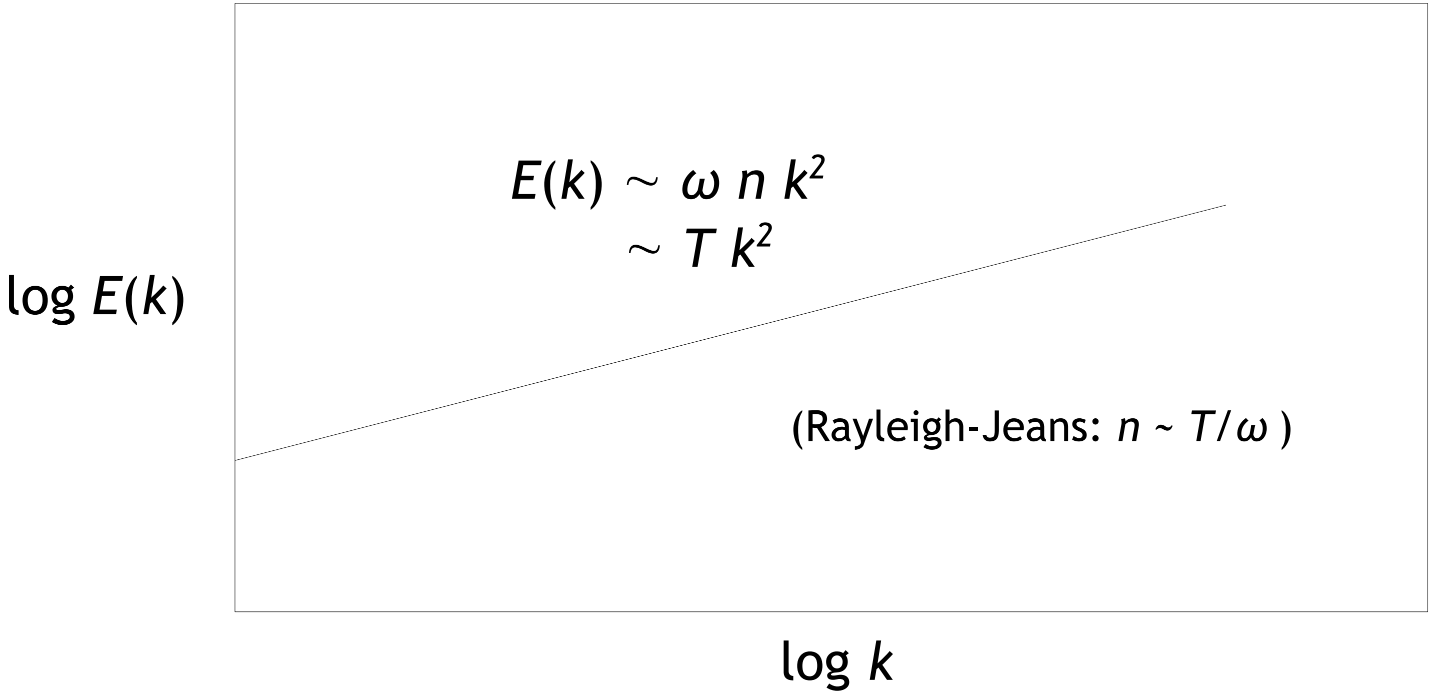
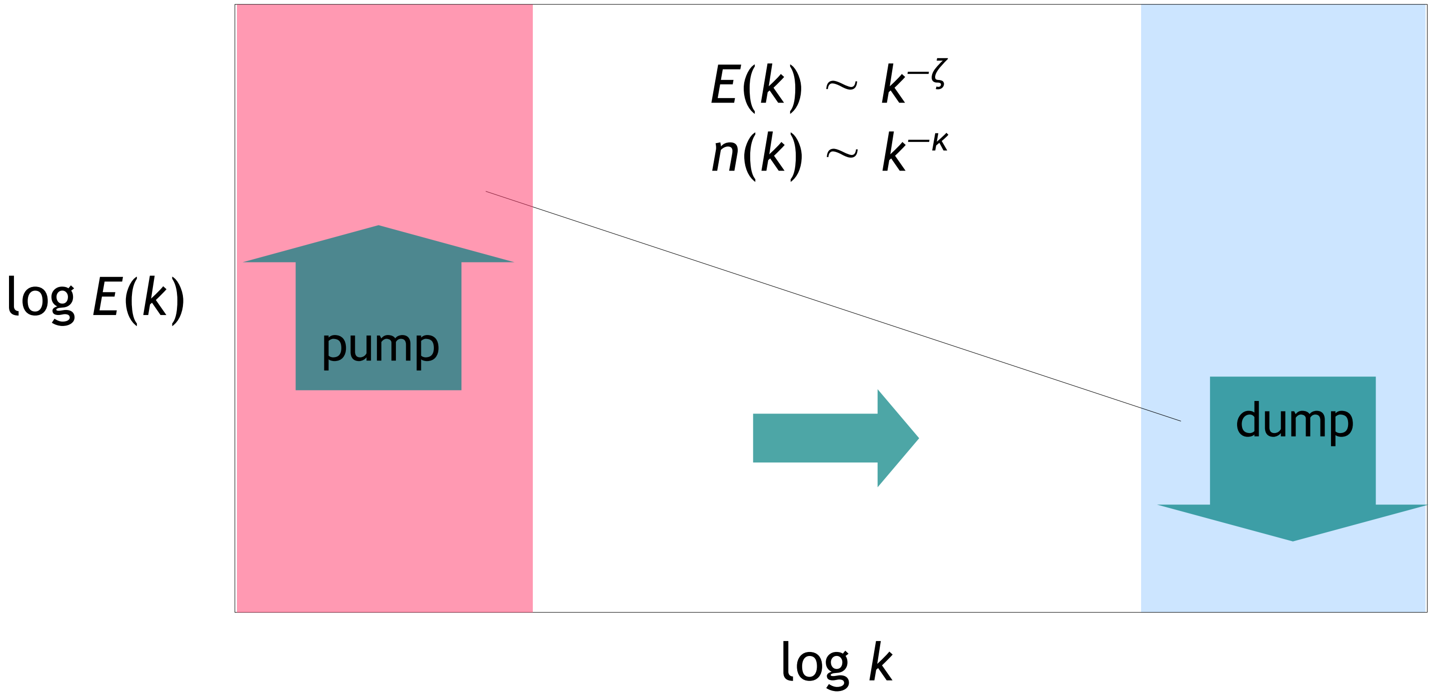


3D:	Radial energy density	E	[kg s^{-2}]

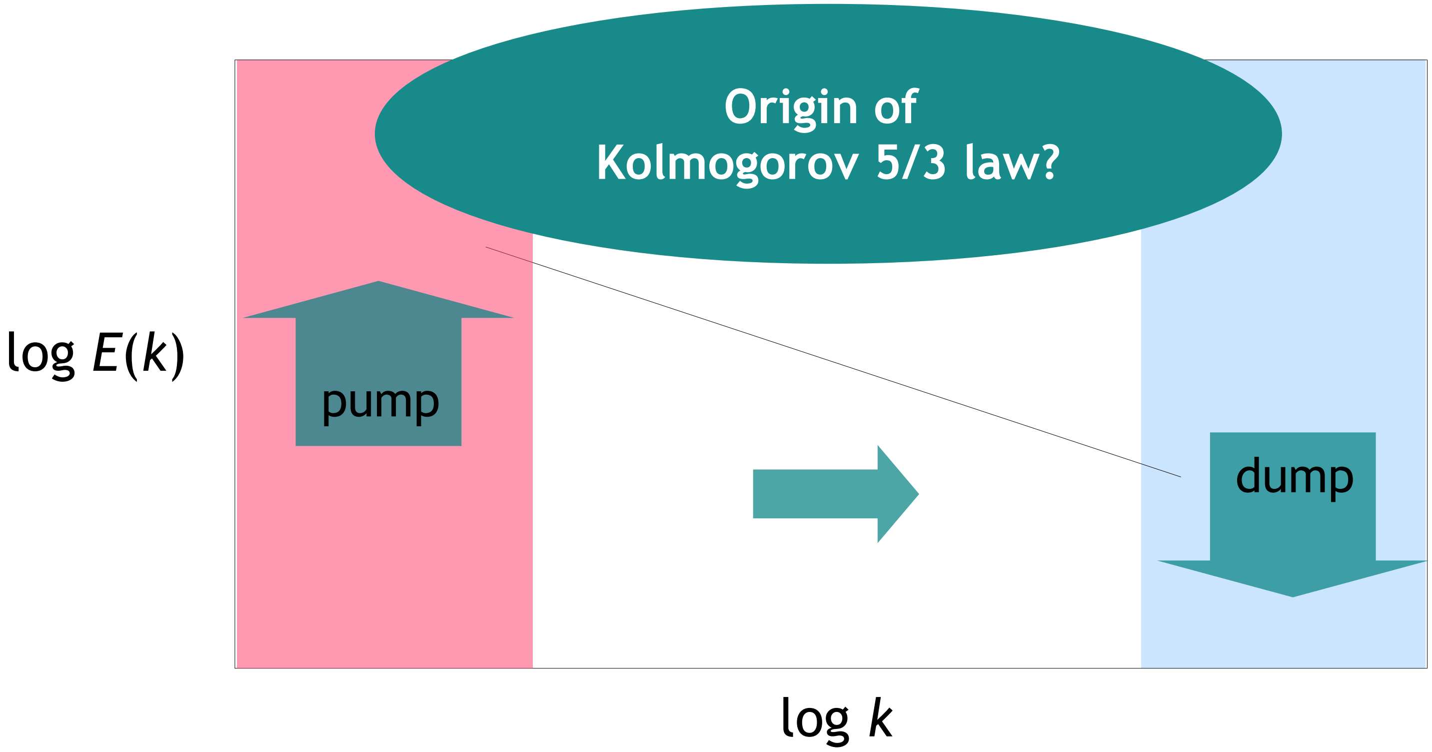
Summary

Scaling stationary solutions:



Summary

Scaling stationary solutions:



Thomas Gasenzer

Kinetic equation

Flow in momentum space from kinetic (Boltzmann) equation:

$$\partial_t n_{\mathbf{k}} = J(\mathbf{k}, t)$$

Scattering integral:

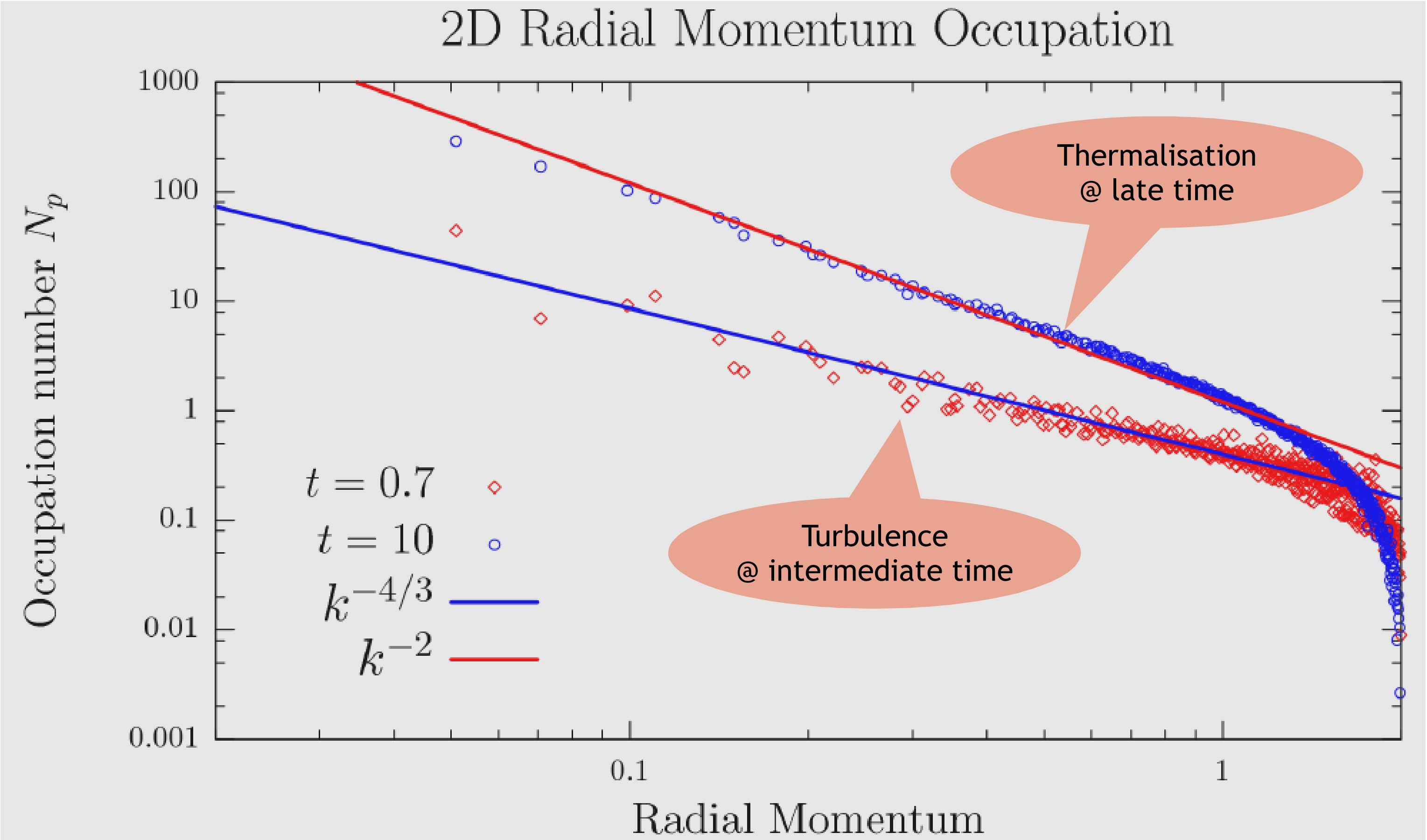
$$J(\mathbf{k},t) = g^2 \int \mathrm{d}^d p \, \mathrm{d}^d q \, \mathrm{d}^d r \, |T_{\mathbf{kpqr}}|^2 \delta(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r})$$
$$\times \; \delta(\omega_{\mathbf{k}} + \omega_{\mathbf{p}} - \omega_{\mathbf{q}} - \omega_{\mathbf{r}})$$
$$\times \; [(n_{\mathbf{k}}+1)(n_{\mathbf{p}}+1)n_{\mathbf{q}}n_{\mathbf{r}} - n_{\mathbf{k}}n_{\mathbf{p}}(n_{\mathbf{q}}+1)(n_{\mathbf{r}}+1)]$$

Zakharov: o derive scaling, familiarize the...

reader with a little miracle of the theory of wave turbulence theory, the so-called Zakharov transformations. They factorize the collision integral. As a result one

Simulations in 2+1 D (semi-classical)

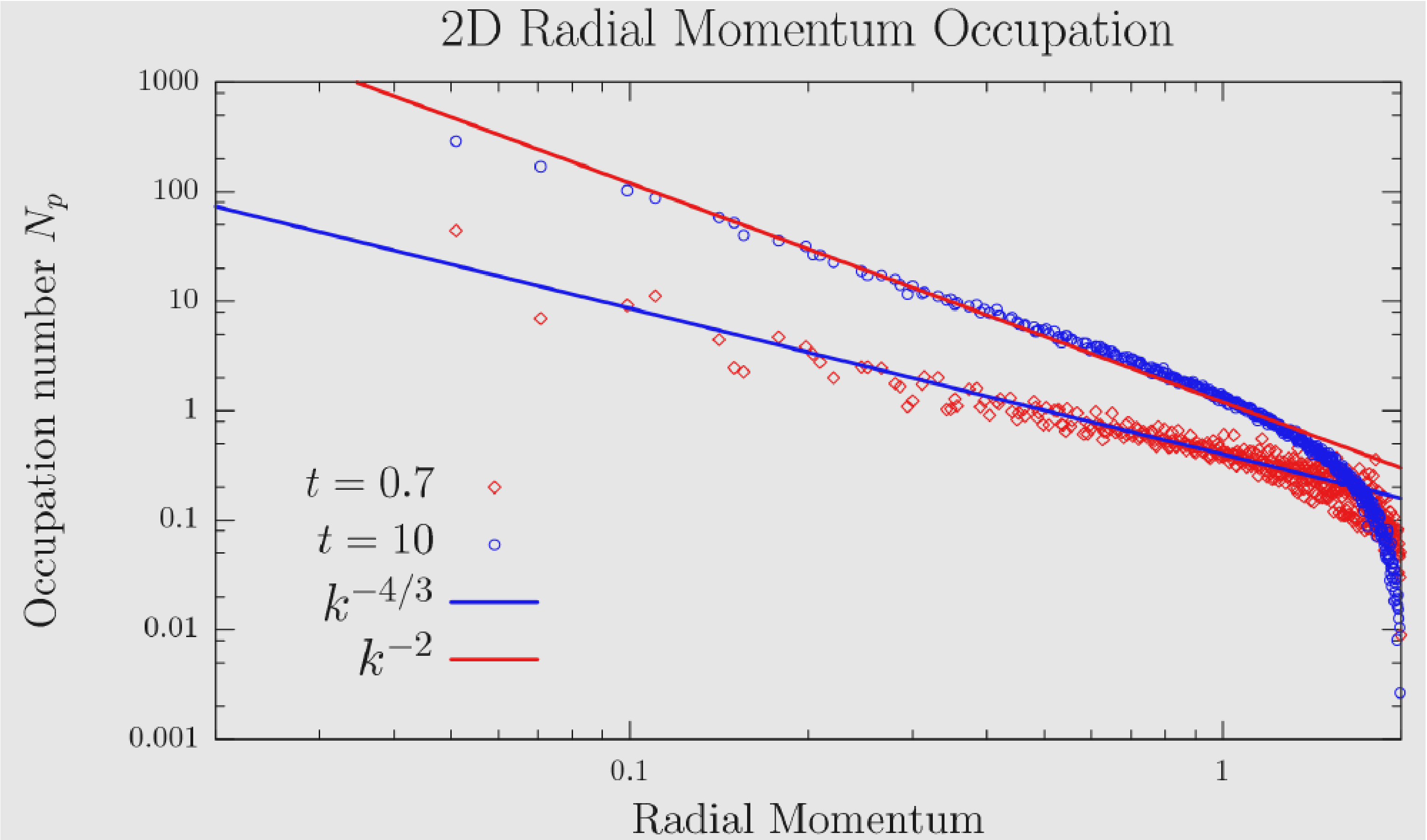
Remember:



B. Nowak, D. Sexty, TG (unpublished)

Simulations in 2+1 D (semi-classical)

Evolution of mom. distr. of a 2D Bose gas following an interaction quench

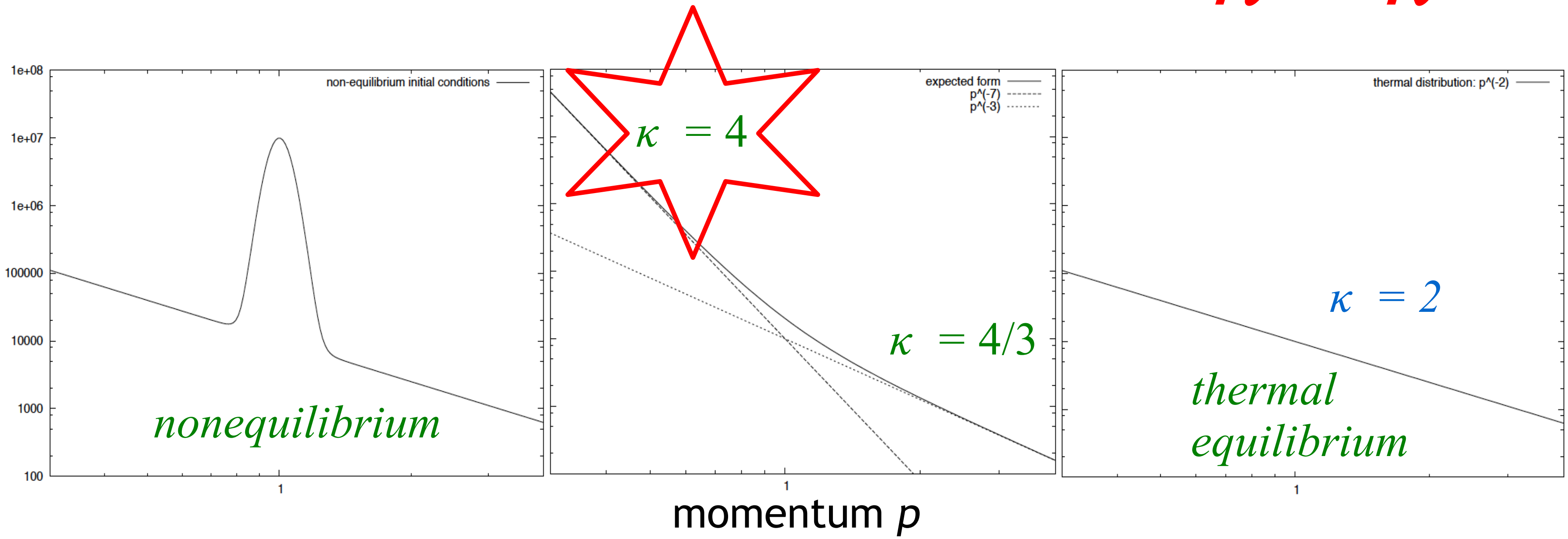


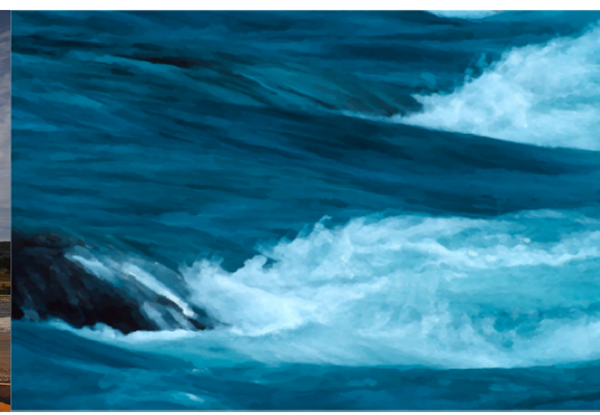
B. Nowak, D. Sexty, TG (unpublished)

Strong Turbulence:
Beyond Kinetic Theory

Strong Turbulence in 2D

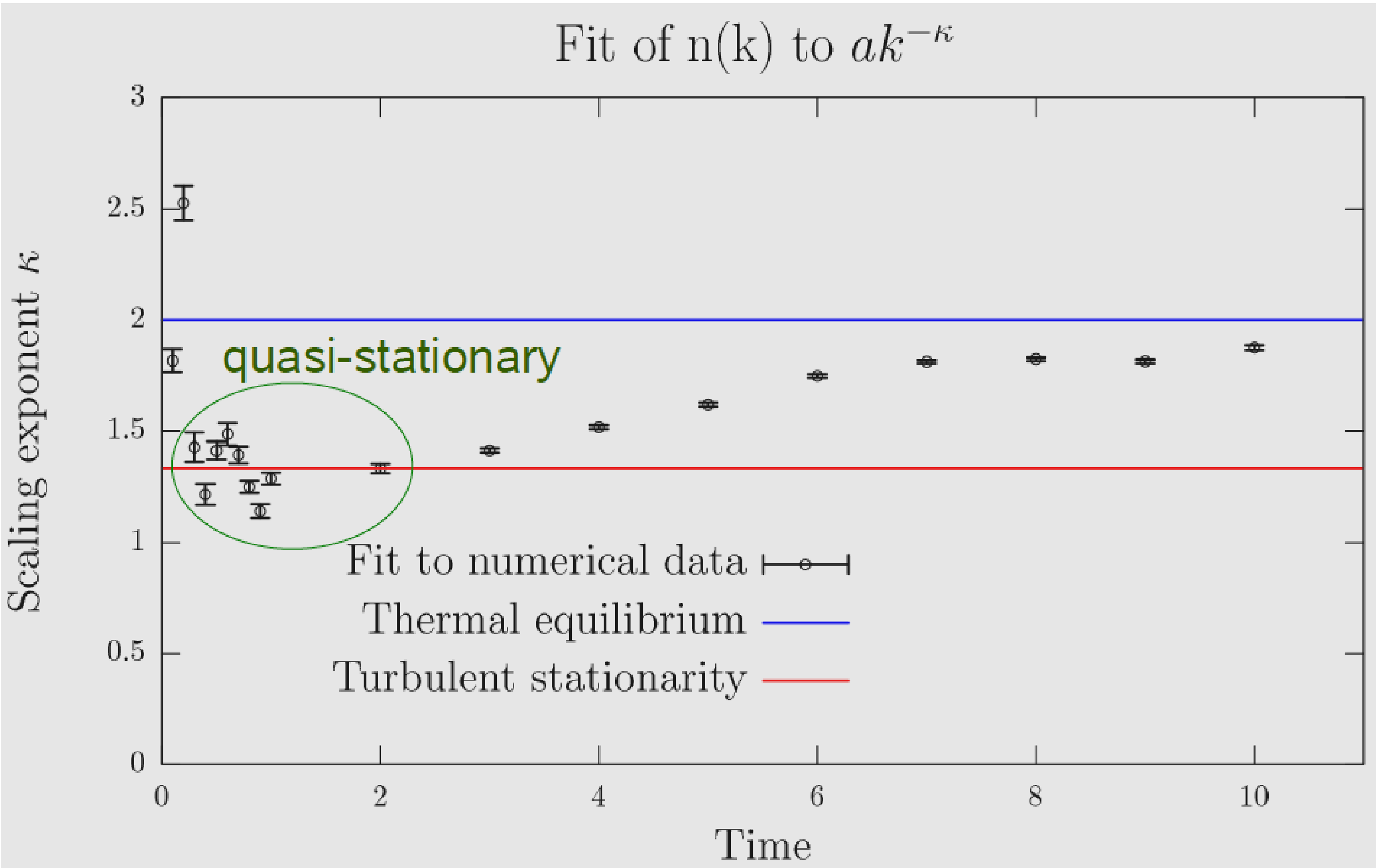
$$n \sim k^{-\kappa}$$



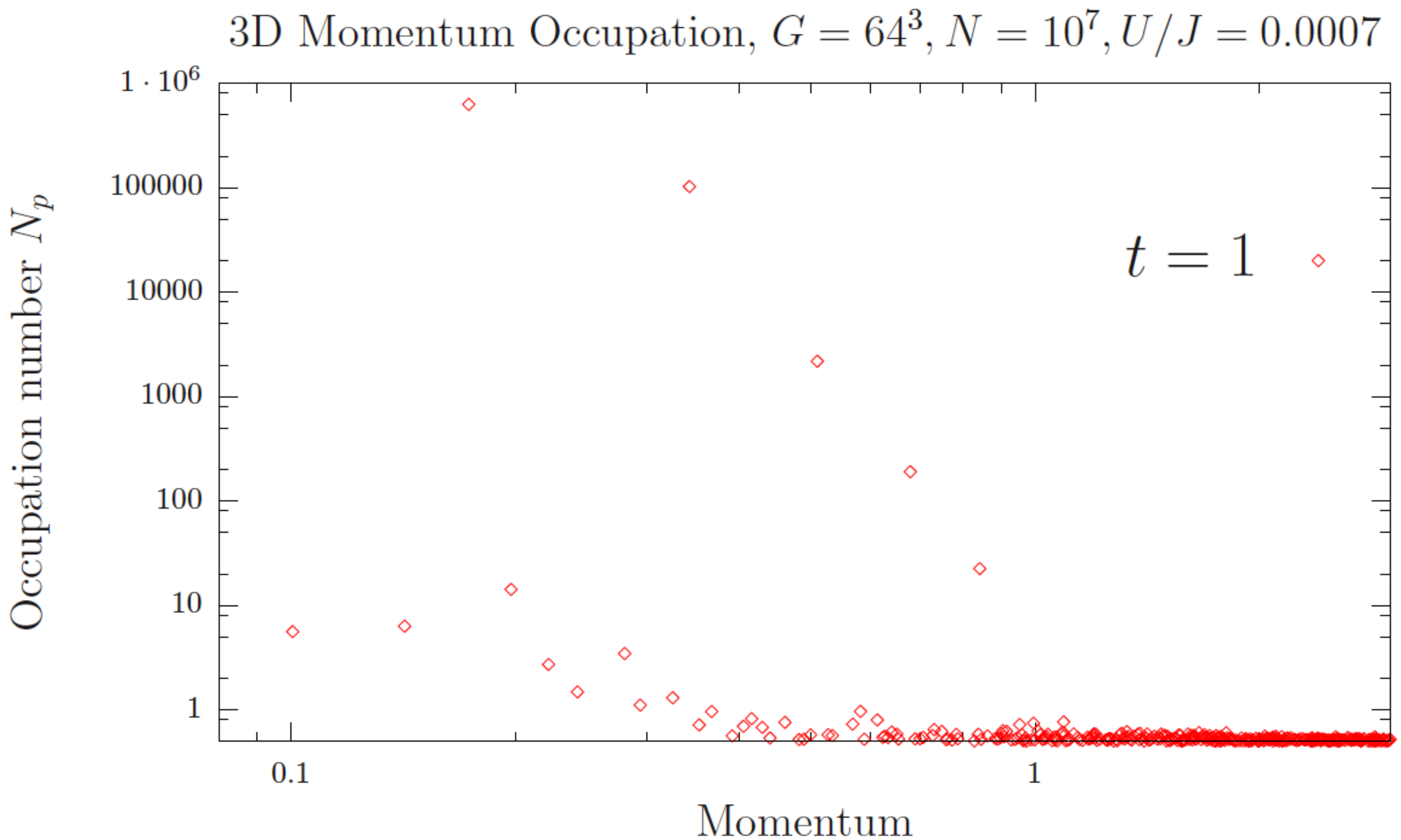

J. Berges et al., PRL **101** (08) 041603
C. Scheppach, J. Berges, TG PRA **81** (10) 033611

Simulations in 2+1 D (semi-classical)



B. Nowak, D. Sexty, TG (unpublished)

Simulations in 3+1 D (semi-classical)



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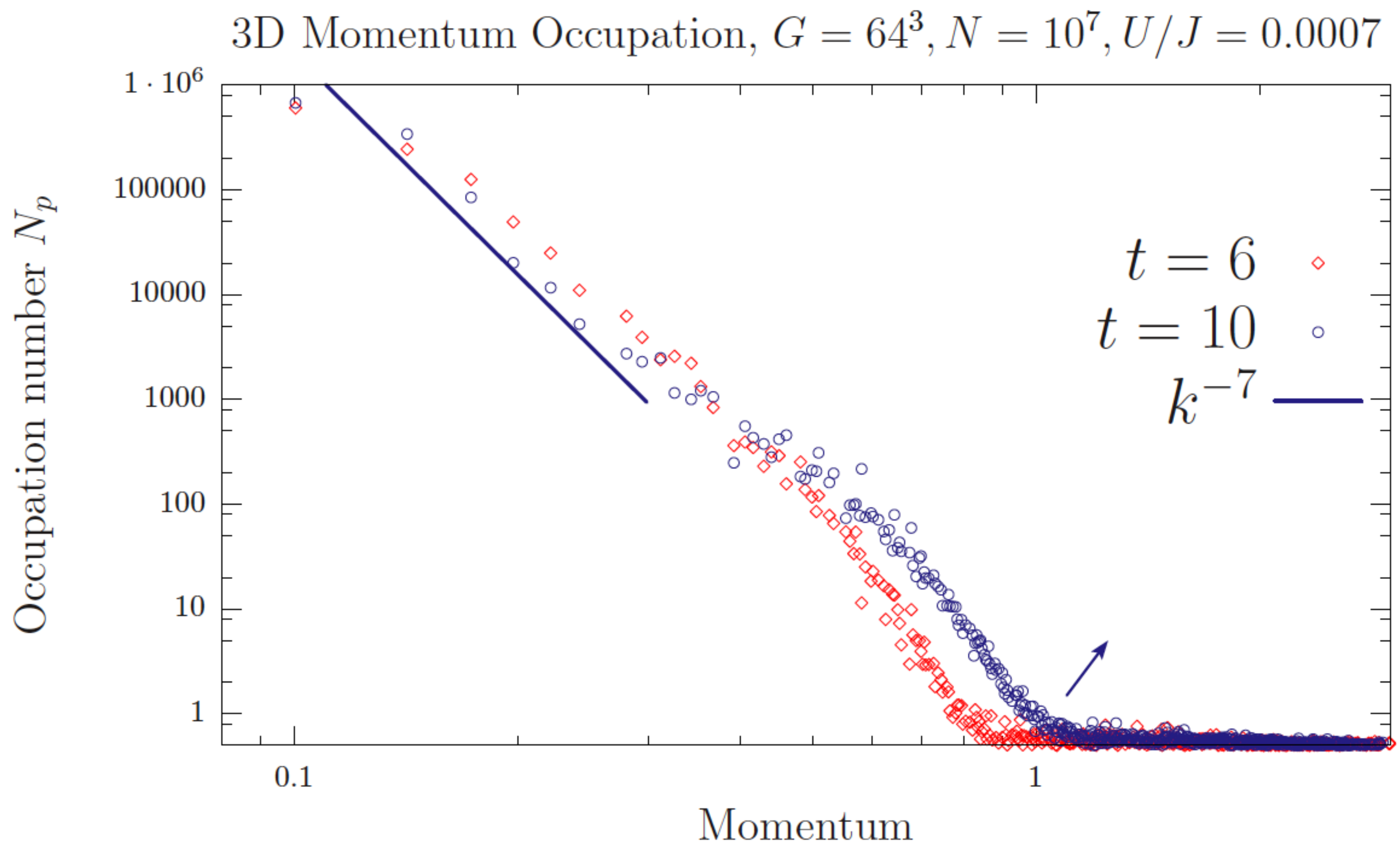

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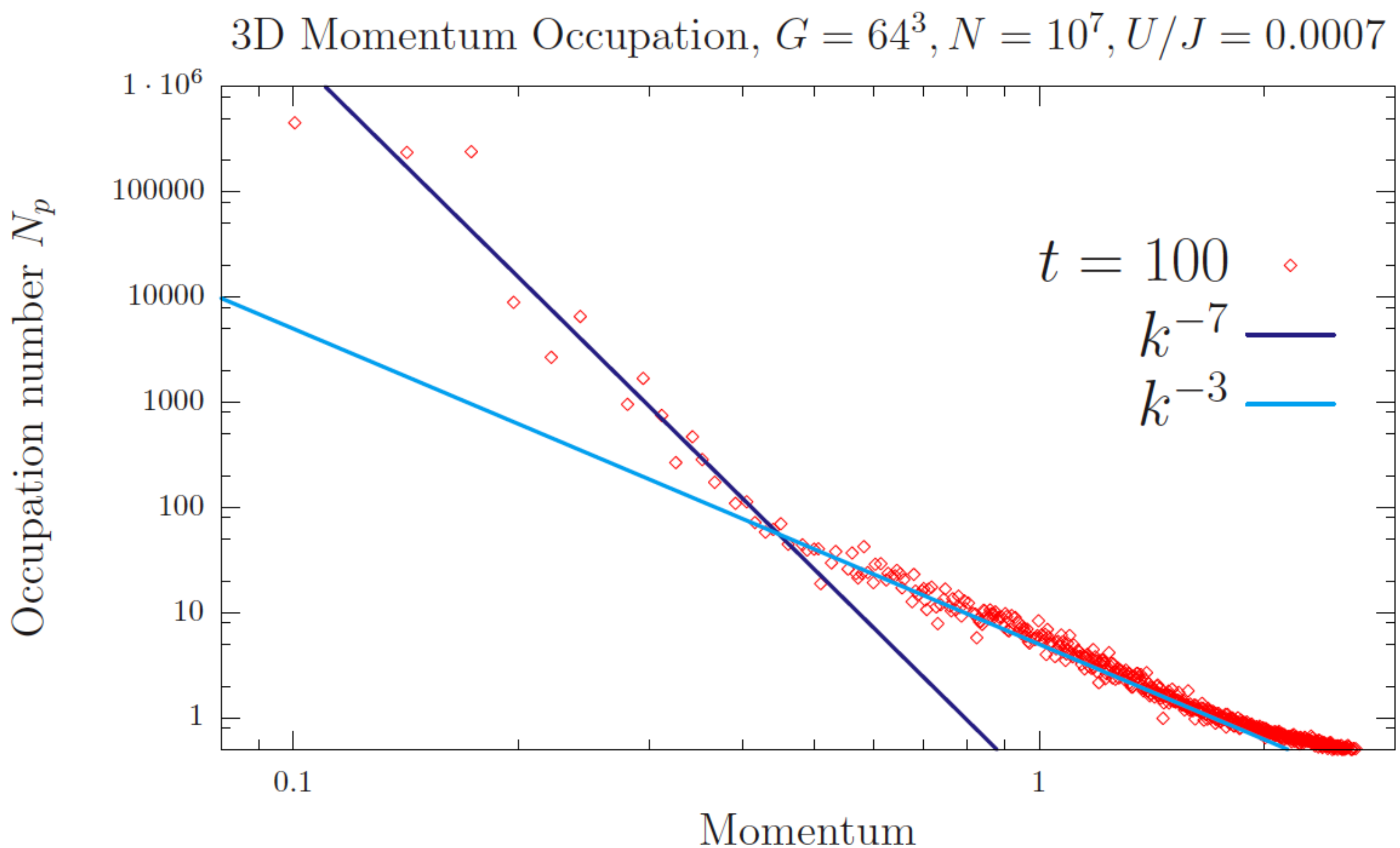
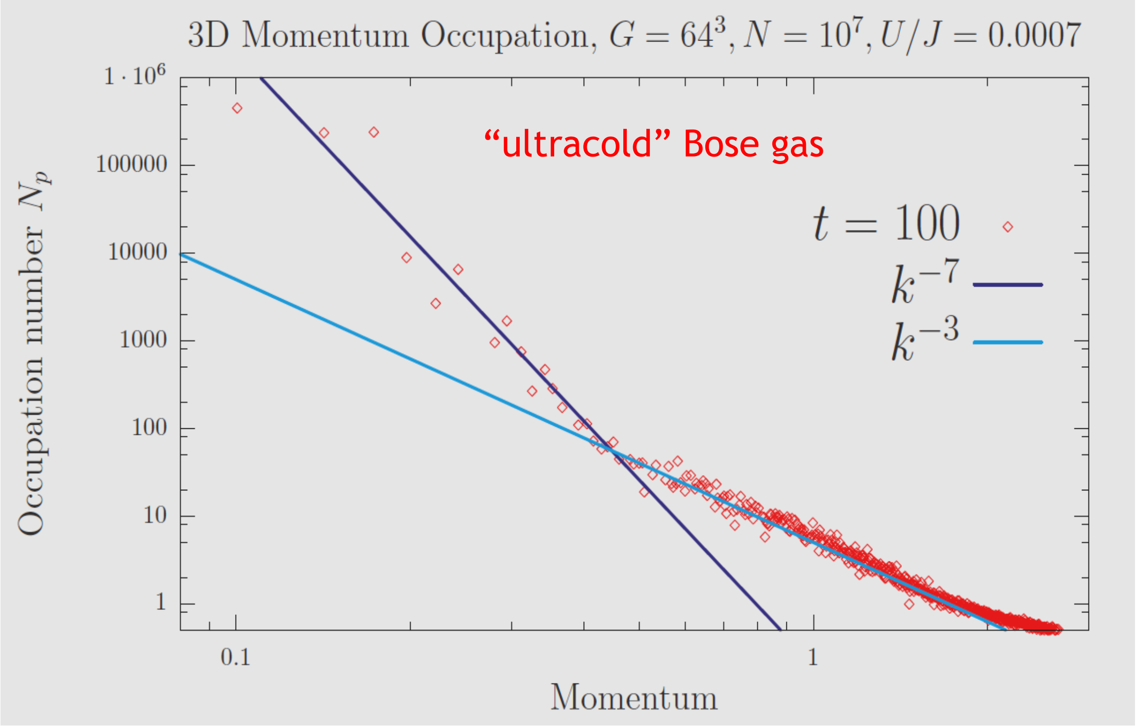
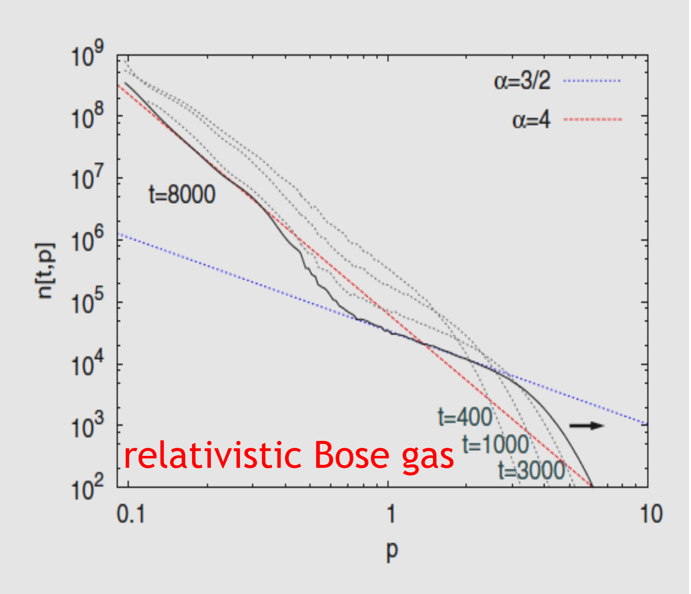


B. Nowak, D. Sexty, TG (unpublished)

3D Simulation

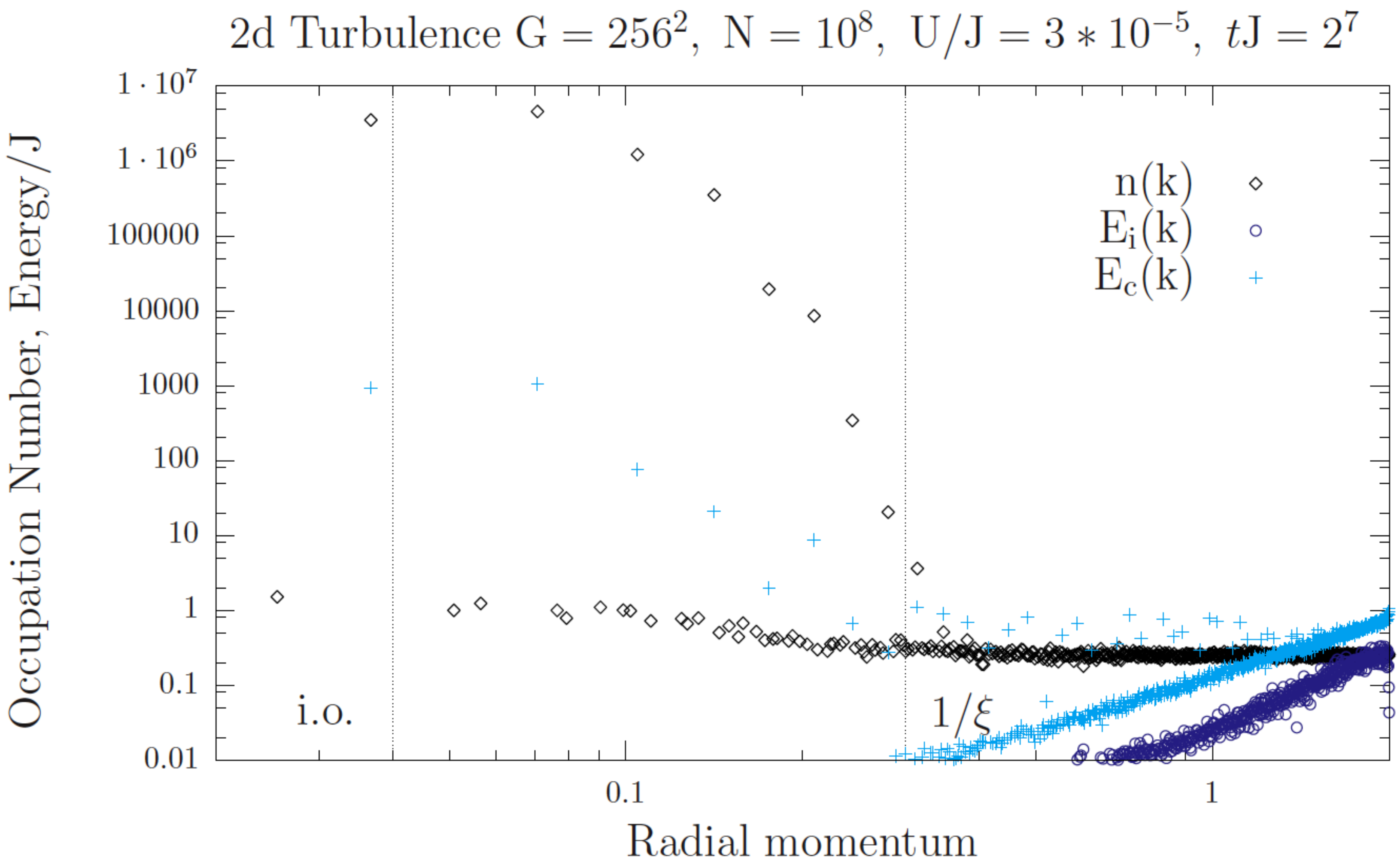


B. Nowak, D. Sexty, TG (unpublished)



J. Berges, A. Rothkopf, and J. Schmidt, PRL **101** (08) 041603

Simulations in 2+1 D (semi-classical)



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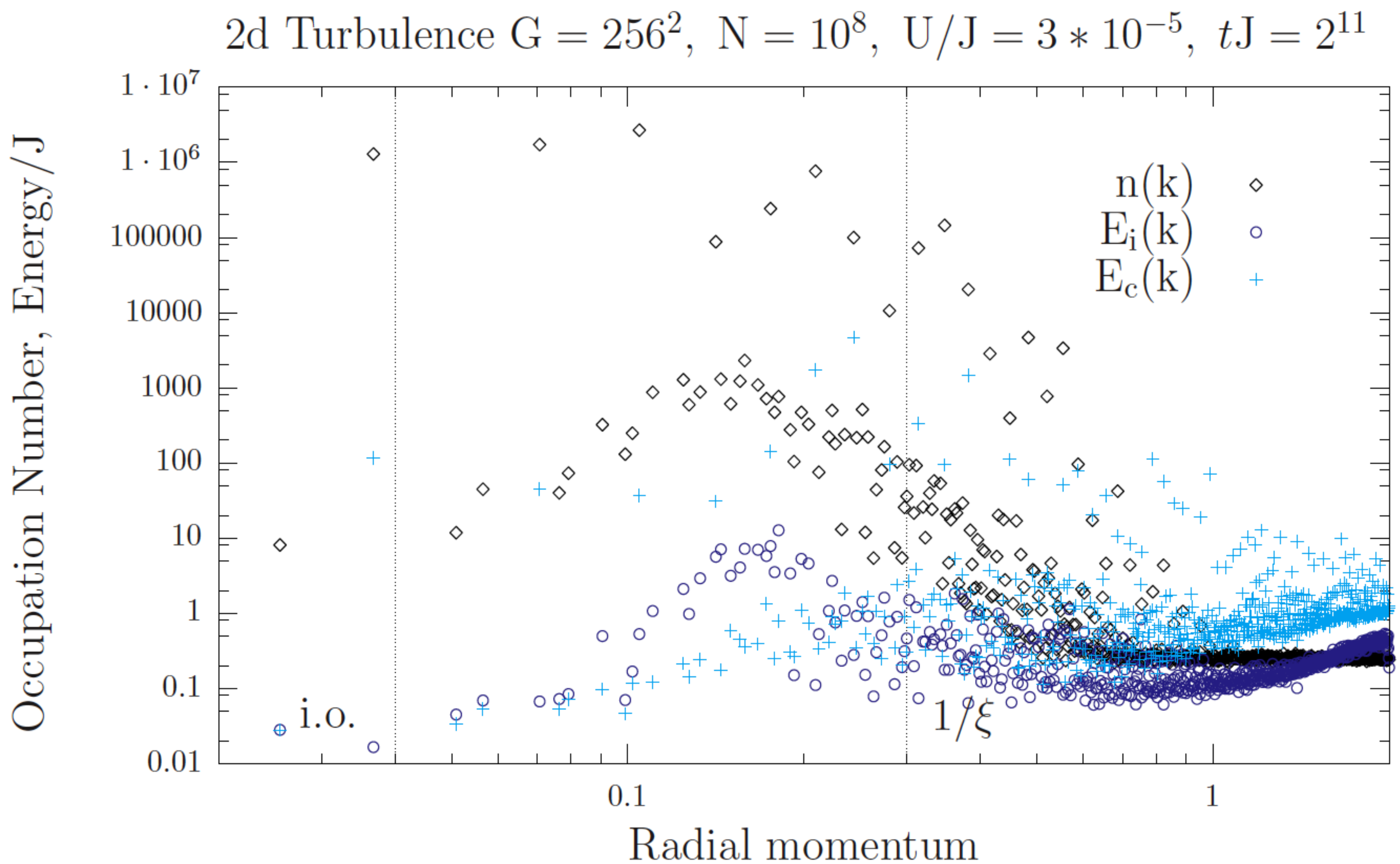
Simulations in 2+1 D (semi-classical)



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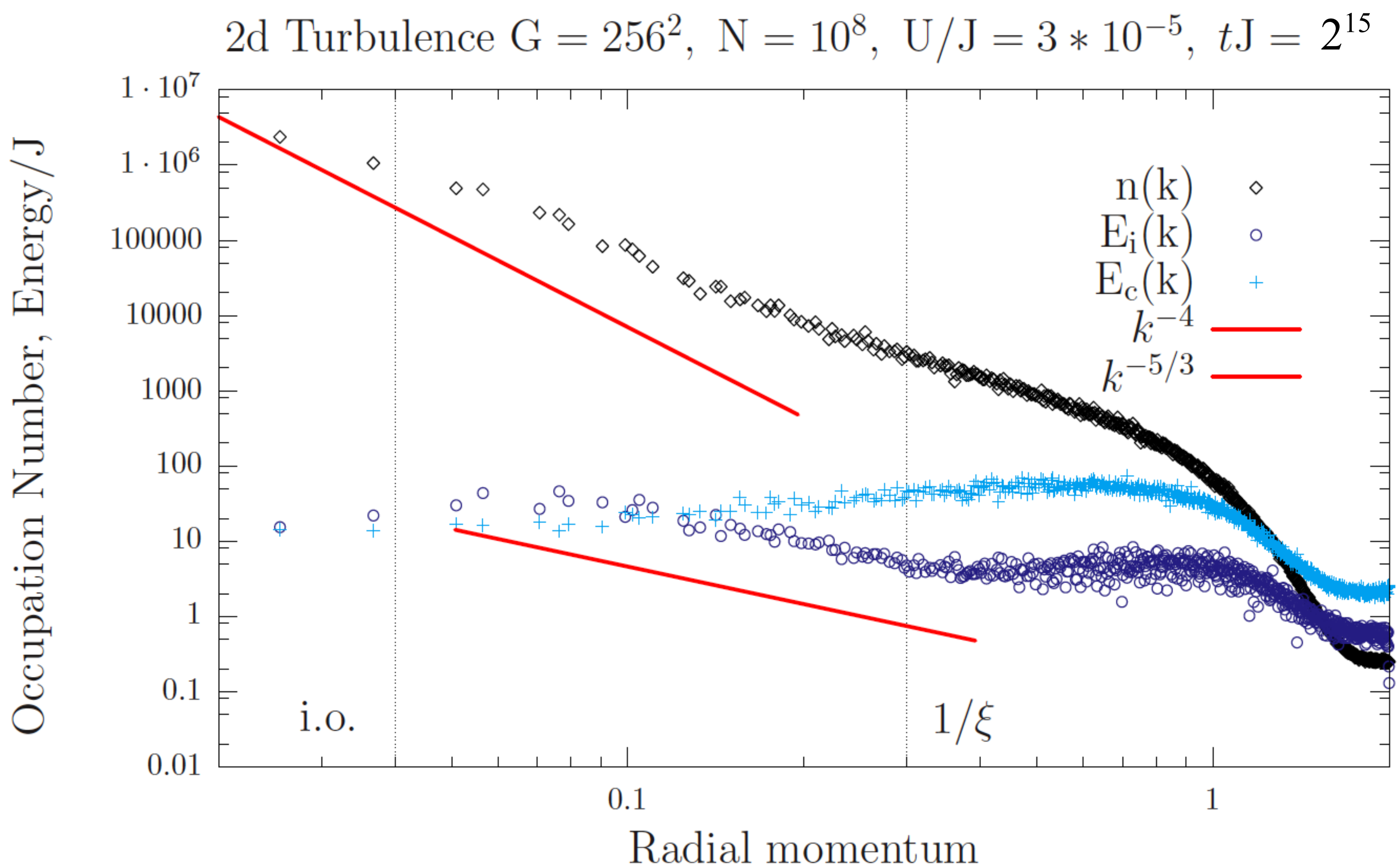
Simulations in 2+1 D (semi-classical)



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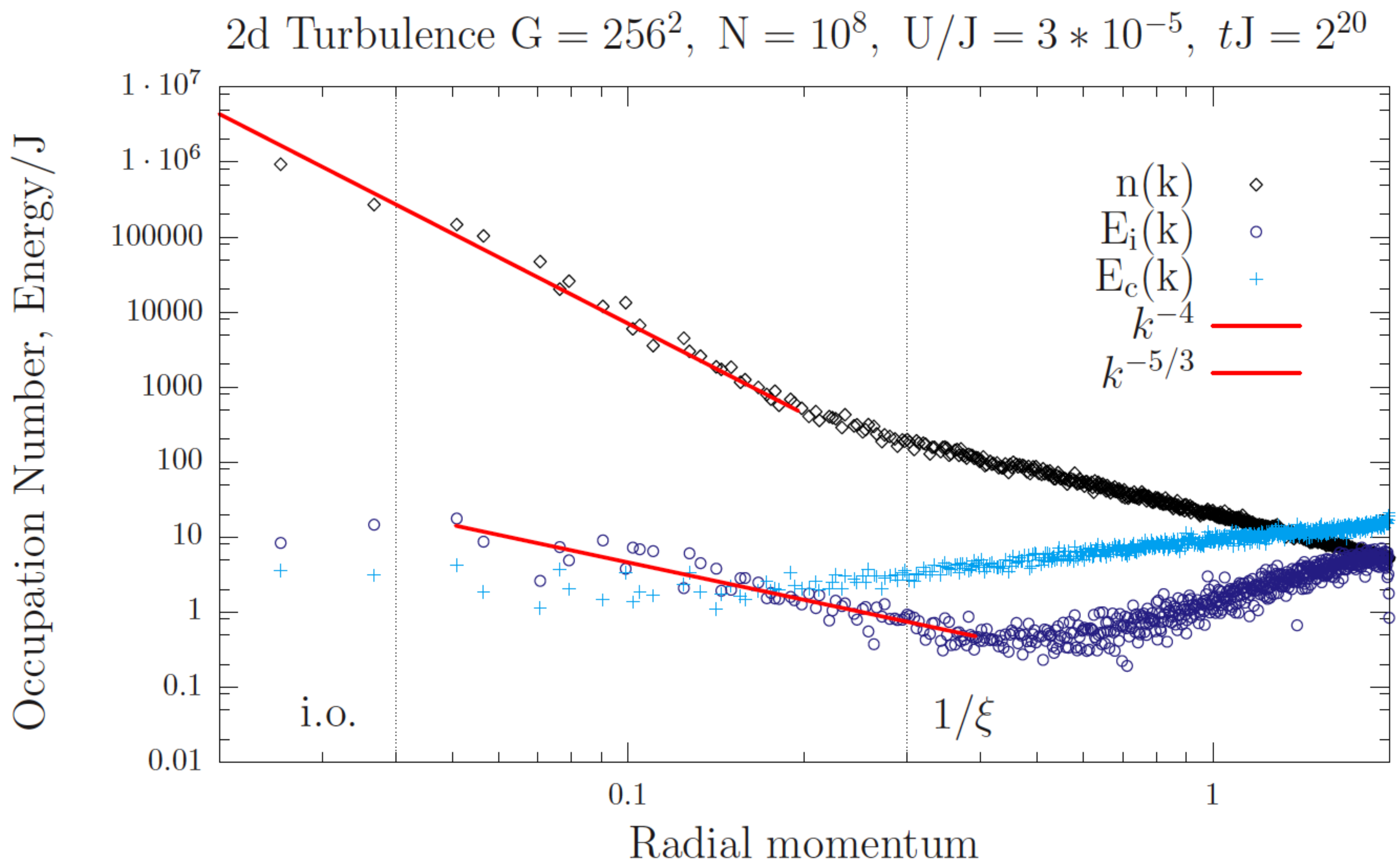


B. Nowak, D. Sexty, TG (unpublished)

Scaling exponents
(in *d* dimensions)

C. Scheppach, J. Berges, T. Gasenzer PRA **81** (10) 033611

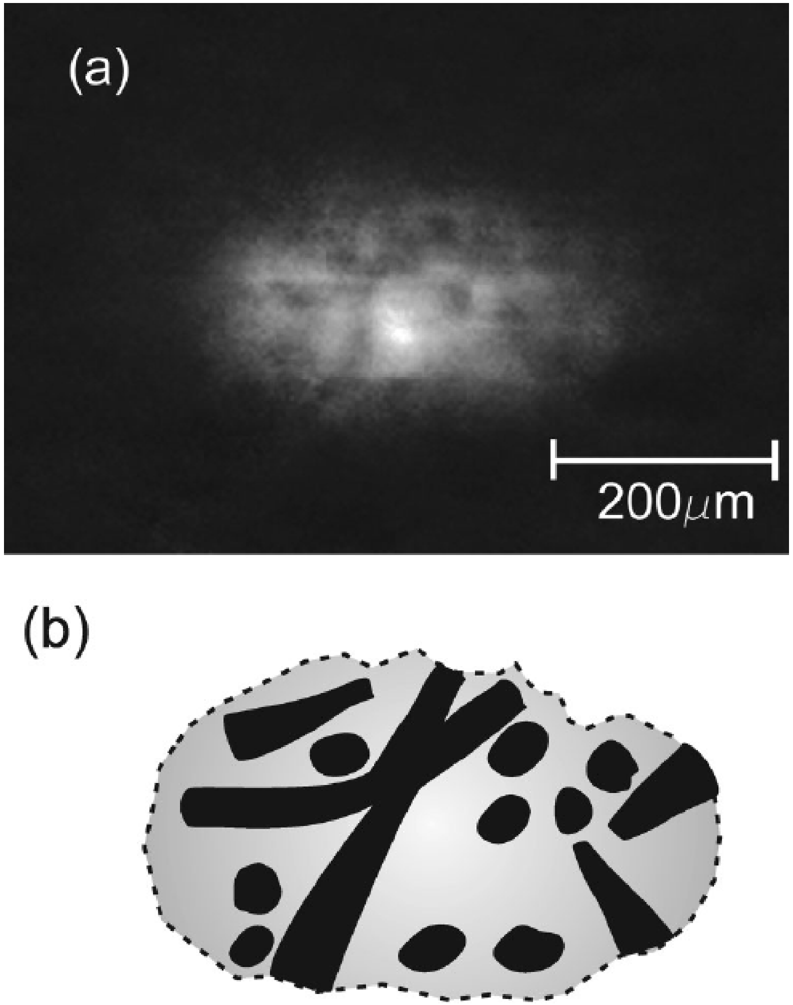
UV: $\kappa = d$ $\kappa = d - 2/3$

Constant $P(k) \equiv P$ Constant $Q(p)$

IR: $\kappa = d + 4$ $\kappa = d + 2$

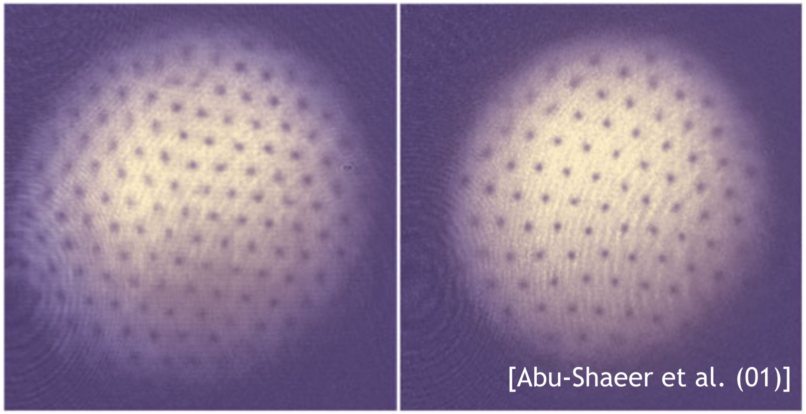
$$n \sim k^{-\kappa}$$

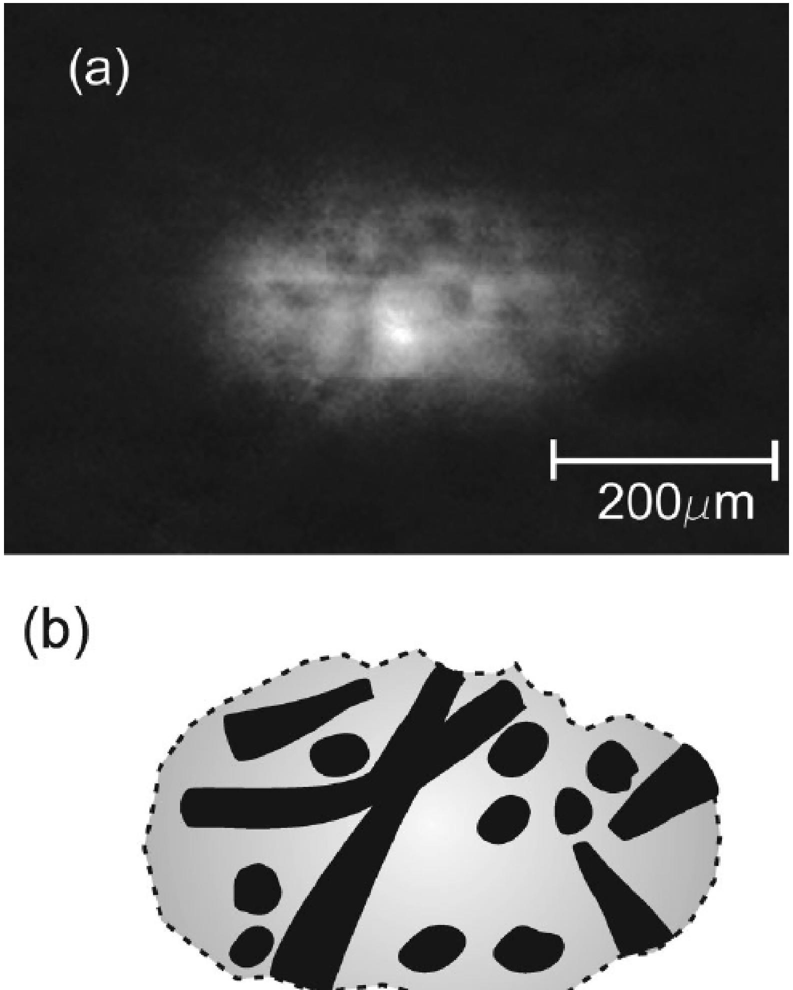
Turbulence in a Bose Einstein Condensate



[E.A.L. Henn et al. PRL (09)]

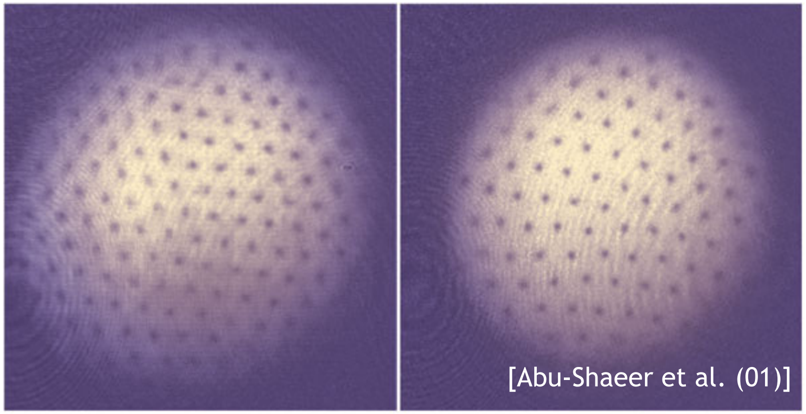
Turbulence in a Bose Einstein Condensate



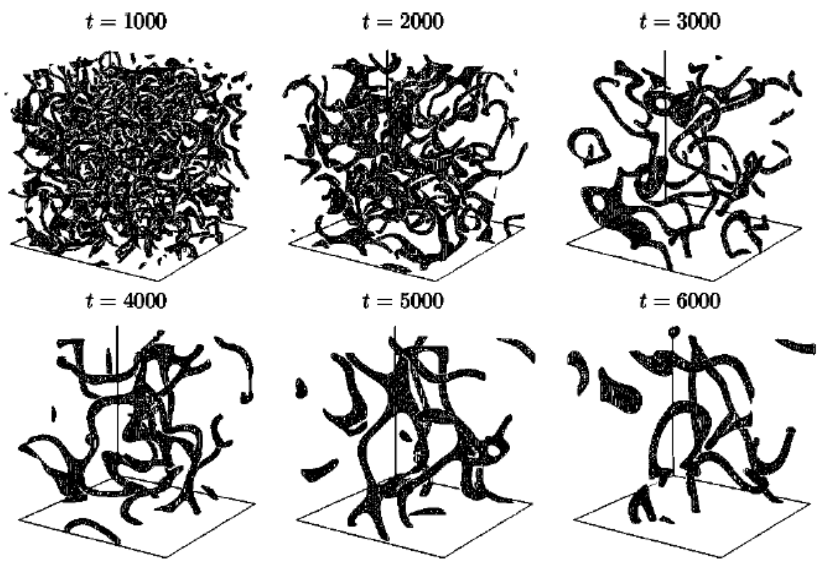


[E.A.L. Henn et al. PRL (09)]

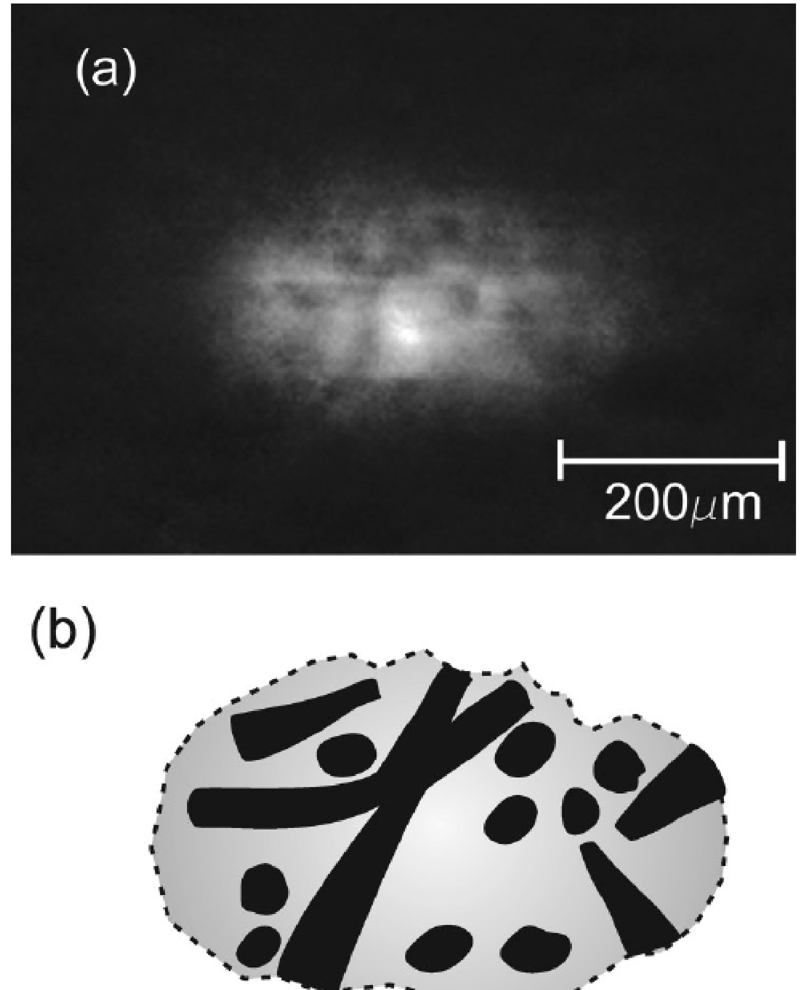
Turbulence in a Bose Einstein Condensate



[Abu-Shaeer et al. (01)]

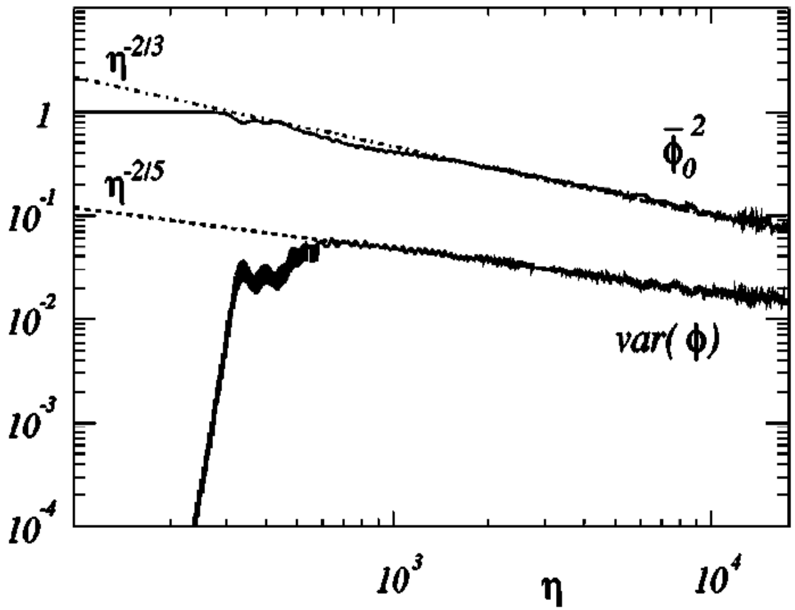


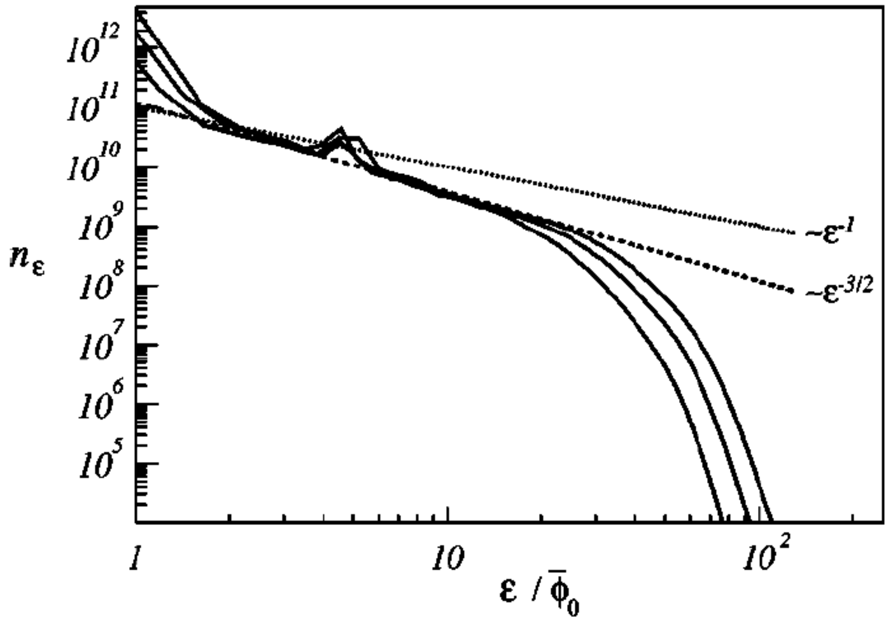
[N. Berloff & B. Svistunov, PRA (02)]

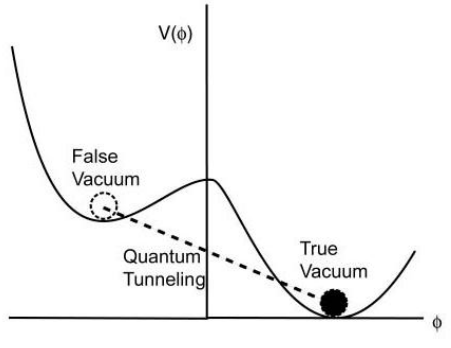


[E.A.L. Henn et al. PRL (09)]

Wave turbulence



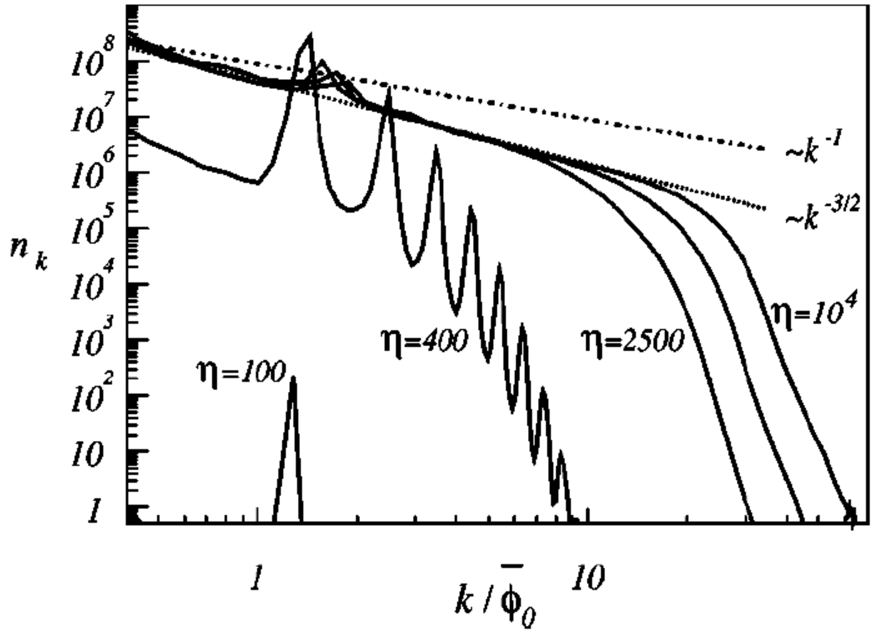


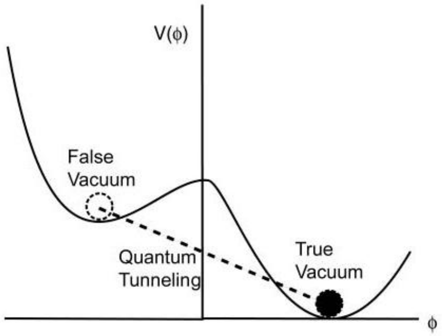


Turbulent thermalisation after universe inflation

[Micha & Tkachev, PRL **90** (03) 121301, PRD **70** (04) 043538]

Wave turbulence





$$\Box\varphi + \lambda\varphi^3 = 0$$

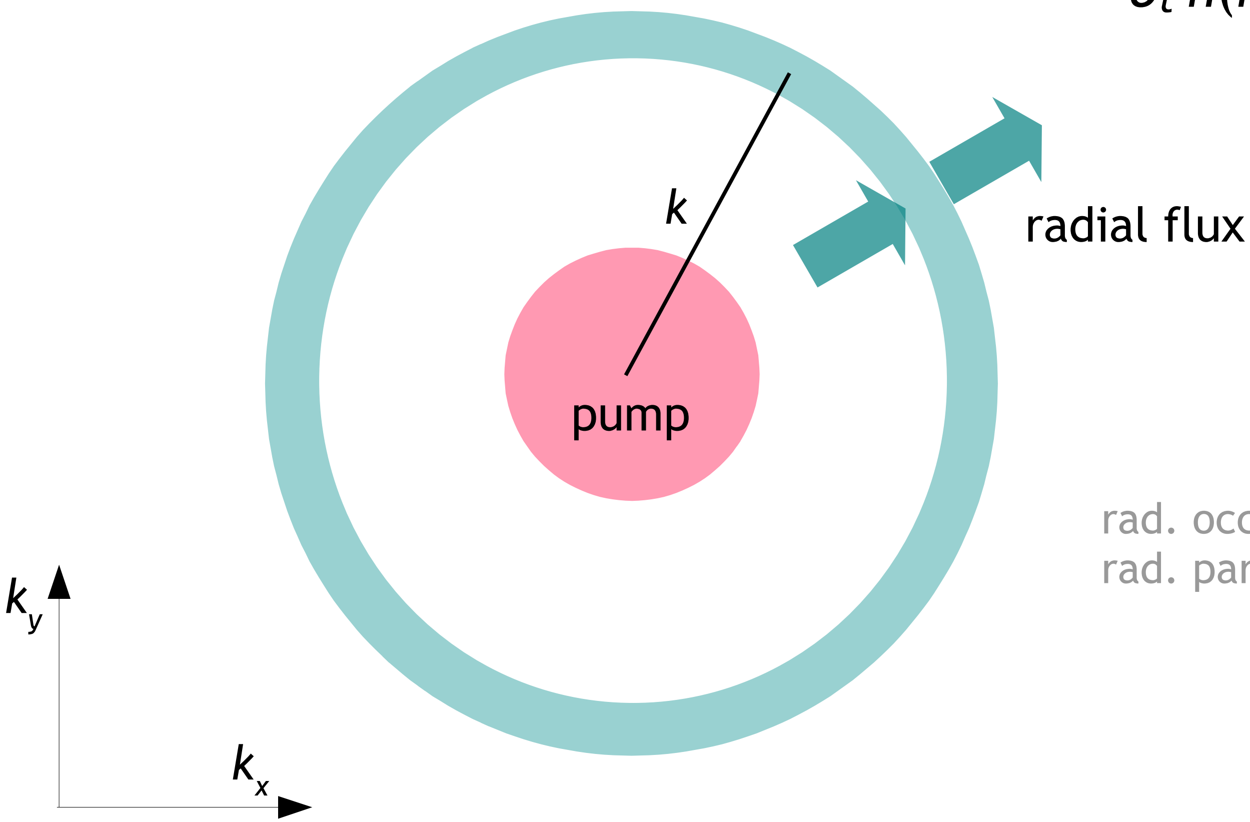
[Micha & Tkachev, PRL **90** (03) 121301, PRD **70** (04) 043538]

Local radial flux only

With kinetic (Boltzmann) eq.

Boltzmann scattering integral

$$\partial_t n(k) = -\partial_k Q(k)$$
$$\sim k^2 J(k)$$



rad. occupation no. n
rad. particle flux Q

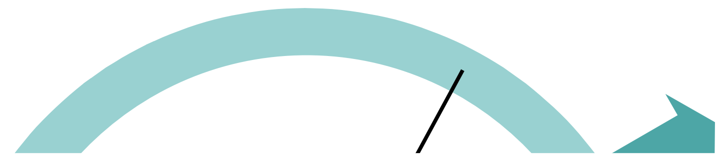
Local radial flux only

With kinetic (Boltzmann) eq.

Boltzmann scattering integral

$$\partial_t\, n(k) = -\,\partial_k\, Q(k)$$

$$\sim\ k^2\, J(k)$$



Scattering integral:

$$J(\mathbf{k},t) = g^2 \int \mathrm{d}^d p\, \mathrm{d}^d q\, \mathrm{d}^d r\, |T_{\mathbf{kpqr}}|^2 \delta(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r})$$
$$\times\ \delta(\omega_{\mathbf{k}}+\omega_{\mathbf{p}}-\omega_{\mathbf{q}}-\omega_{\mathbf{r}})$$
$$\times\ [(n_{\mathbf{k}}+1)(n_{\mathbf{p}}+1)n_{\mathbf{q}}n_{\mathbf{r}} - n_{\mathbf{k}}n_{\mathbf{p}}(n_{\mathbf{q}}+1)(n_{\mathbf{r}}+1)]$$

rad. particle flux Q

Dynamical field theory

Kinetic (Quantum-Boltzmann) eq.: $\partial_t n_{\mathbf{k}} = J(\mathbf{k}, t)$

Scattering integral:

$$J(\mathbf{k},t) = g^2 \int \mathrm{d}^d p\, \mathrm{d}^d q\, \mathrm{d}^d r\, |T_{\mathbf{kpqr}}|^2 \delta(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r}) \\ \times\ \delta(\omega_{\mathbf{k}} + \omega_{\mathbf{p}} - \omega_{\mathbf{q}} - \omega_{\mathbf{r}}) \\ \times\ [(n_{\mathbf{k}}+1)(n_{\mathbf{p}}+1)n_{\mathbf{q}}n_{\mathbf{r}} - n_{\mathbf{k}}n_{\mathbf{p}}(n_{\mathbf{q}}+1)(n_{\mathbf{r}}+1)]$$

Here: Dynamic (Schwinger-Dyson/Kadanoff-Baym) eq.:

$$G^{-1}_{ab}(x,y) \ = \ G^{-1}_{0,ab}(x,y;\phi) - \Sigma_{ab}(x,y;\phi,G)$$

or...

Dynamical field theory

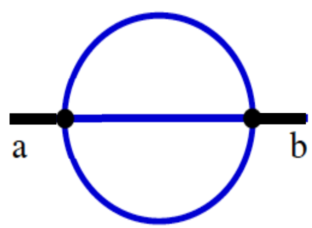
Kinetic (Quantum-Boltzmann) eq.: $\partial_t n_{\mathbf{k}} = J(\mathbf{k},t)$

Scattering integral:

$$J(\mathbf{k},t) = g^2 \int \mathrm{d}^d p\, \mathrm{d}^d q\, \mathrm{d}^d r\, |T_{\mathbf{kpqr}}|^2 \delta(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r}) \times \delta(\omega_{\mathbf{k}}+\omega_{\mathbf{p}}-\omega_{\mathbf{q}}-\omega_{\mathbf{r}}) \times [(n_{\mathbf{k}}+1)(n_{\mathbf{p}}+1)n_{\mathbf{q}}n_{\mathbf{r}} - n_{\mathbf{k}}n_{\mathbf{p}}(n_{\mathbf{q}}+1)(n_{\mathbf{r}}+1)]$$

Here: Dynamic eq. from **2PI effective action**

$$\partial_t n(p) = \Sigma^{\rho}_{ab}(p) F_{ba}(p) - \Sigma^{F}_{ab}(p)\rho_{ba}(p)$$

$\Sigma_{ab}(x,y) =$ 

$p = (p_0, \mathbf{p})$:

To derive scaling

Familiarize the...

reader with a little miracle of the theory of wave turbulence theory, the so-called Zakharov transformations. They factorize the collision integral. As a result one

$$I(\omega) = \int_{\Omega} d\omega_2 d\omega_3 U(\omega, \omega_2 + \omega_3 - \omega, \omega_2, \omega_3)$$
$$\times\, n(\omega_2) n(\omega_2 + \omega_3 - \omega) n(\omega) n(\omega_3)$$
$$\times \left[n^{-1}(\omega) + n^{-1}(\omega_2 + \omega_2 - \omega) - n^{-1}(\omega_2) - n^{-1}(\omega_3) \right]$$

$$\omega_2 = \frac{\omega \omega_2'}{\omega_2' + \omega_3' - \omega}, \quad \omega_3 = \frac{\omega \omega_3'}{\omega_2' + \omega_3' - \omega}$$

V.E. Zakharov, V.S. L'vov, G. Falkovich, *Kolmogorov Spectra of Turbulence I* (Springer, Berlin, 1992)

To derive scaling

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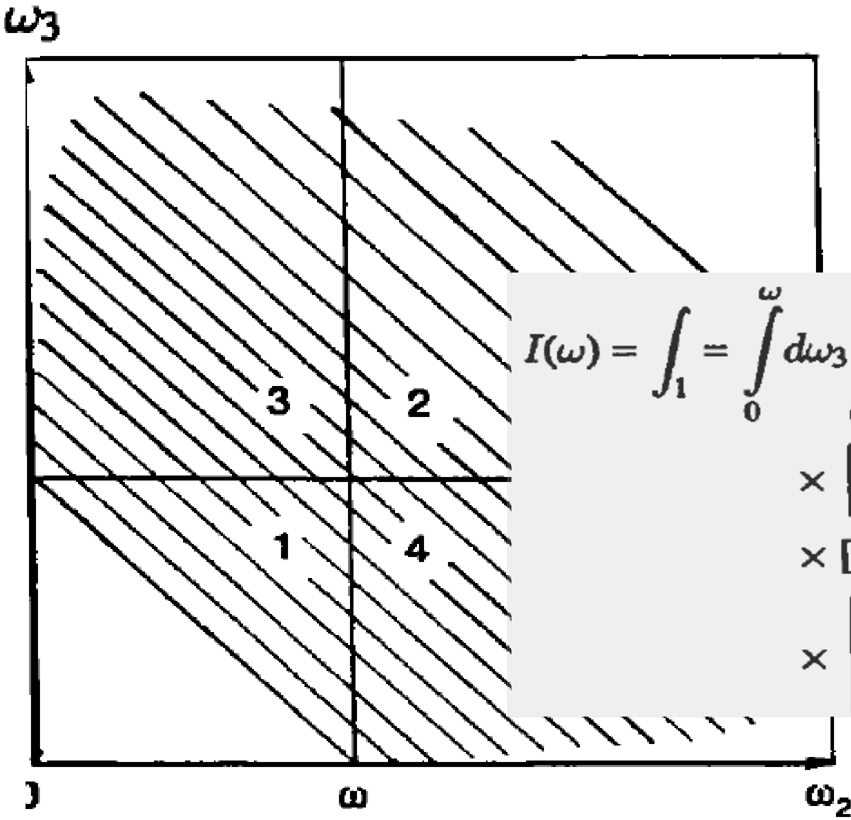


$$I(\omega) = \int_\Omega d\omega_2 d\omega_3 U(\omega, \omega_2+\omega_3-\omega, \omega_2, \omega_3) \times n(\omega_2)n(\omega_2+\omega_3-\omega)n(\omega)n(\omega_3) \times \left[n^{-1}(\omega)+n^{-1}(\omega_2+\omega_2-\omega)-n^{-1}(\omega_2)-n^{-1}(\omega_3)\right]$$

$$I(\omega) = \int_1 = \int_0^\omega d\omega_3 \int_{\omega-\omega_3}^{\omega} d\omega_2 U(\omega, \omega_2+\omega_3-\omega, \omega_2, \omega_3) \times \left[\omega^x + (\omega_2+\omega_3-\omega)^x - \omega_2^x - \omega_3^x)\right] \times [\omega(\omega_2+\omega_3-\omega)\omega_2\omega_3]^{-x} \times \left[1+\left(\frac{\omega_2+\omega_3-\omega}{\omega}\right)^y - \left(\frac{\omega_2}{\omega}\right)^y - \left(\frac{\omega_3}{\omega}\right)^y\right] = 0$$

$$y = 3x - 3 - \left[\frac{2m+3d}{\alpha} - 4\right]$$

V.E. Zakharov, V.S. L'vov, G. Falkovich, *Kolmogorov Spectra of Turbulence I* (Springer, Berlin, 1992)

(Weak) wave turbulence

Flow in momentum space from kinetic (Boltzmann) equation:

$$\partial_t n_{\mathbf{k}} = J(\mathbf{k}, t)$$

Scattering integral:

$$J(\mathbf{k},t) = g^2 \int \mathrm{d}^d p\, \mathrm{d}^d q\, \mathrm{d}^d r\, |T_{\mathbf{kpqr}}|^2 \delta(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r})$$
$$\times\ \delta(\omega_{\mathbf{k}}+\omega_{\mathbf{p}}-\omega_{\mathbf{q}}-\omega_{\mathbf{r}})$$
$$\times\ [(n_{\mathbf{k}}+1)(n_{\mathbf{p}}+1)n_{\mathbf{q}}n_{\mathbf{r}} - n_{\mathbf{k}}n_{\mathbf{p}}(n_{\mathbf{q}}+1)(n_{\mathbf{r}}+1)]$$

$\partial_k P(k) \sim J(k)$ ⇒ Stationarity if $J(k) = 0$

(Weak) wave turbulence

Flow in momentum space from kinetic (Boltzmann) equation:

$$\partial_t n_{\mathbf{k}} = J(\mathbf{k}, t)$$

Scattering integral:

$$J(\mathbf{k},t) = g^2 \int \mathrm{d}^d p\, \mathrm{d}^d q\, \mathrm{d}^d r\, |T_{\mathbf{kpqr}}|^2 \delta(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r}) \times \delta(\omega_{\mathbf{k}}+\omega_{\mathbf{p}}-\omega_{\mathbf{q}}-\omega_{\mathbf{r}}) \times [(n_{\mathbf{k}}+1)(n_{\mathbf{p}}+1)n_{\mathbf{q}}n_{\mathbf{r}} - n_{\mathbf{k}}n_{\mathbf{p}}(n_{\mathbf{q}}+1)(n_{\mathbf{r}}+1)]$$

$$\partial_k P(k) \sim J(k) \sim n_k^{\,3} \quad \Rightarrow \quad n_k \sim P^{1/3}$$

$$\Rightarrow \quad E(k) \sim (P\rho^2)^{1/3}\, \omega_k\, k^{-7/3}$$

(Weak) wave turbulence

Scaling solutions

Constant energy flux P:

$$E(k) \sim (P\rho^2)^{1/3}\,\omega_k\,k^{-7/3}$$

(Weak) wave turbulence

Scaling solutions

Constant energy flux P:

$$E(k) \sim (P\rho^2)^{1/3}\,\omega_k\,k^{-7/3}$$

Constant particle flux Q $\quad (P \to P_k = \omega_k Q)$:

$$E(k) \sim (Q\rho^2)^{1/3}\,\omega_k^{4/3}\,k^{-7/3}$$

⇒ 2 distinct scaling laws!

Scaling solutions

Implies scaling of the single-particle momentum distribution ($\eta = 0$):

$$\rho_{ab}(s^z p_0, s\mathbf{p}) = s^{-2+\eta} \rho_{ab}(p_0, \mathbf{p})$$
$$F_{ab}(s^z p_0, s\mathbf{p}) = s^{-2+\eta-\kappa} F_{ab}(p_0, \mathbf{p})$$

($n(\vec{p}) = \frac{1}{2} \int \frac{dp_0}{2\pi} (F_{11}(p) + iF_{12}(p) - iF_{21}(p) + F_{22}(p))$)

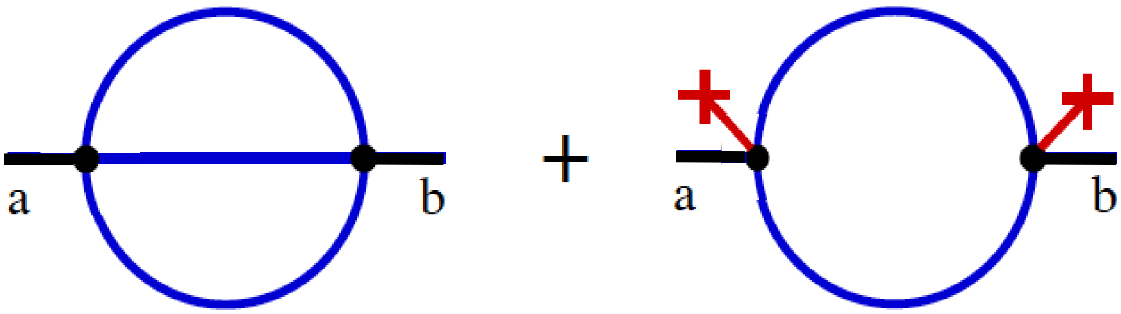
$$n(s\mathbf{p}) = s^{z-2-\kappa+\eta} n(\mathbf{p})$$

Stationarity Condition

in the full dynamical theory

$p = (p_0, \mathbf{p})$:

$$J(p) := \Sigma^{\rho}_{ab}(p) F_{ba}(p) - \Sigma^{F}_{ab}(p) \rho_{ba}(p) \stackrel{!}{=} 0$$

$\Sigma_{ab}(x,y) =$ 

(Bosons with 3- and 4-interactions, 2PI diagrams to $O(g^2)$)

Ultraviolet scaling exponent

(large p in d dim^s)

[C. Scheppach, J. Berges, & TG, to be submitted]

$$\kappa = d$$

Corresponds to constant $P(p)$

$$n(s\vec{p}) = s^{-\kappa} n(\vec{p})$$

$$\kappa = d + 4(z - 2 + \eta)/3$$

Ultraviolet scaling exponent

(large p in d dim^s)

[C. Scheppach, J. Berges, & TG, to be submitted]

$$\kappa = d - 2/3$$

Corresponds to constant $Q(p)$

$$n(s\vec{p}) = s^{-\kappa} n(\vec{p})$$

$$\kappa = d + z - 4(2-\eta)/3$$

Ultraviolet scaling exponent

(large p in d dim^s)

[C. Scheppach, J. Berges, & TG, to be submitted]

$$\kappa = d - 2/3$$

Corresponds to constant $Q(p)$

Not present in general dynamical case!

$$n(s\vec{p}) = s^{-\kappa} n(\vec{p})$$

$$\kappa = d + z - 4(2-\eta)/3$$

To derive scaling

Familiarize the...

reader with a little miracle of the theory of wave turbulence theory, the so-called Zakharov transformations. They factorize the collision integral. As a result one

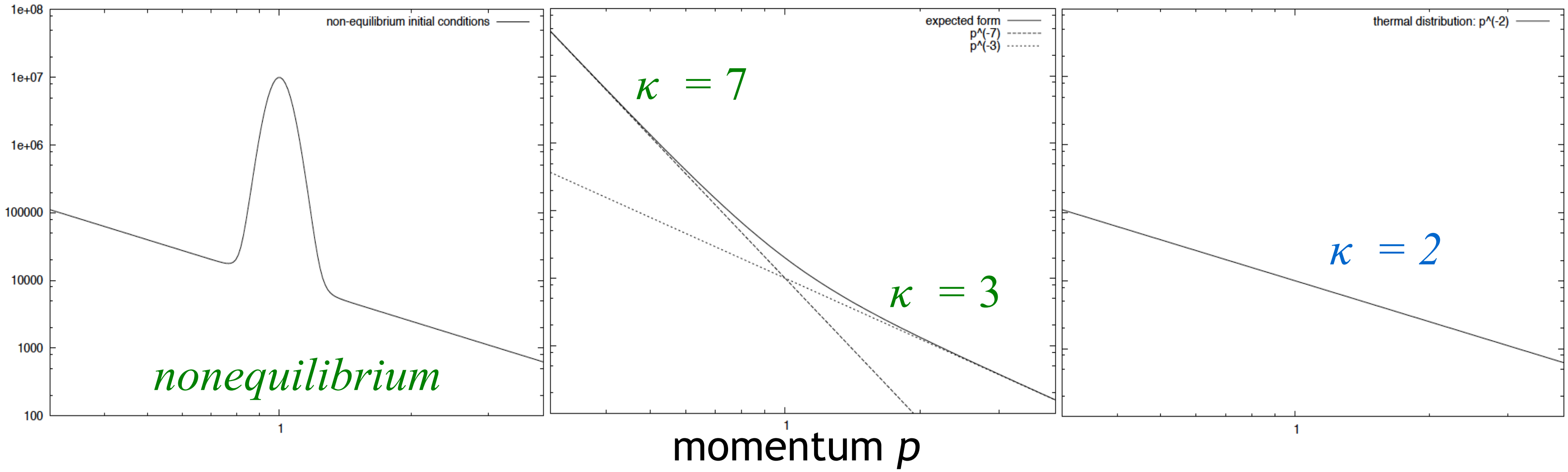
Scattering integral

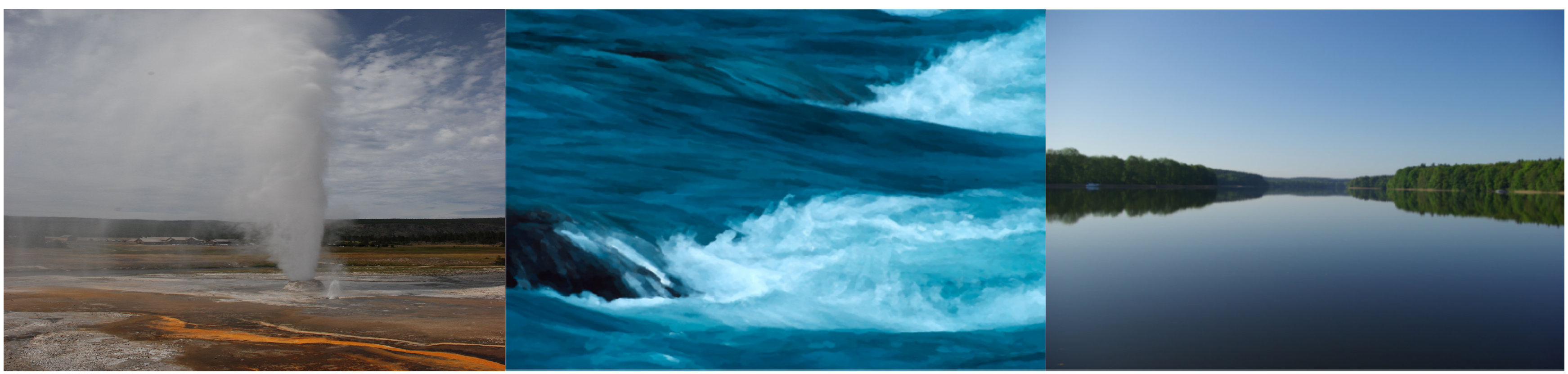
reader with a little miracle of the theory of wave turbulence theory, the so-called Zakharov transformations. They factorize the collision integral. As a result one

$$
\begin{aligned}
J^0(p_0) \;=\; & \int_{k_0>0,q_0>0,r_0>0} \mathrm{d}^3\vec{p}\,\mathrm{d}^4k\,\mathrm{d}^4q\,\mathrm{d}^4r \\
& \Big\{\delta_{p+k-q-r}\lambda^{\mathrm{eff}}_{p+k}\,\overset{ab}{\rho}_p\overset{ab}{F}_k\overset{cd}{F}_q\overset{cd}{F}_r\cdot p_0^{-\beta}(p_0^\beta+k_0^\beta-q_0^\beta-r_0^\beta) \\
& +\;\delta_{p-k+q+r}\lambda^{\mathrm{eff}}_{p-k}\,\overset{ab}{\rho}_p\overset{ba}{F}_k\overset{cd}{F}_q\overset{cd}{F}_r\cdot p_0^{-\beta}(p_0^\beta-k_0^\beta+q_0^\beta+r_0^\beta) \\
& +\;\delta_{p-k-q-r}\lambda^{\mathrm{eff}}_{p-k}\,\overset{ab}{\rho}_p\overset{ba}{F}_k\overset{cd}{F}_q\overset{cd}{F}_r\cdot p_0^{-\beta}(p_0^\beta-k_0^\beta-q_0^\beta-r_0^\beta) \\
& +2\delta_{p+k-q+r}\lambda^{\mathrm{eff}}_{p+k}\,\overset{ab}{\rho}_p\overset{ab}{F}_k\overset{cd}{F}_q\overset{dc}{F}_r\cdot p_0^{-\beta}(p_0^\beta+k_0^\beta-q_0^\beta+r_0^\beta) \\
& +2\delta_{p-k-q+r}\lambda^{\mathrm{eff}}_{p-k}\,\overset{ab}{\rho}_p\overset{ba}{F}_k\overset{cd}{F}_q\overset{dc}{F}_r\cdot p_0^{-\beta}(p_0^\beta-k_0^\beta-q_0^\beta+r_0^\beta)\Big\}
\end{aligned}
$$

Strong turbulence

Infrared scaling modification $n(s\vec{p}) = s^{-\kappa} n(\vec{p})$



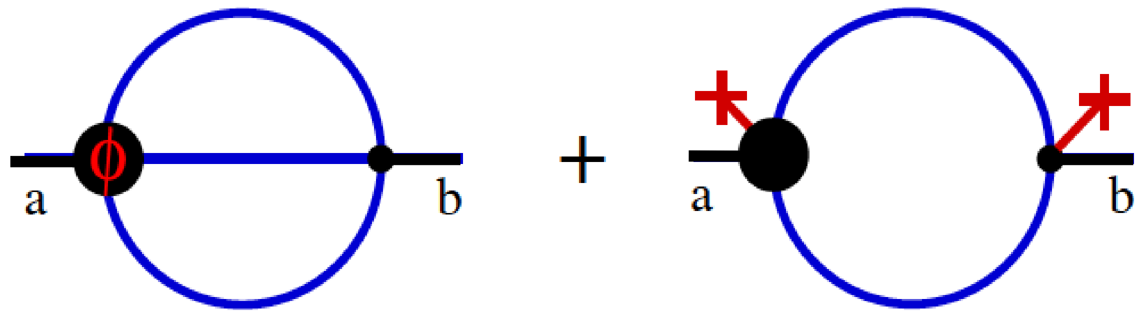


Stationarity Condition
in the full dynamical theory

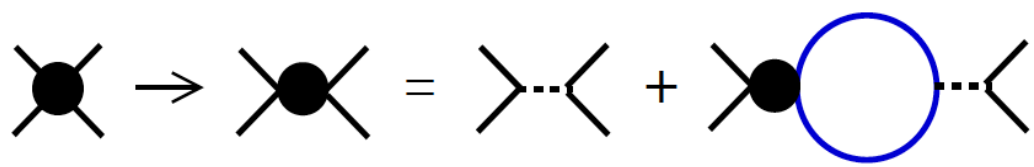
$p = (p_0, \mathbf{p})$:

$$J(p) := \Sigma^{\rho}_{ab}(p)F_{ba}(p) - \Sigma^{F}_{ab}(p)\rho_{ba}(p) \overset{!}{=} 0$$

$\Sigma_{ab}(x,y)$ =



2PI to NLO in $1/N$: Vertex:



IR scaling exponent

(small p in d dim^s)

[Berges et al., PRL 101 (2008) 041603,
C. Scheppach, J. Berges, & TG, to be submitted]

$$\kappa = d + 4$$

Corresponds to constant $P(p)$

$$n(s\vec{p}) = s^{-\kappa} n(\vec{p})$$

$$\kappa = d + 2z$$

IR scaling exponent

(small p in d dim^s)

[Berges et al., PRL 101 (2008) 041603,
C. Scheppach, J. Berges, & TG, to be submitted]

$$\kappa = d + 2$$

Corresponds to constant $Q(p)$

$$n(s\vec{p}) = s^{-\kappa} n(\vec{p})$$

$\kappa = d + z$

IR scaling exponent

(small p in d dim^s)

[Berges et al., PRL 101 (2008) 041603,
C. Scheppach, J. Berges, & TG, to be submitted]

$$\kappa = d + 2$$

Corresponds to constant $Q(p)$

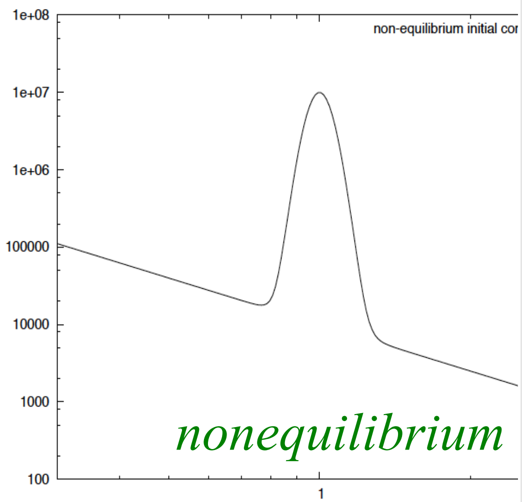
Should not be present in general dynamical case either!

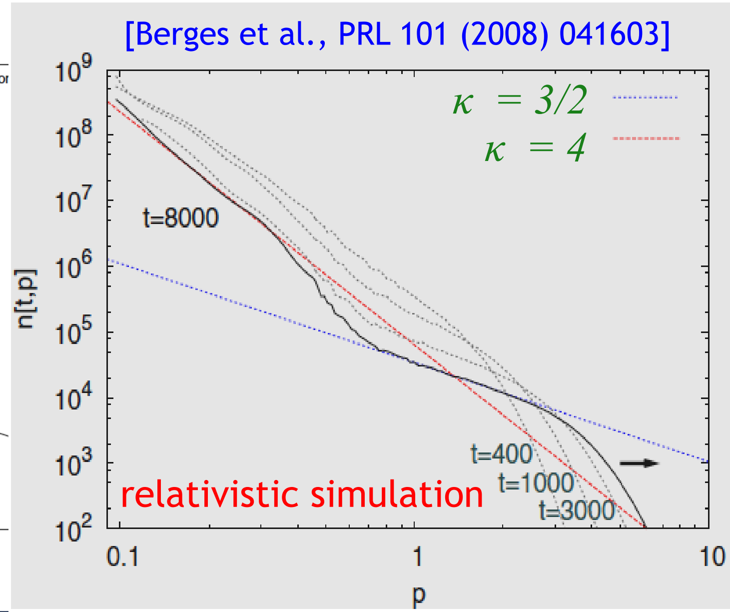
$$n(s\vec{p}) = s^{-\kappa} n(\vec{p})$$

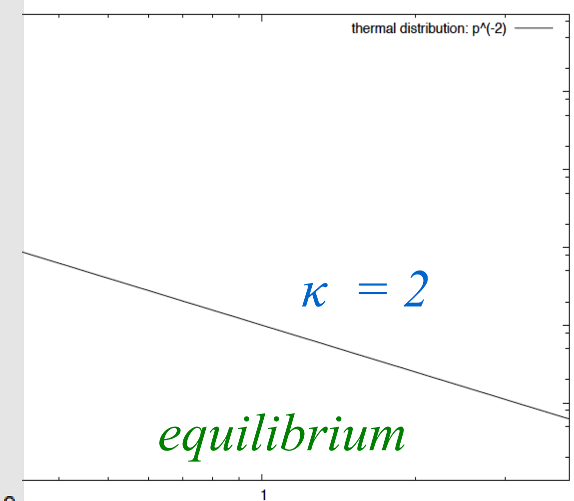
$$\kappa = d + z$$

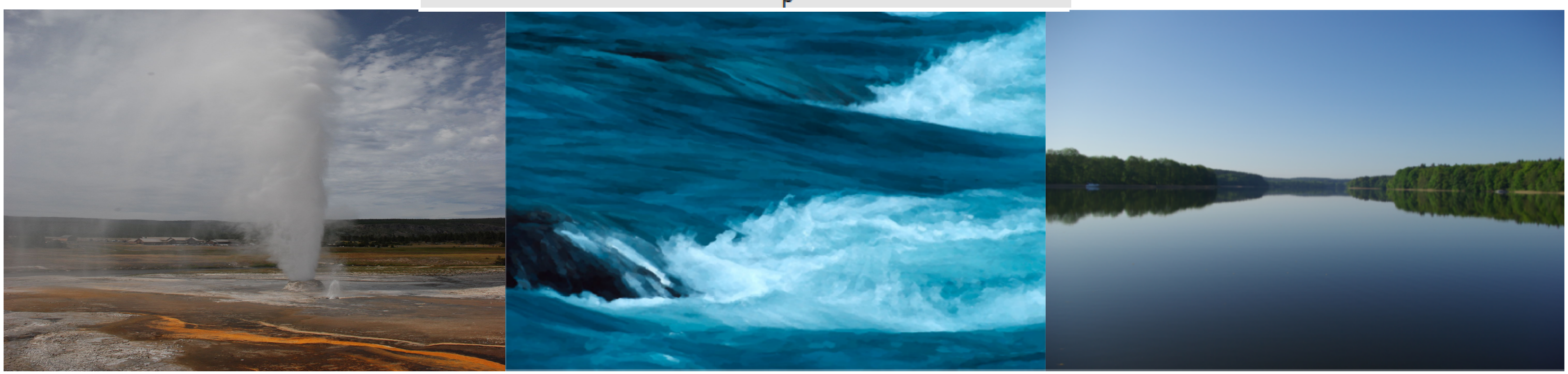
Summary

$$n(s\vec{p}) = s^{-\kappa} n(\vec{p})$$



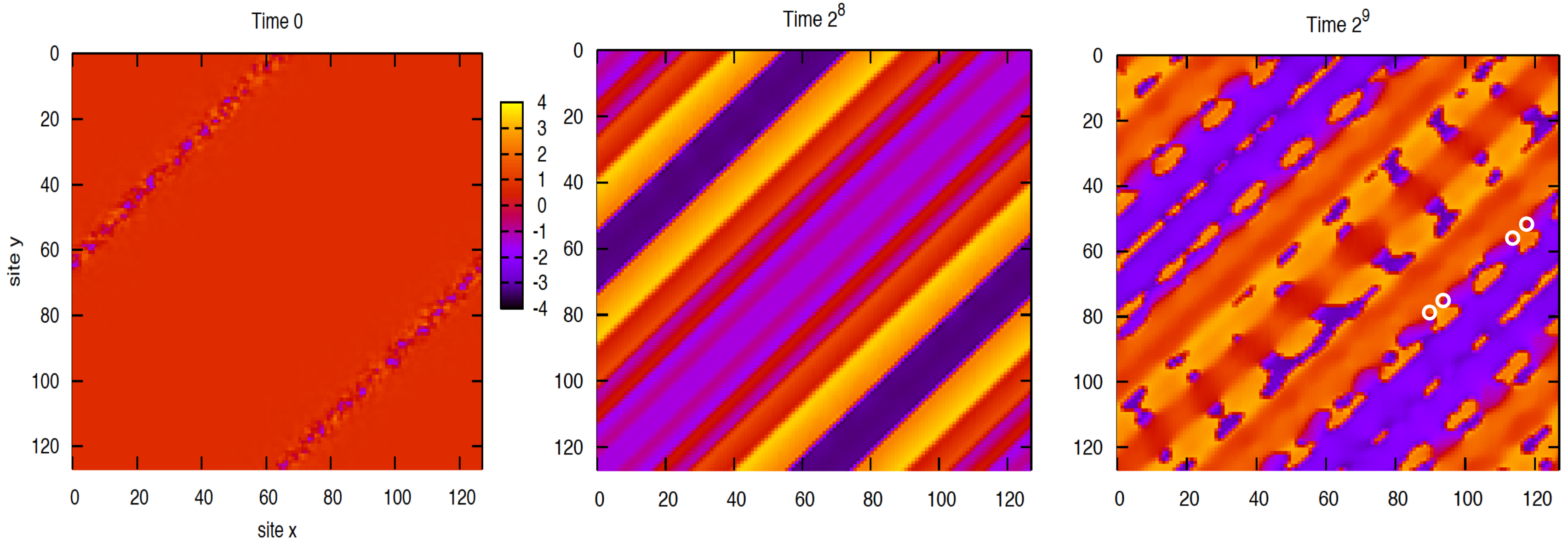






B. Nowak, D. Sexty, TG (unpublished)

Simulations in 2+1 D (semi-classical)



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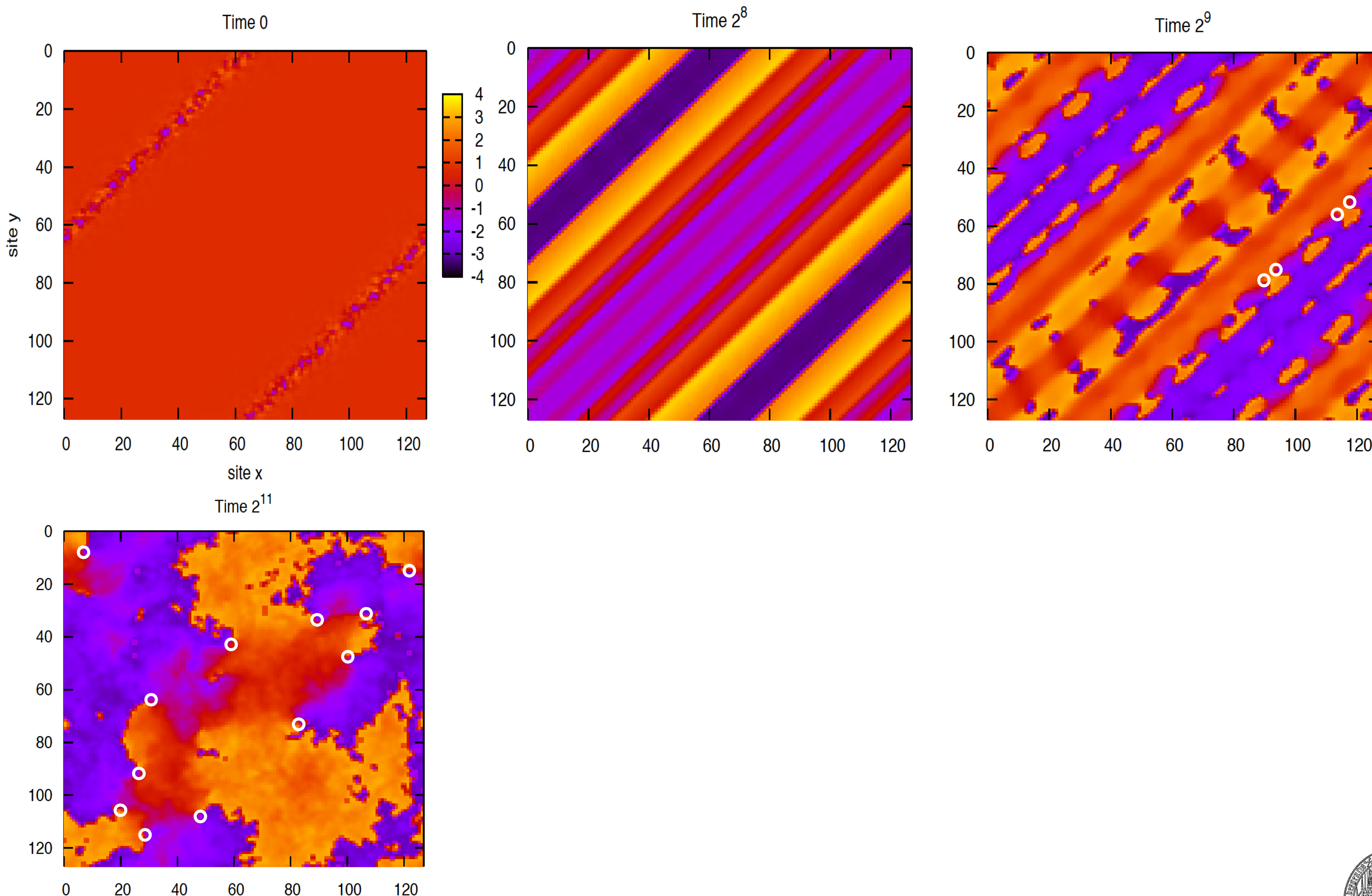

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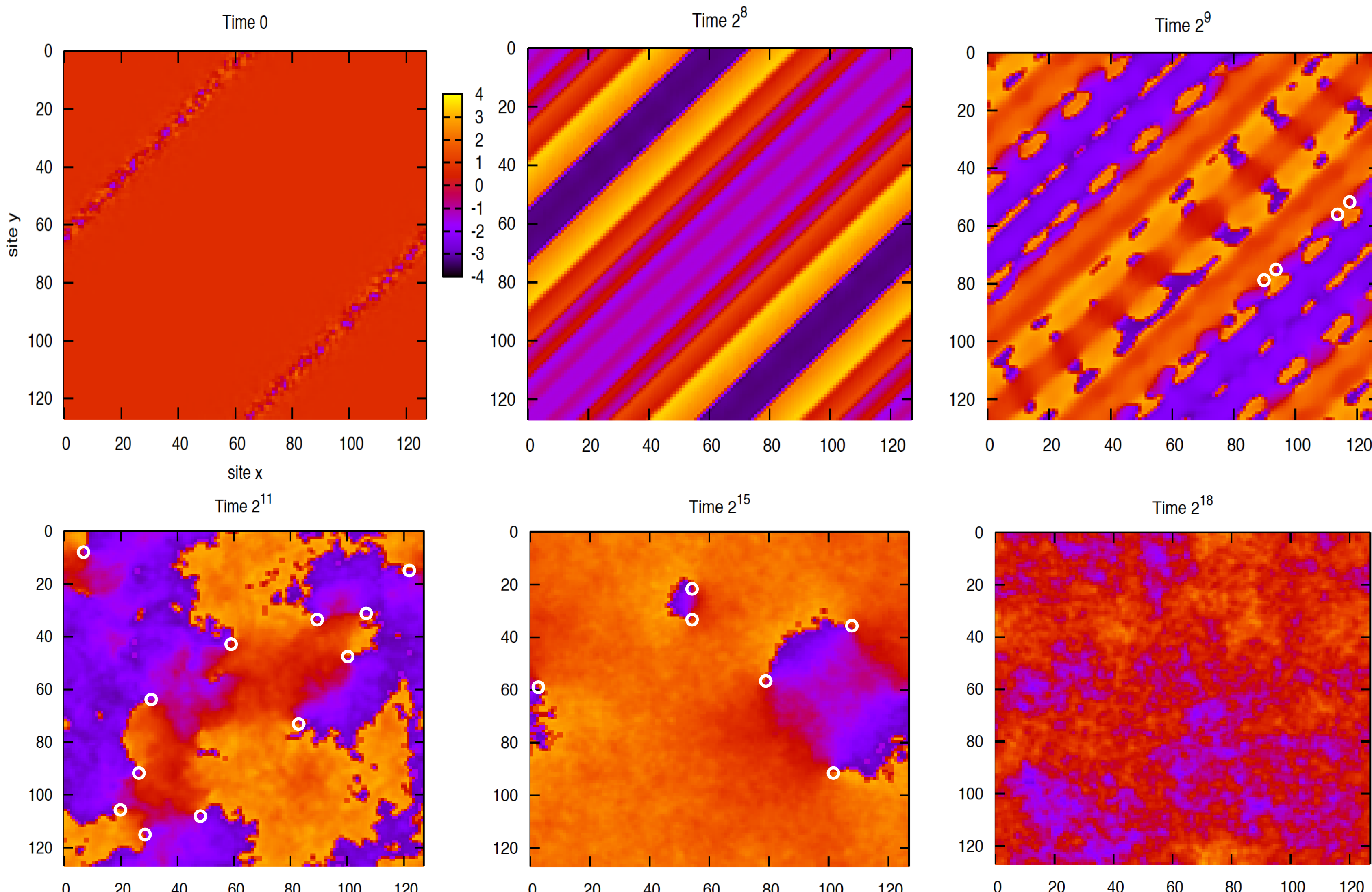

Lewis Fry Richardson, FRS (1881-1953)

Big whirls have little whirls that feed on their velocity,
and little whirls have lesser whirls and so on to viscosity.

(L.F. Richardson, *The supply of energy from and to Atmospheric Eddies*, 1920)

Great fleas have little fleas upon their backs to bite 'em,
And little fleas have lesser fleas, and so ad infinitum.
And the great fleas themselves, in turn, have greater fleas to go on;
While these again have greater still, and greater still, and so on.

(Augustus de Morgan, *A Budget of Paradoxes, 1872, p. 370*)

So, naturalists observe, a flea
Has smaller fleas that on him prey;
And these have smaller still to bite 'em;
And so proceed ad infinitum.

(Jonathan Swift: *Poetry, a Rhapsody*, 1733)