Renormalization-group studies of cold fermion atoms and BEC mixtures

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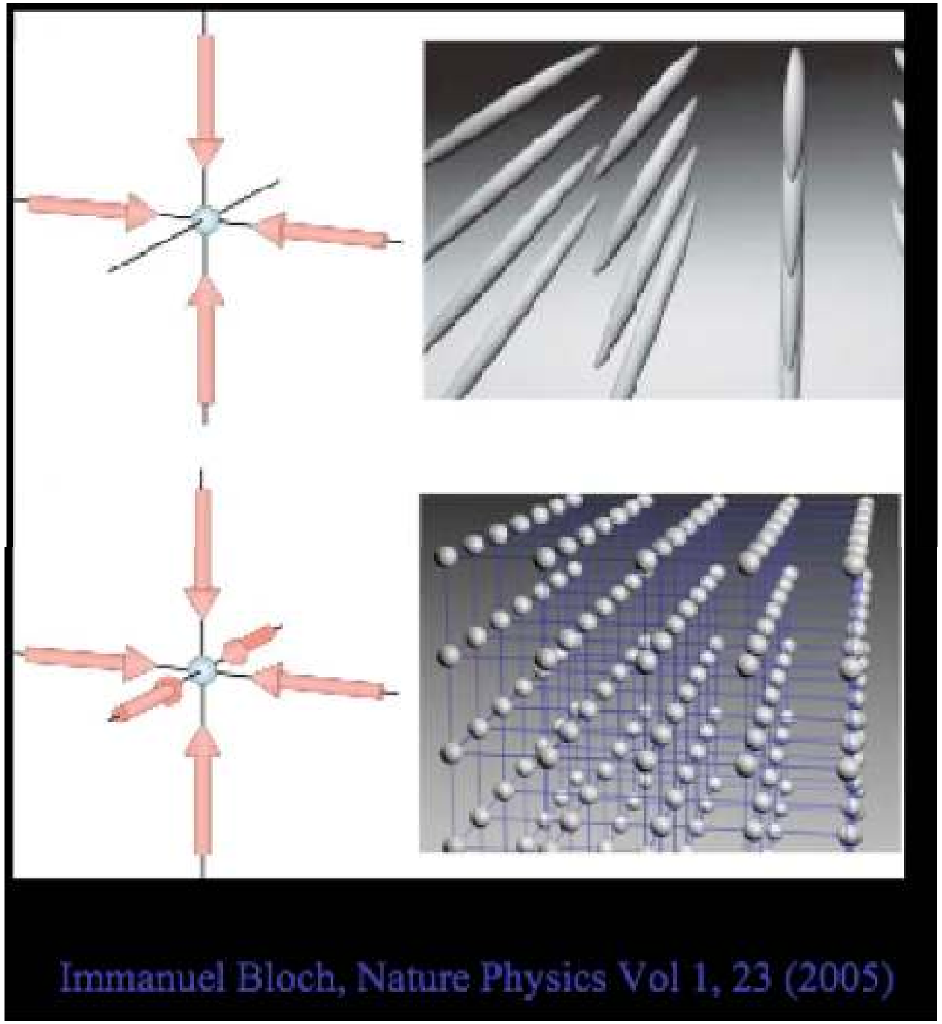
(UC - Riverside)

Thanks: NSF, UC-Lab Fees Research Program

ERG2010 - Corfu, Greece, September 2010

Artificial lattices: cold atoms in optical lattices



Expmt: M. T. de Pue *et al*., PRL **82**, 2262 (1999); M. Greiner *et al*., PRL **87**, 160405 (2001); M. Greiner *et al*., Nature **415**, 39 (2002); T. Storferle *et al*., PRL **92**, 130403 (2004), etc.

Approx. as a Hubbard model: D. Jaksch *et al*., PRL **81**, 3108 (1998).

This talk:
- neglect finite-size effects
- use simplified models
- consider fermions + boson BEC

$$V = V_0[\sin^2(kx) + \sin^2(ky) + \sin^2(kz)] \;+\; \text{magnetic trap}$$

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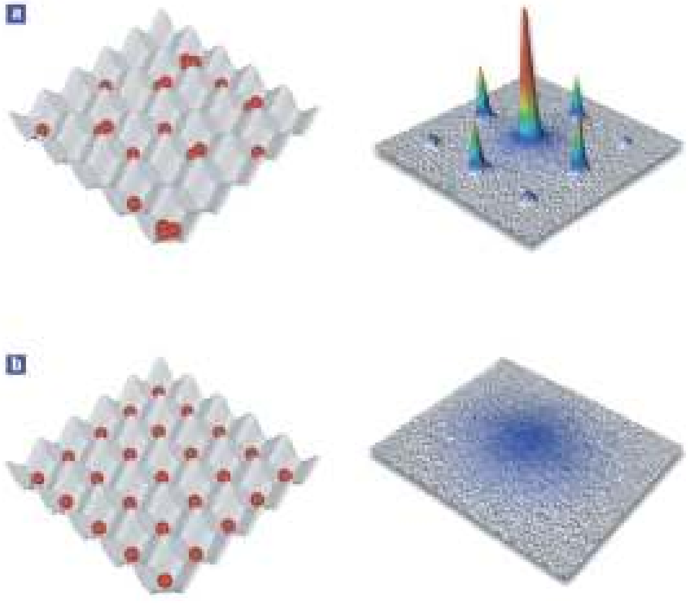
QuickTime™ and a
TIFF (LZW) decompressor
are needed to see this picture.

This talk:
- neglect finite-size effects
- use simplified models
- consider fermions + boson BEC

$$V = V_0\left[\sin^2(kx) + \sin^2(ky) + \sin^2(kz)\right] \text{ + magnetic trap}$$

Bosons in artificial lattices:



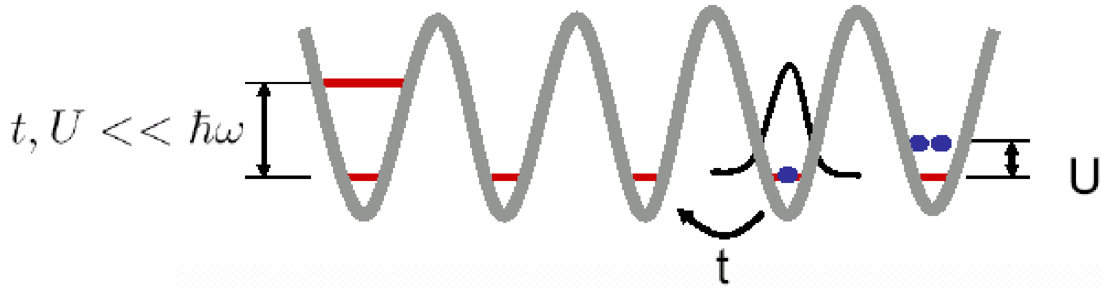
Rubidium-87 (^{87}Rb) = 37p + 37e + 50n = boson

• two-components: O. Mandel *et al.*, Nature **425**, 937 (2003)

^{87}Rb: $|\uparrow\rangle$ = $|$ F = 1, m_F = -1 $\rangle$

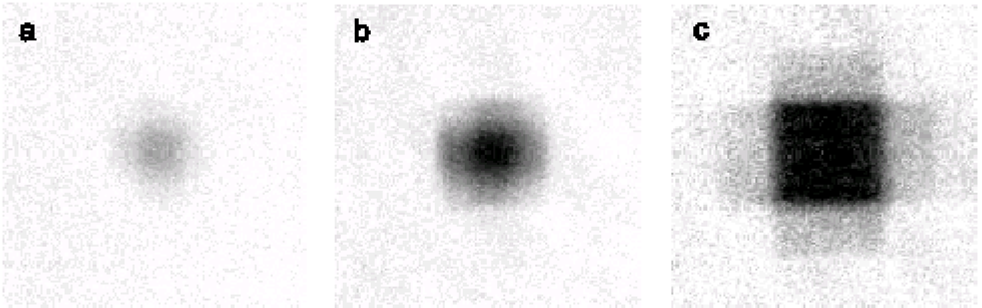
$|\downarrow\rangle$ = $|$ F = 1, m_F = -2 $\rangle$

• Bosonic Hubbard model: J. Hubbard, Proc. R. Soc. Lond. **A276**, 238 (1963).



$$\mathcal{H} = -t \sum_{\langle ij \rangle} b_i^\dagger b_j + U \sum_i n_i^2 - \mu \sum_i n_i$$

Fermions in artificial lattices:



G. Modugno *et al*., PRA **68**, 011601 (2003);
M. Kohl, *et al*., PRL 95, 080403 (2005)

"Snapshots of the Fermi surface"

Stoferle *et al*., cond-mat/0601045, ETH group

Bose-Fermi mixtures in artificial lattices:

PRL **97,** 120403 (2006) PHYSICAL REVIEW LETTERS week ending 22 SEPTEMBER 2006

Tuning of Heteronuclear Interactions in a Degenerate Fermi-Bose Mixture

S. Ospelkaus, C. Ospelkaus, L. Humbert, K. Sengstock, and K. Bongs
Institut für Laserphysik, Luruper Chaussee 149, 22761 Hamburg, Germany

Modugno *et al*., Science **297**, 2240 (2002);
Goldwin *et al*., PRA **70**, 021601 (2004);
Ospelkaus *et al*.,PRL 96, 020401 (2006), …

Fermi-Bose mixtures in artificial lattices:

- Interacting many-body system where the microscopic Hamiltonian is well known and can be controlled and tuned experimentally.

- The system can be loaded on lattices with different geometries, in different dimensions.

- There are no problems with cut-offs (IR ~ 1/L, UV ~ 1/a)

- Use RG approach to obtain effective low-energy theories and determine the phase diagram for some limits of this problem.

Fermi-Bose Hubbard model:

$$H = -t_f \sum_{\langle ij \rangle, s} f_{i,s}^{\dagger} f_{j,s} - t_b \sum_{\langle ij \rangle} b_i^{\dagger} b_j - \sum_i (\mu_f n_{f,i} + \mu_b n_{b,i}) + \sum_i \left[U_{ff} n_{f,i,\uparrow} n_{f,i,\downarrow} + \frac{U_{bb}}{2} n_{b,i} n_{b,i} + U_{bf} n_{b,i} n_{f,i} \right]$$

- eg: ^{40}K - ^{87}Rb, ^{6}Li – ^{7}Li
- Fermi-Bose Hubbard model: contact interactions only (U_{ff}, U_{bb}, U_{fb})
- Hopping terms t_f, t_b. Dispersion relations: $\epsilon_{b/f,\mathbf{k}} = -2t_{b/f}(\cos k_x + \cos k_y)$
- Square lattice, two species of fermions, neglect finite-size effects
- Will focus on the limit where the bosonic atoms have condensed (Santamore, Gaudio, Timmermans 04, Santamore & Timmermans 05, 08, Wang et al 05, ...) and study the resulting correlations for the fermions.

- **Even with contact interactions only, the system can develop effective long-range interactions leading to: charge order, spin order, BCS pairing in s-wave, p-wave, d-wave.**

Focus on special limit where bosonic atoms are condensed.

D.-W. Wang et al., PRA 72, 051604 (2005).

Bosons: BEC + fluctuations

$$\hat{b}_{\vec{k}=0} \to \sqrt{N_B}$$

$$\hat{b}^{\dagger}_{\vec{k}=0} \to \sqrt{N_B}$$

$$\hat{n}^B_{\vec{q}} \simeq \sqrt{N_B}(\hat{b}^{\dagger}_{\vec{q}} + \hat{b}_{-\vec{q}}), \quad (\vec{q} \neq 0)$$

Bogoliubov modes:

$$\begin{aligned} \hat{b}_{-\vec{q}} &= u_q \hat{a}_{-\vec{q}} + v_q \hat{a}^{\dagger}_{\vec{q}} \\ \hat{b}^{\dagger}_{\vec{q}} &= u_q \hat{a}^{\dagger}_{\vec{q}} + v_q \hat{a}_{-\vec{q}} \end{aligned}$$

$$\omega_{\mathbf{k}} = \sqrt{\epsilon_{b,\mathbf{k}}(\epsilon_{b,\mathbf{k}} + 2U_{bb}n_b)}$$

$$v_b = \sqrt{2t_b U_{bb} n_b} \qquad \xi = \sqrt{t_b / 2n_b U_{bb}}$$

Similar to electron-phonon problem in crystals.

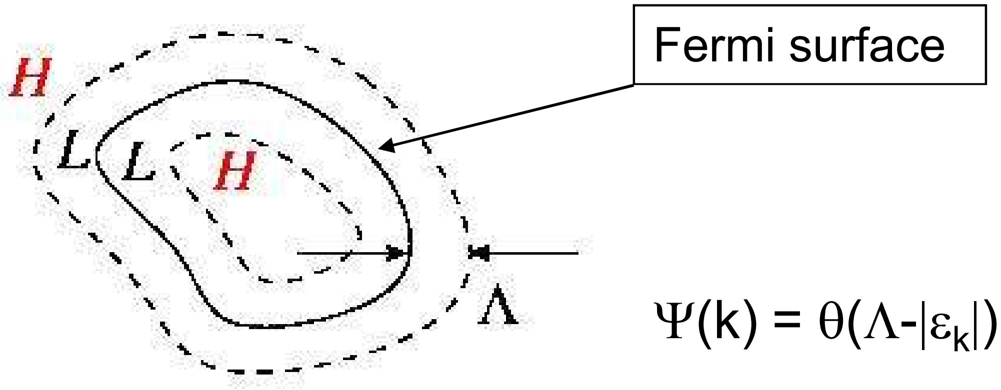
$$V_{ind,\mathbf{k}} = -\tilde{V}/(1 + \xi^2(4 - 2\cos k_x - 2\cos k_y))$$

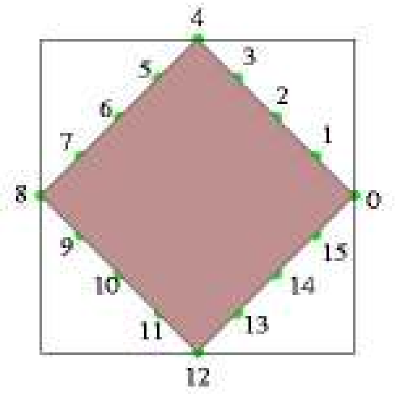
for $v_b >> v_F$

$$\tilde{V} = U_{bf}^2 / U_{bb}$$

L. Mathey, SWT, A. H. Castro Neto, PRL 97, 030601 (2006)

Renormalization-group calculation



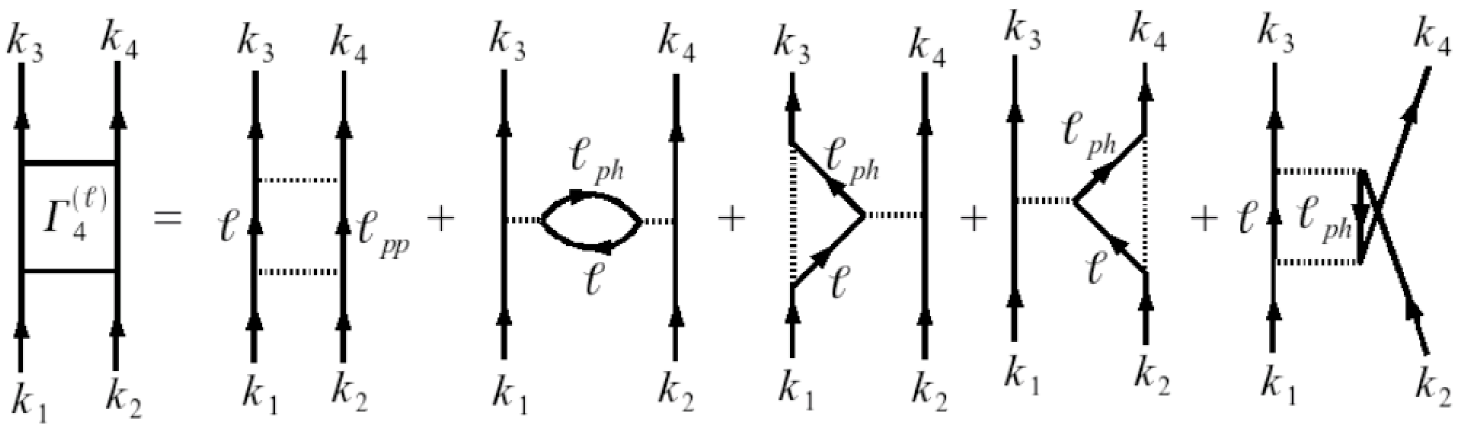


$$\Psi(k) = \theta(\Lambda-|\varepsilon_k|)\,\Psi_L(k) + \theta(|\varepsilon_k|-\Lambda)\,\Psi_H(k)$$

$$S_\Lambda(\Psi_L) = S(\Psi_L) + \Omega_H + \delta S(\Psi_L)$$

Correction to the four-point vertex (Polchinski equation):

Polchinski '84



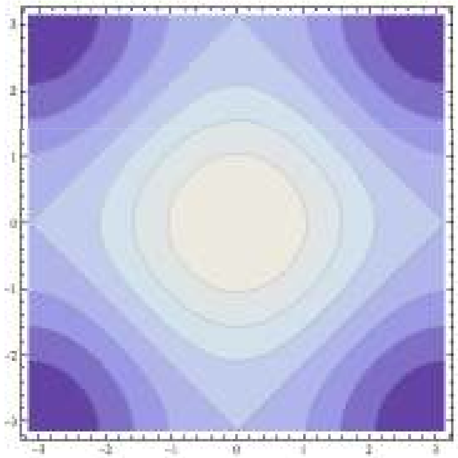
Feldman & Trubowitz '90, '91, Benfatto & Gallavotti '90, Shankar '91, Polchinski '92,...

Interplay of: on-site instantaneous repulsive interaction U_{ff}

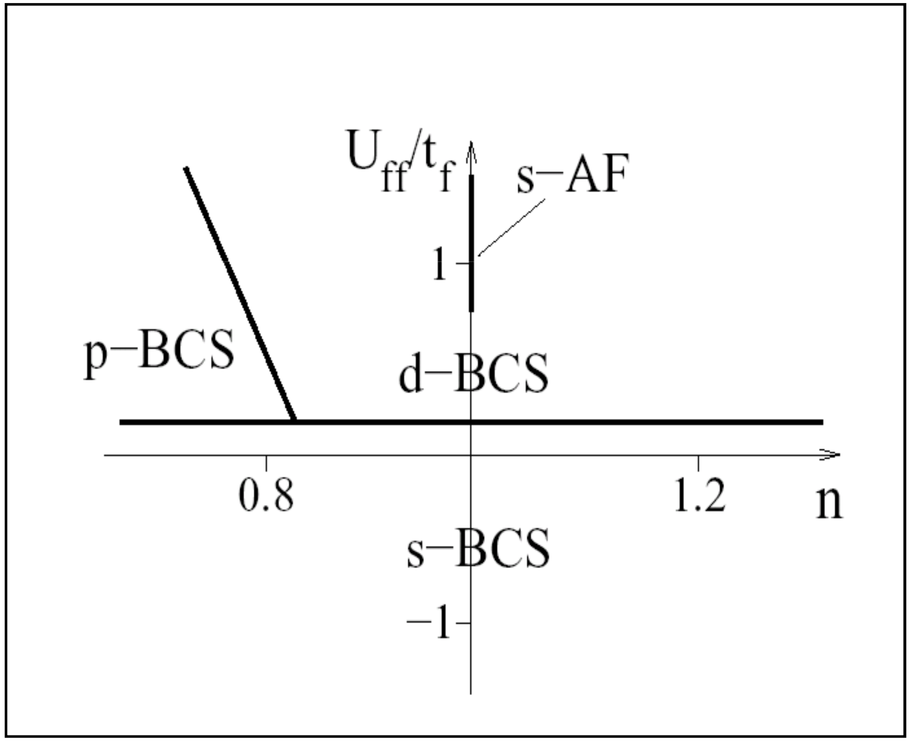
+

long-range retarded attractive interaction V_{ind}

• the phase diagram ($v_b >> v_f$):

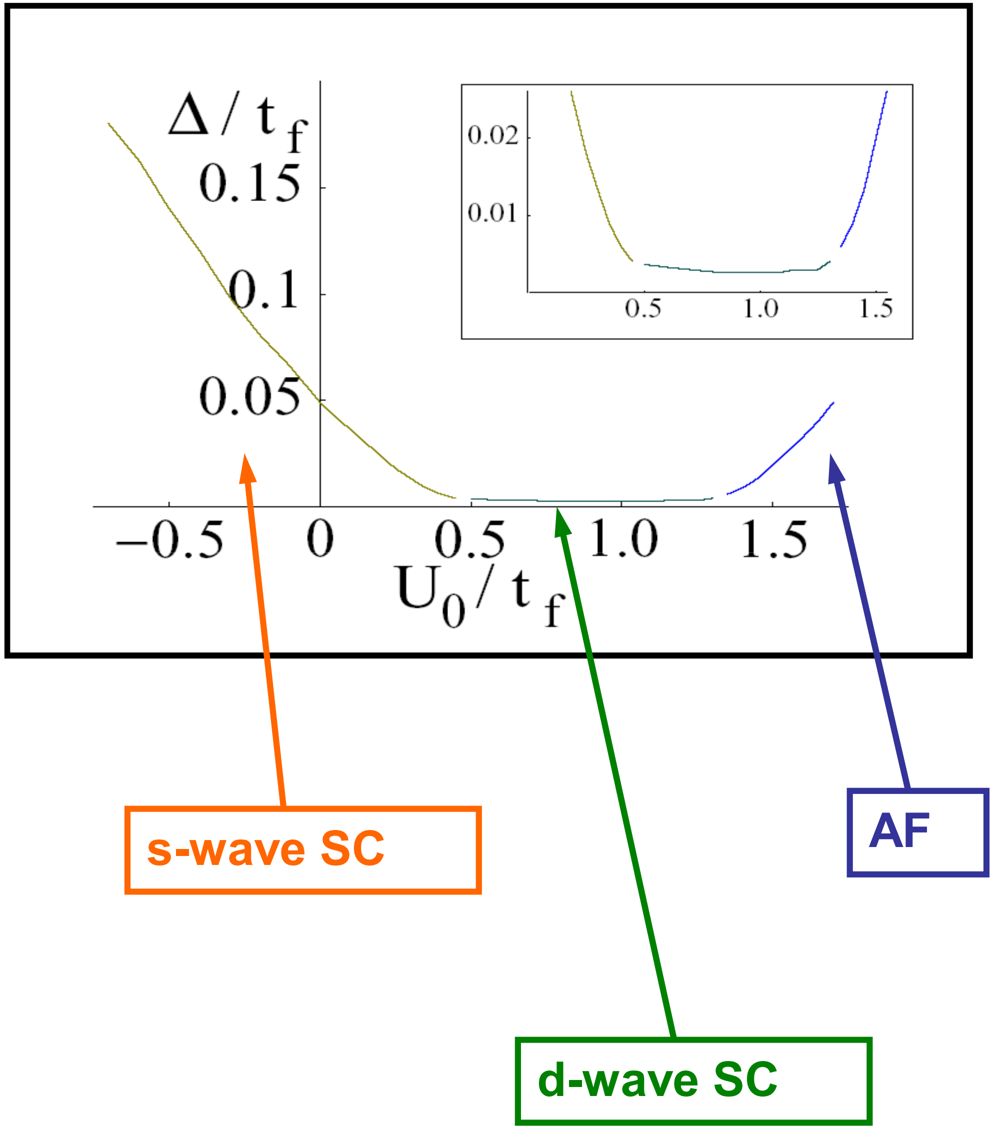


Fermi surfaces for a square lattice.

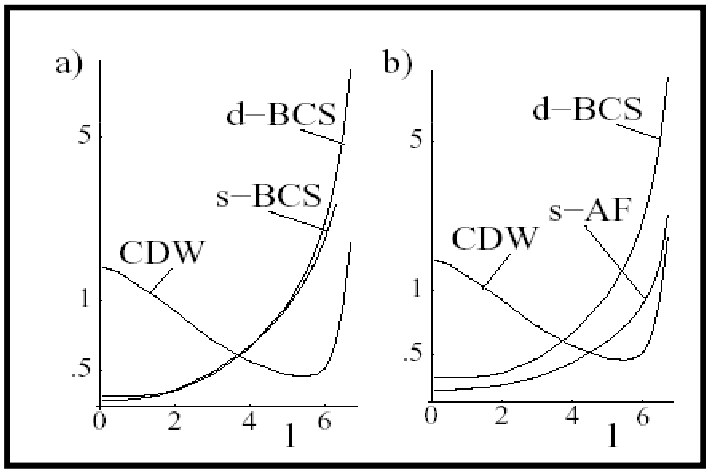


L. Mathey, SWT, A. H. Castro Neto,
PRL 97, 030601 (2006)

• Instability gaps:



• subdominant correlations:



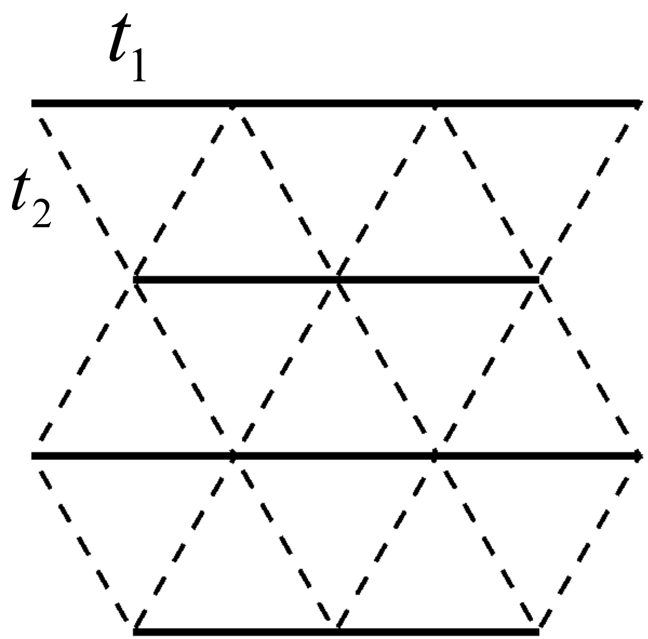
CDW at short length scales

Boson-Fermion Hubbard model:

L. Mathey, et al, PRB 2007

$$H = -\sum_{\langle ij\rangle_{n=1,2},s} t_{f,n} f^\dagger_{i,s} f_{j,s} - \sum_{\langle ij\rangle_{n=1,2}} t_{b,n} b^\dagger_i b_j$$
$$-\sum_i (\mu_f n_{f,i} + \mu_b n_{b,i}) + \sum_i \left[U_{ff} n_{f,i,\uparrow} n_{f,i,\downarrow} + \frac{U_{bb}}{2} n_{b,i} n_{b,i} + U_{bf} n_{b,i} n_{f,i} \right]$$

Anisotropic triangular model (n=1,2)



Fermion isospin: s = ↑,↓

Interactions: U_{ff}, U_{bb}, U_{bf}

Interaction effects + frustration

Bosons in the BEC regime: N_0

Fluctuations of the BEC:

$$\omega_{\mathbf{k}} = \sqrt{(\epsilon_{b,\mathbf{k}} - \epsilon_{b,0})(\epsilon_{b,\mathbf{k}} - \epsilon_{b,0} + 2U_{bb}n_b)}$$ → first order in $1/N_0$

$$\begin{aligned} v_{b,x} &= \sqrt{(2t_{b,1} + t_{b,2})U_{bb}n_b} \\ v_{b,y} &= \sqrt{3t_{b,2}U_{bb}n_b}. \end{aligned}$$

Assume:

$$\{v_{bx}, v_{by}\} >> \{v_{f1}, v_{f2}\}$$

→ retardation effects can be neglected in this limit

Effective Hamiltonian for the fermions:

$$H_{\text{eff.}} = \sum_{\mathbf{k}} \left\{ (\epsilon_{\mathbf{k}} - \mu_f) \sum_s f^{\dagger}_{\mathbf{k},s} f_{\mathbf{k},s} + \frac{U_{ff}}{V} \rho_{f,\mathbf{k},\uparrow} \rho_{f,-\mathbf{k},\downarrow} + \frac{1}{2V} V_{\text{ind.},\mathbf{k}} \; \rho_{f,\mathbf{k},} \rho_{f,-\mathbf{k}} \right\}$$

Free fermion dispersion:

$$\varepsilon_k = -2t_{f1} \cos k_x - 2t_{f2} \left[\cos\left(\frac{k_x}{2} + \frac{\sqrt{3}k_y}{2} \right) + \cos\left(\frac{k_x}{2} - \frac{\sqrt{3}k_y}{2} \right) \right]$$

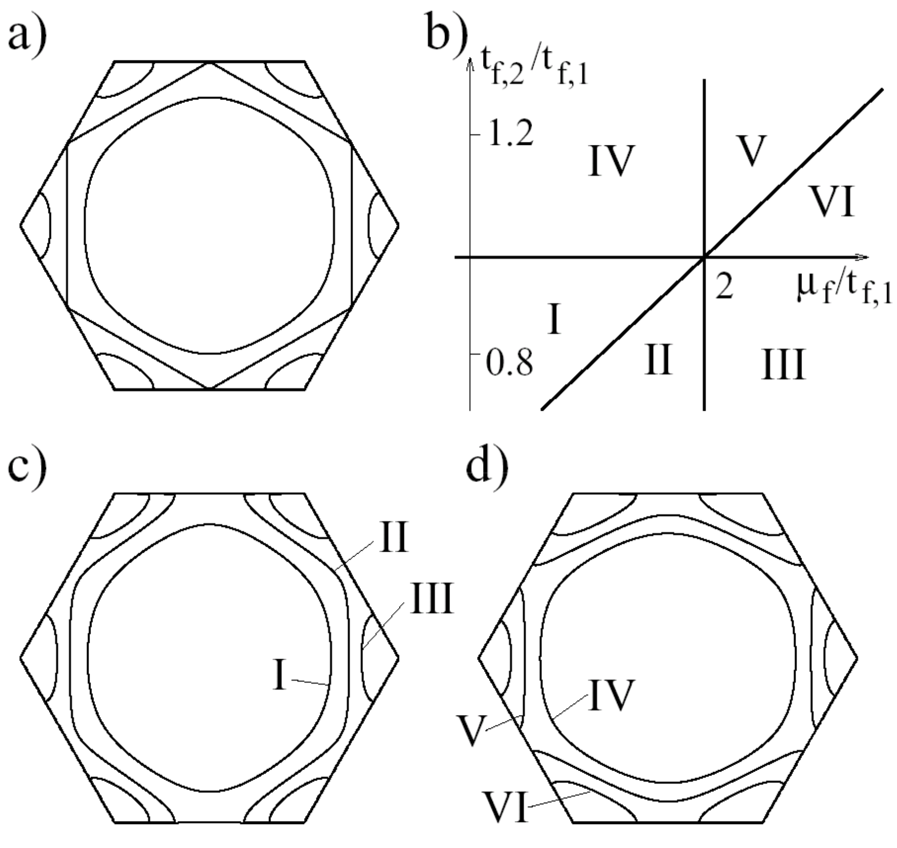
Short-range repulsive interaction: U_{ff}

Long-range attractive interaction:

$$V_{\text{ind.},\mathbf{k}} = -\tilde{V} / (1 + \xi_1^2 (2 - 2\cos k_x) + \xi_2^2 (4 - 4\cos(k_x/2)\cos(\sqrt{3}k_y/2)))$$

where: $\tilde{V} = \dfrac{U_{bf}^2}{U_{bb}}, \xi_n = \sqrt{\dfrac{t_{bn}}{2n_b U_{bb}}}, n = 1,2$

Fermi surfaces (free fermions):



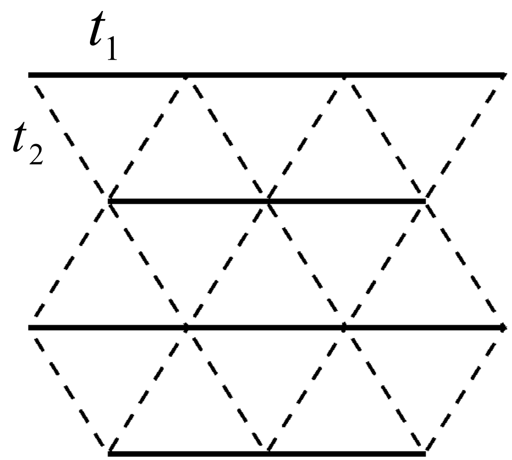
1→2→6 **1→4→6**

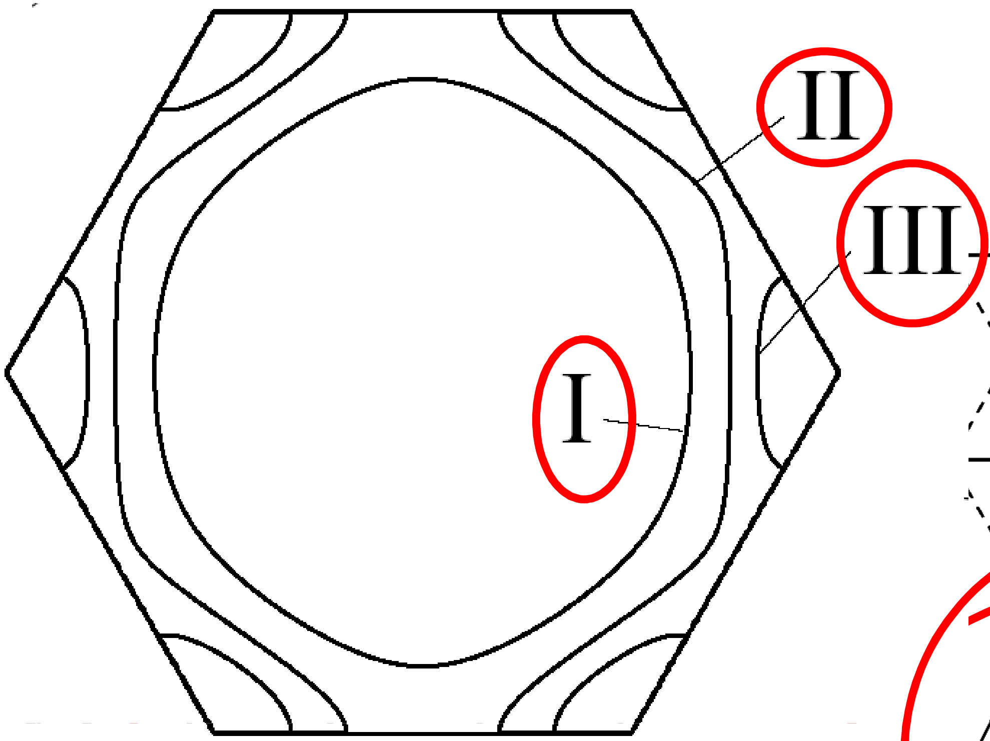
→ study interaction effects using a renormalization-group method

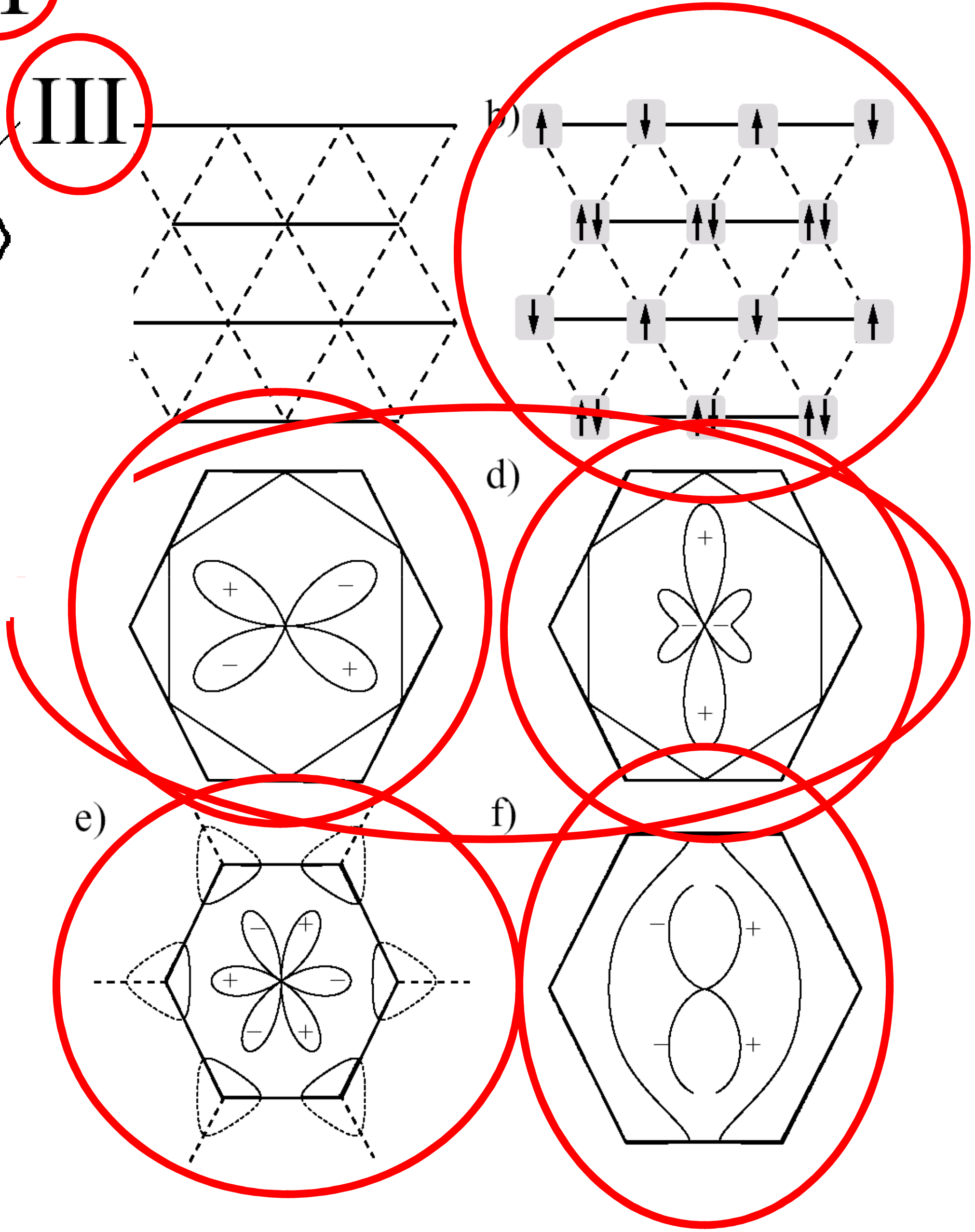
→ discretize the Fermi surface

→ calculate the flow of all interaction processes around the Fermi surface

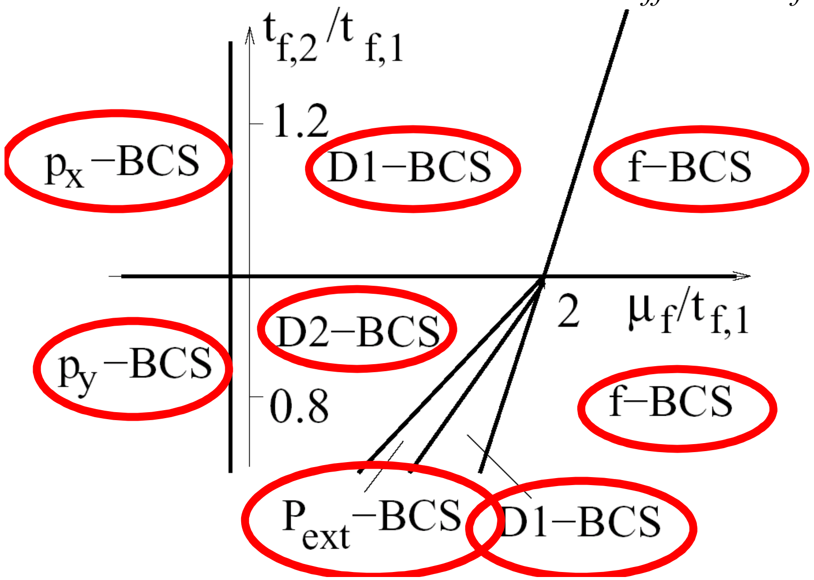
→ from the dominant (most divergent) couplings/susceptibilities determine the instability







anisotropic case: $(U_{ff} = 3t_f)$



Interacting fermions coupled to bosonic modes:

$$S(\psi,\phi) = S_e(\psi) + S_{ph}(\phi) + S_{e\text{-}ph}(\psi,\phi) + S_{e\text{-}e}(\psi)$$

$$S_e = \int_{\omega\mathbf{k}} \psi_k^{\dagger\sigma}(i\omega - \epsilon_{\mathbf{k}})\psi_{k\sigma}$$

$$S_{ph} = \int_{\Omega\mathbf{q}} \phi_q^{\dagger}(i\Omega - w_{\mathbf{q}})\phi_q$$

$$S_{e\text{-}ph} = \int_{\omega\mathbf{k}}\int_{\Omega\mathbf{q}} g(q)\psi_{k+q}^{\dagger\sigma}\psi_{k\sigma}(\phi_q + \phi_{-q}^{\dagger})$$

$$S_{e\text{-}e} = \frac{1}{2}\prod_{i=1}^{3}\int_{\omega_i\mathbf{k}_i} u(k_4,k_3,k_2,k_1)\psi_{k_4}^{\dagger\sigma}\psi_{k_2\sigma}\psi_{k_3}^{\dagger\sigma'}\psi_{k_1\sigma'}$$

$k \equiv (\omega_n, \mathbf{k})$

$k_1 + k_2 = k_3 + k_4$

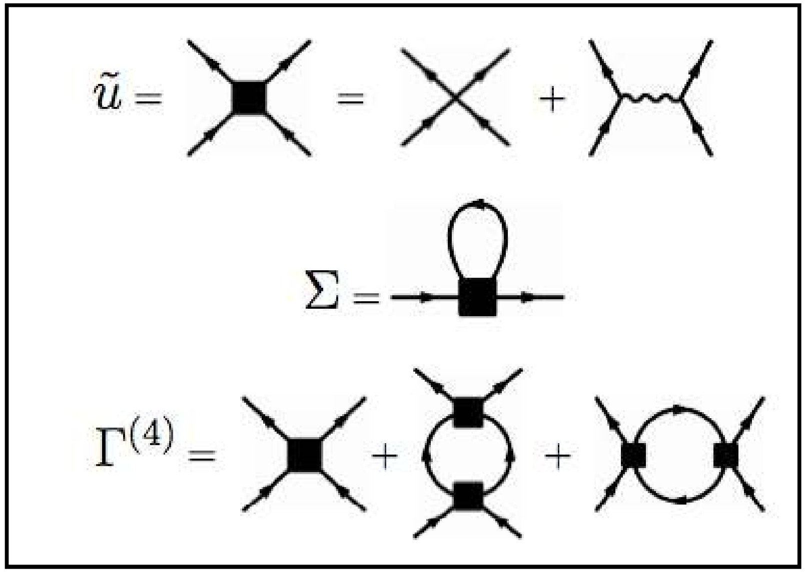
Bosons can be integrated out exactly:

$$U_0(k_1,k_2,k_3) = u_0(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) - 2g(k_1,k_3)g(k_2,k_4)D(k_1-k_3)$$

$$\lambda = 2N(0)g^2/\omega_E$$

SWT, AH Castro Neto, R Shankar, DK Campbell, PRB **72**, 054531(2005)

Circular Fermi surface, isotropic interaction, BCS channel:



matrix equation: $U_{ij} = U(\omega_i, \omega_j)$

$$\frac{d\mathbf{U}}{d\ell} = -\mathbf{U}\cdot\mathbf{M}\cdot\mathbf{U}$$

Exact solution:

$$\begin{aligned}\mathbf{U}(\ell) &= [1+\mathbf{U}(0)\cdot\mathbf{P}(\ell)]^{-1}\,\mathbf{U}(0)\\ \mathbf{P}(\ell) &= \int_0^\ell d\ell'\mathbf{M}(\ell')\,.\end{aligned}$$

Coupling diverges at $\ell = \ell_c$, where:

$$\det\left[1+\mathbf{U}(0)\cdot\mathbf{P}(\ell_c)\right] = 0$$

which is equivalent to:

$$[1+\mathbf{U}(0)\cdot\mathbf{P}(\ell_c)]\cdot\mathbf{f} = 0$$

→ **gives ELIASHBERG's equations at $T=T_c$**

Weak-intermediate coupling
McMillan, '68

$$T^* \approx 1.13\,\omega_E \exp\{-(1+\lambda)/(\lambda-\mu^*(1+\lambda))\}$$

Strong Coupling
Allen-Dynes, '75

$$T^* \approx 0.16\sqrt{\lambda}\omega_E$$

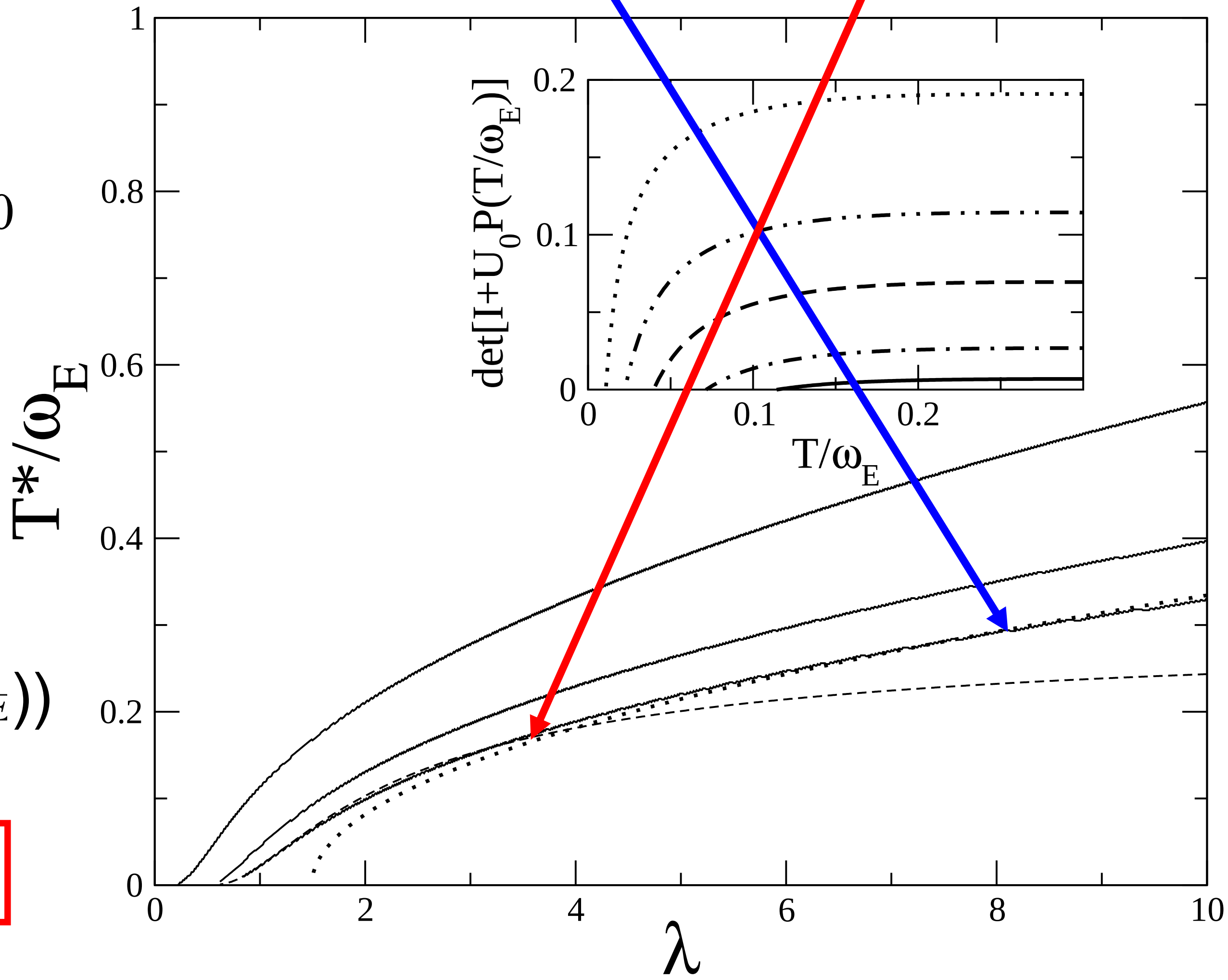
T* calculated from:

det [**1** + **U**(0). **P**(T*/ω_E)] = 0

$$\lambda = 2N(0)g^2/\omega_E$$

Effective e-e interaction:

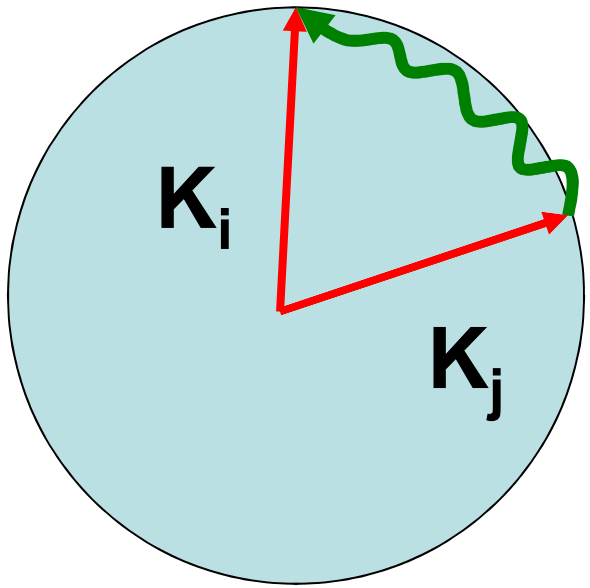
$$\mu^* = u_0/(1+u_0\ln(\Lambda_0/\omega_E))$$



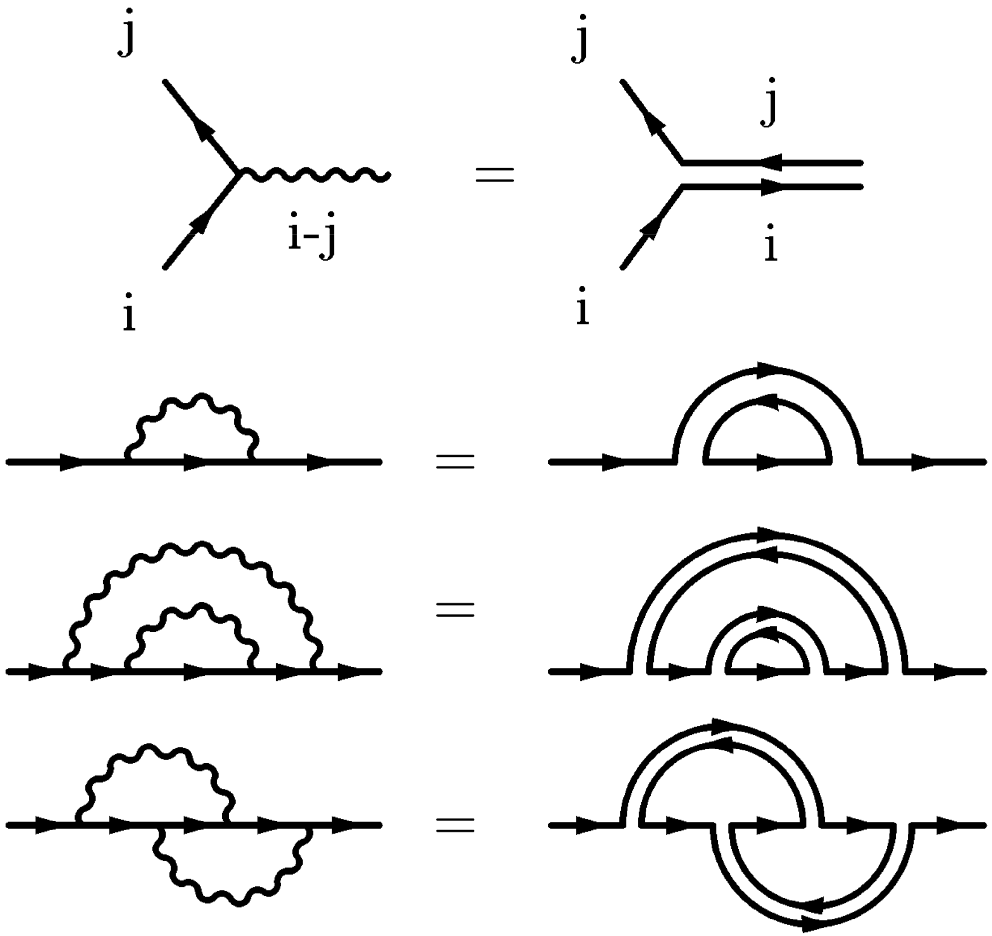
SWT, AH Castro Neto, R Shankar, DK Campbell, PRB **72**, 054531(2005)

Migdal's theorem ('58)

$$u \sim g^2 \sim 1/N$$



t'Hooft 1974



$\propto \; g/\sqrt{N}$

$\propto \; \frac{g^2}{N} N$

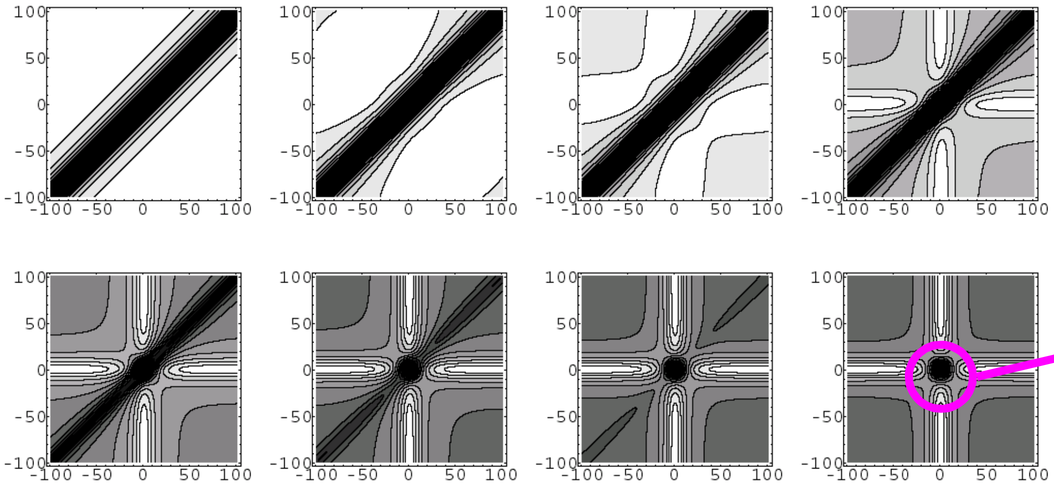
$\propto \; g^4$

Self-energy

$\propto \; g^4/N^2$

No e-ph vertex corrections

RG evolution of the couplings in the BCS channel (λ = 0.3, 4.0)

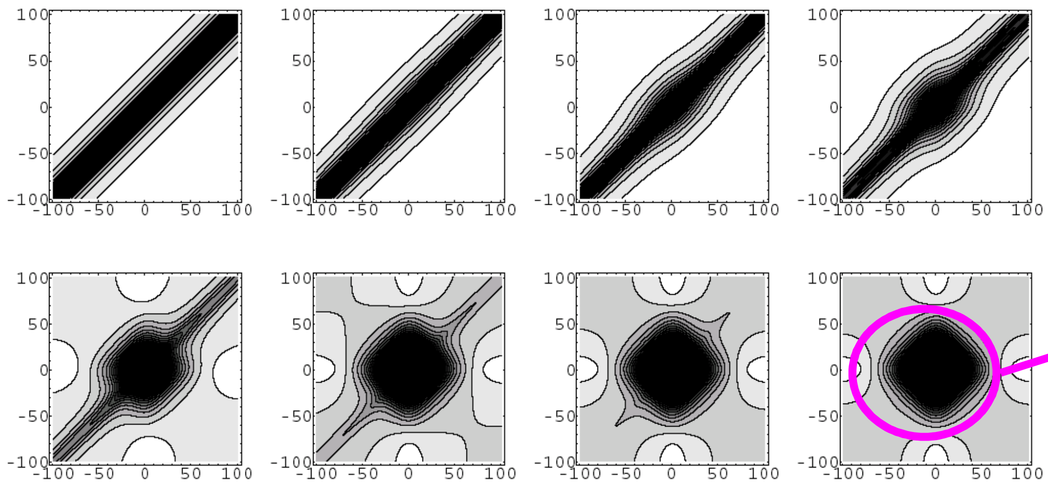


“weak-coupling”
λ < 1

~ ω_E

“two-step” RG is OK

Figure 1: Plots of the NxN matrix U at different RG scales ℓ. Here the number of frequency divisions $N=200$, and the value of the parameters used are $\lambda=0.3$, $\Lambda_0=100$, $\omega_E=10$, $u_0=0.1$. Panels correspond to ℓ = 0, 2.5, 3, 5, 6.5, 6.9, 7.1, and 7.19.



“strong-coupling”
λ > 1

~ T_c

SWT *et al.*, Philos. Mag. **86**, 2631 (2006)

Figure 2: Plots of the NxN matrix U at different RG scales ℓ. Here the number of frequency divisions $N=200$, and the value of the other parameters are $\lambda=4$, $\Lambda_0=100$, $\omega_E=10$, and $u_0=0.1$. Panels correspond to ℓ = 0, 1, 2, 2.5, 3, 3.13, 3.157, 3.172. The scale $2W_c \approx 40$ distinguishes the high and low frequencies close to ℓ_c.

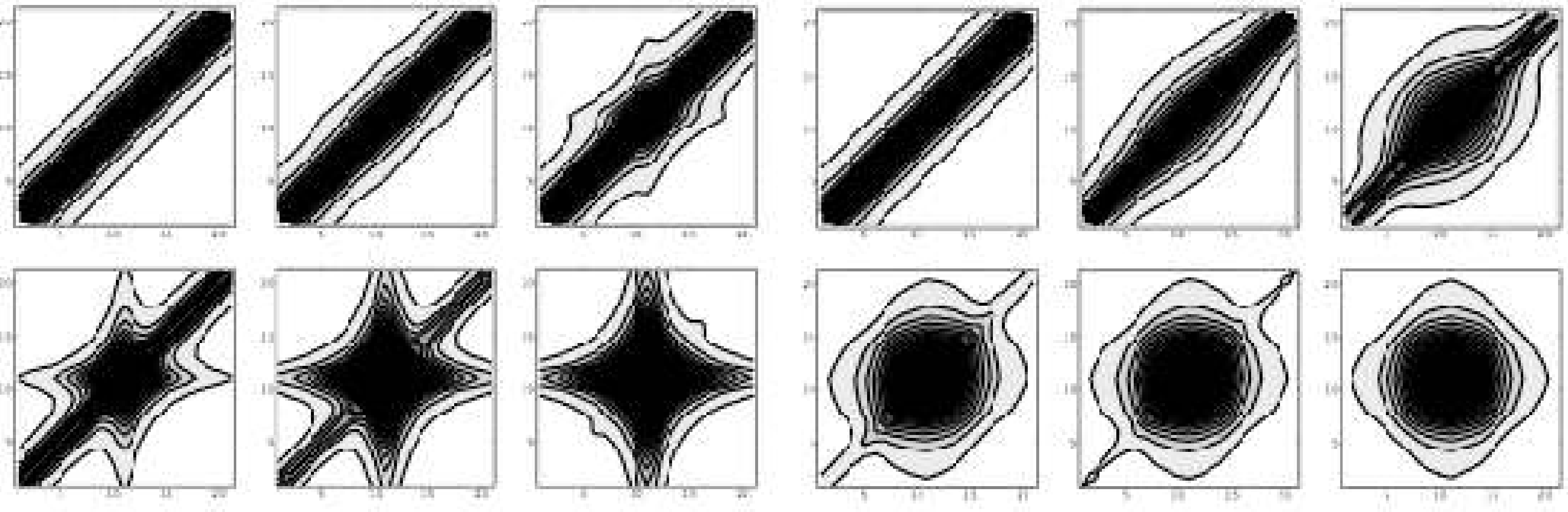
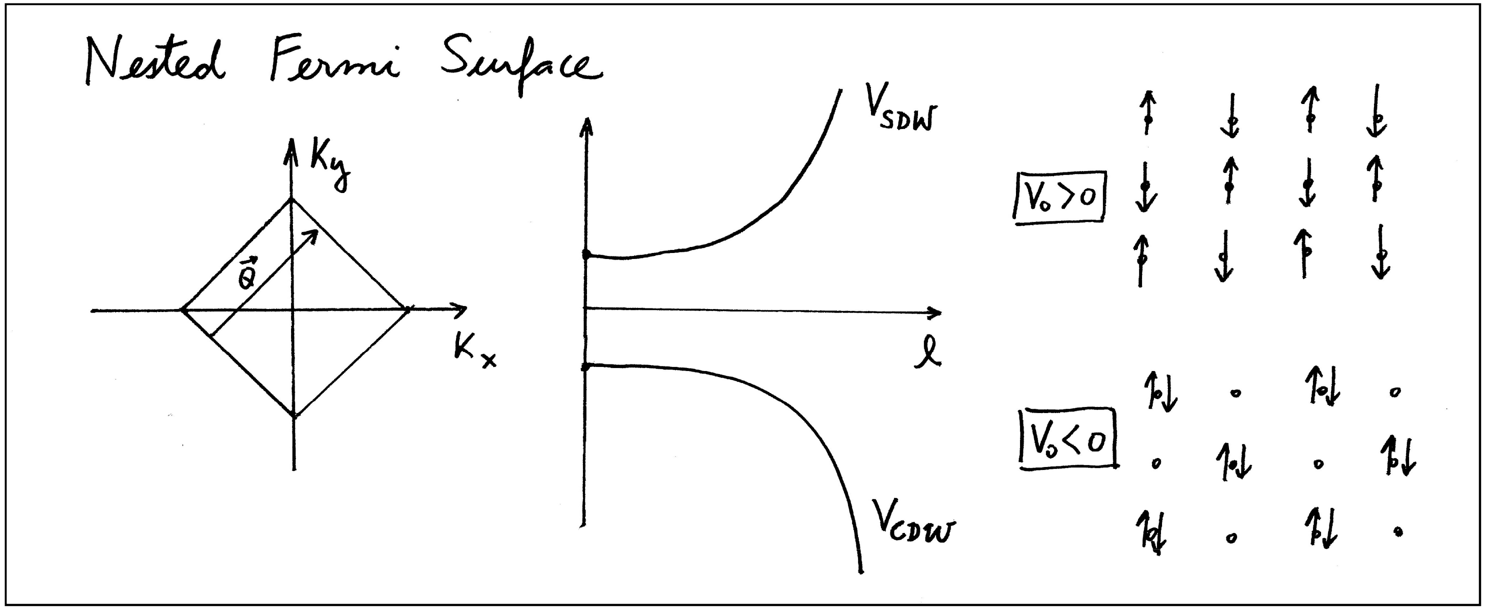


FIG. 2: Same as Fig.[1] but for the $d_{x^2-y^2}$-channel, $\gamma = 2$. The six panels on the left side are the evolution of the matrix elements at weak coupling, $\lambda = 0.4$ and the panels on the right side are the same but for the strong coupling regime, $\lambda = 4.0$

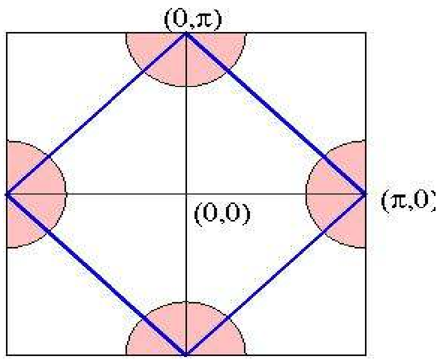
R. Roldan, SWT, MP Sancho-Lopez, PRB 2009

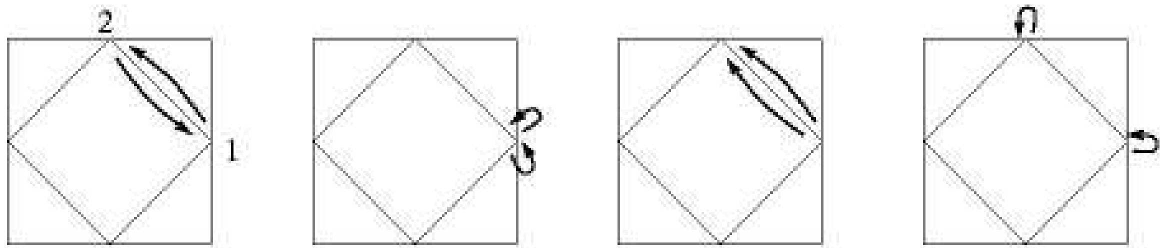
Square lattice at half-filling (Hubbard U only):



Two-patch model for van Hove problem:

H. Schulz, Europhys. Lett. (1987)





$g_1 = u(1221)$ $g_2 = u(1111)$ $g_3 = u(2211)$ $g_4 = u(1212)$

Fermions with spin:

van Hove problem with only Hubbard U has been extensively studied, *e. g.,* J. Gonzalez, F. Guinea, and M. A. H. Vozmediano 1997, N. Furukawa, T. M. Rice and M. Salmhofer 1998, C. Honerkamp, M. Salmhofer, N. Furukawa, and T. M. Rice 2001, B. Binz, D. Baeriswyl, and B. Doucot, 2002, …

- What is the interplay between effects of nesting and phonons?
- Are phonons always pair-breaking in the d-wave superconducting channel?
- Can phonons and AF fluctuations cooperate to enhance T_c for d-wave superconducivity ?

$$\frac{\partial g_1}{\partial \ell} = -2g_1(g_1 - g_4),$$

$$\frac{\partial g_2}{\partial \ell} = -g_2^2 - g_3^2,$$

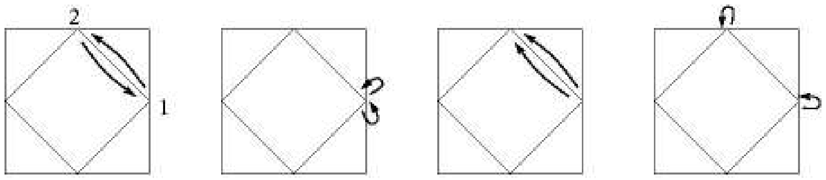
$$\frac{\partial g_3}{\partial \ell} = -2g_3(g_1 + g_2 - 2g_4),$$

$$\frac{\partial g_4}{\partial \ell} = g_3^2 + g_4^2.$$

$$\begin{aligned} u_{(\mathrm{sd})\mathrm{SC}} &= g_2 \pm g_3 \\ u_{(\mathrm{sd})\mathrm{SDW}} &= -(g_4 \pm g_3) \\ u_{(\mathrm{sd})\mathrm{CDW}} &= 2g_1 \pm g_3 - g_4 \end{aligned}$$

Go back to fermion-boson problem and include retardation effects:

$$\tilde{U}_{\mathrm{ff}}(k_1, k_2, k_3) = U_{\mathrm{ff}} - \frac{U_{\mathrm{fb}}^2/U_{\mathrm{bb}}}{1 + 4\xi^2 - 2\xi^2[\cos(\boldsymbol{k}_1 - \boldsymbol{k}_3)_x + \cos(\boldsymbol{k}_1 - \boldsymbol{k}_3)_y]} \frac{\omega_{\boldsymbol{k}_1-\boldsymbol{k}_3}^2}{(\omega_1 - \omega_3)^2 + \omega_{\boldsymbol{k}_1-\boldsymbol{k}_3}^2}$$



$g_1 = u(1221)$ $g_2 = u(1111)$ $g_3 = u(2211)$ $g_4 = u(1212)$

$$g_{1,3}(\omega_1, \omega_2, \omega_3) = U_{\mathrm{ff}} - \frac{U_{\mathrm{fb}}^2 8t_b^2/U_{\mathrm{bb}}\xi^2}{(\omega_1 - \omega_3)^2 + 8t_b^2(1 + 8\xi^2)/\xi^2}$$

$$g_{2,4}(\omega_1, \omega_2, \omega_3) = U_{\mathrm{ff}} - \frac{U_{\mathrm{fb}}^2}{U_{\mathrm{bb}}} \delta_{\omega_1,\omega_3}$$

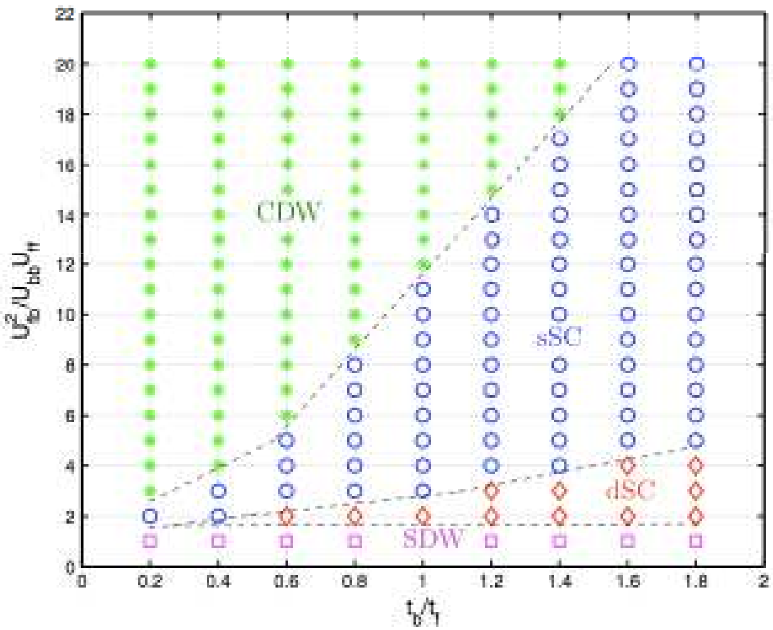
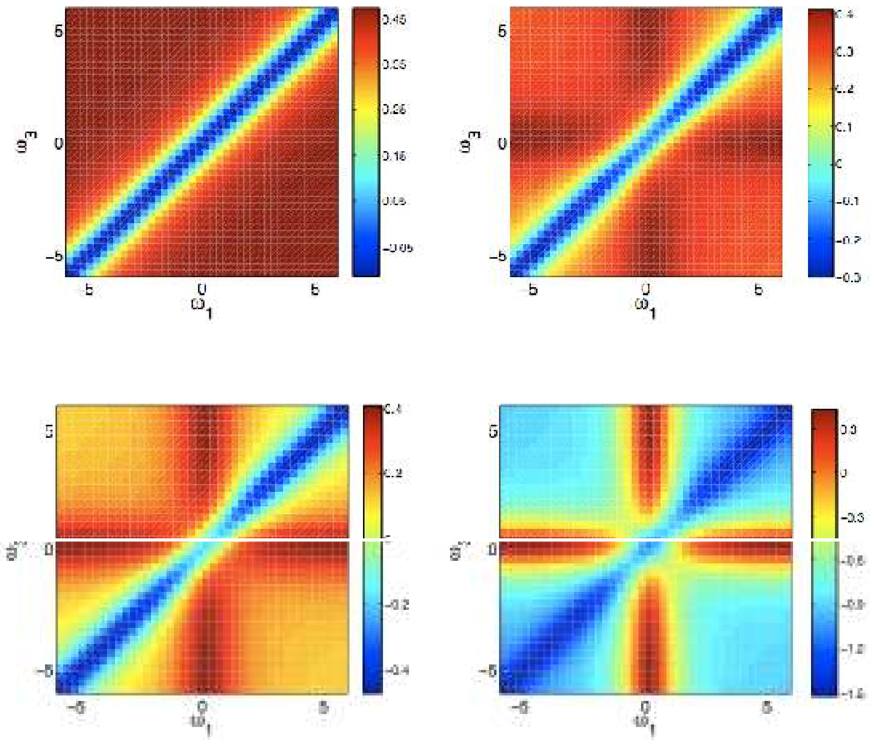


FIG. 1 (color online). Phase diagram for $U_{\mathrm{ff}} = 0.4t_f$, $U_{\mathrm{bb}} = 0.8t_f$, and $n_b = 2.5$. Blue circles indicate sSC, red rhombuses indicate dSC, magenta squares SDW, and green stars CDW type of ordering. Dashed lines are guides to the eye.

F. D. Klironomos & SWT, PRL (2007)

RG evolution of $g_2(\omega_1, -\omega_1, \omega_3, -\omega_3)$:



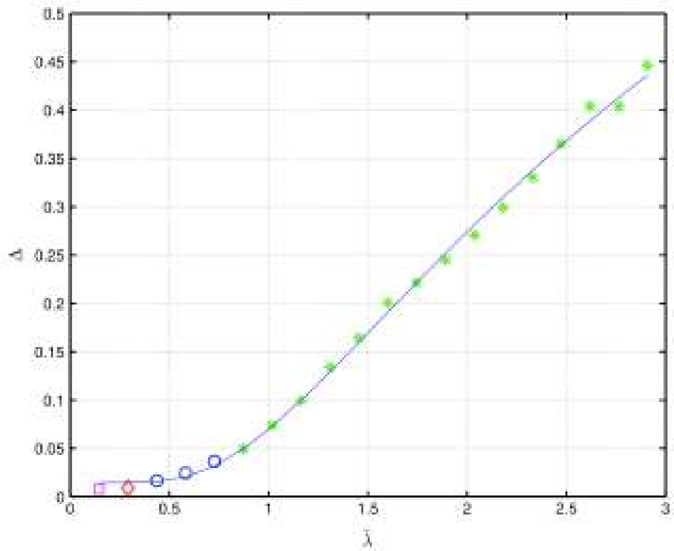


FIG. 3 (color online). Evolution of the gap along $t_b/t_f = 0.6$ of Fig. 1 with identical symbol scheme for the different orders. The blue line fitting was according to $\Delta = 0.015 + 3.326\exp[-(3.101 + \bar{\lambda})/\bar{\lambda}]$.

$$\bar{\lambda} = \frac{U_{\rm fb}^2}{2U_{\rm bb}}\left(1 + \frac{1}{1 + 8\xi^2}\right)$$

There can be dominant BCS pairings even at half-filling due to a separation of scales: a given coupling may have a different sign at low and high frequencies

F. D. Klironomos and SWT,
PRB **74**, 205109 (2006)

Summary:

Bose-Fermi mixtures in optical lattices

- *bosons in BEC state + fluctuations*
- *fermion BCS states*
- *retardation effect*

On-site repulsion + long-range fluctuation-mediated attraction + lattice geometry

- *exotic pairing symmetries*
- *phase diagrams*
- *subdominant orders*
- *estimate for the gaps*

+ retardation effects

- *change in critical energy scales*
- *possible new phases*

Collaborators:

Ludwig Mathey ***(NIST)***
Filippos Klironomos ***(UCR, now at U. Freiburg)***

David K. Campbell *(BU)*
Antonio H. Castro Neto *(BU)*
Maria Pilar Lopez Sancho *(ICMM, Madrid)*
J. Brad Marston *(Brown)*
Rafael Roldan *(UCR, now at U. Nijmegen)*
Ramamurti Shankar *(Yale)*
Ka-Ming Tam *(U. Waterloo)*

Kyle Irwin *(UCR)*
Ryan Kalas *(LANL)*
Nicolas Lopez *(UCR)*
Chuntai Shi *(UCR)*
Eddy Timmermans *(LANL)*
Ling Yang *(UCR)*

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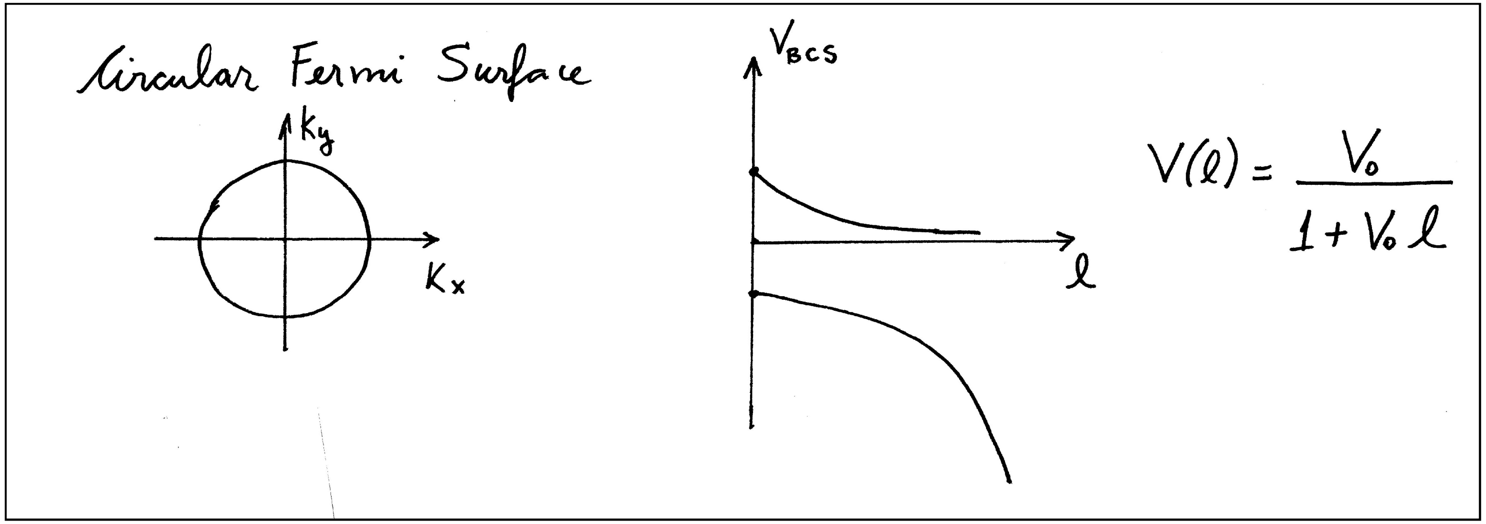
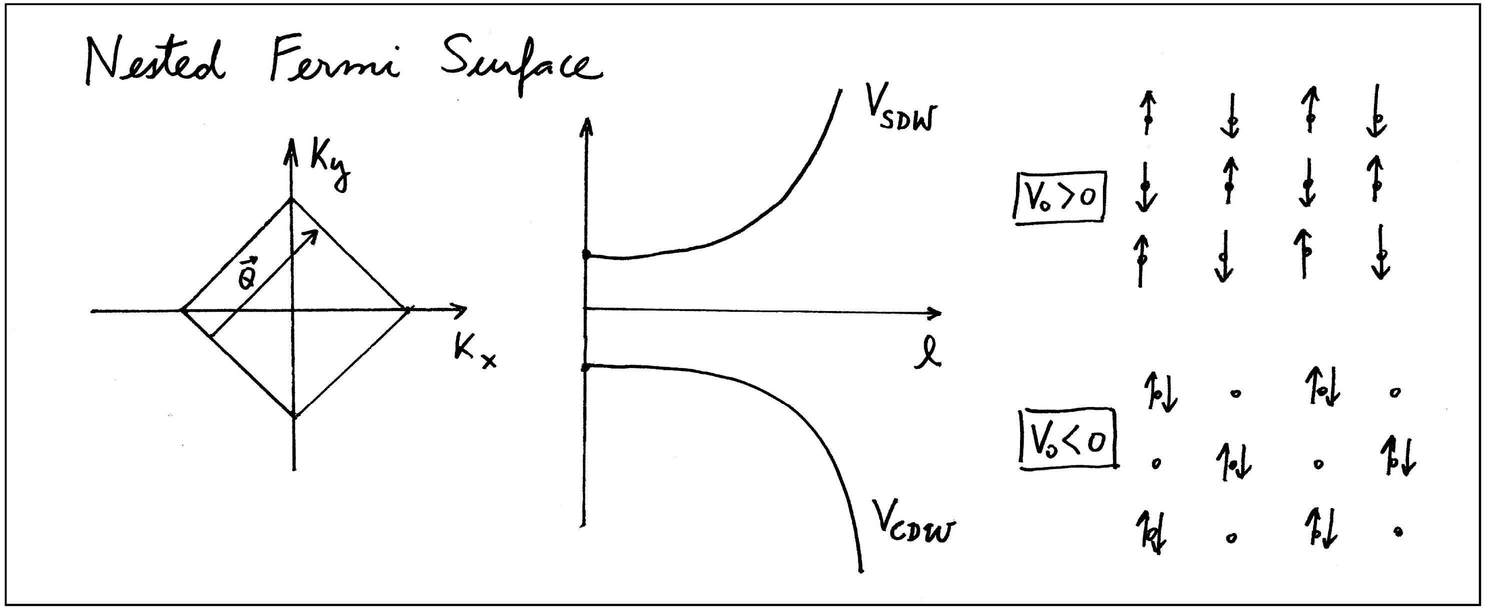
Maria Pilar Lopez-Sancho *(ICMM-CSIS, Madrid)*
Tun Wang *(IQOQI, U. Innsbruck)*

David K. Campbell *(BU)*
Antonio H. Castro Neto *(BU)*
Ramamurti Shankar *(Yale)*
J. Brad Marston *(Brown)*

$$\frac{d}{d\ell}\tilde{v} = -\tilde{v}^2$$





Allow for anisotropic phonons, calculate flow of susceptibilities:

$u_0 = 0.5,\ \omega_E = 1.0$

$\lambda_0 < \lambda_\pi$ $\lambda_0 > \lambda_\pi$

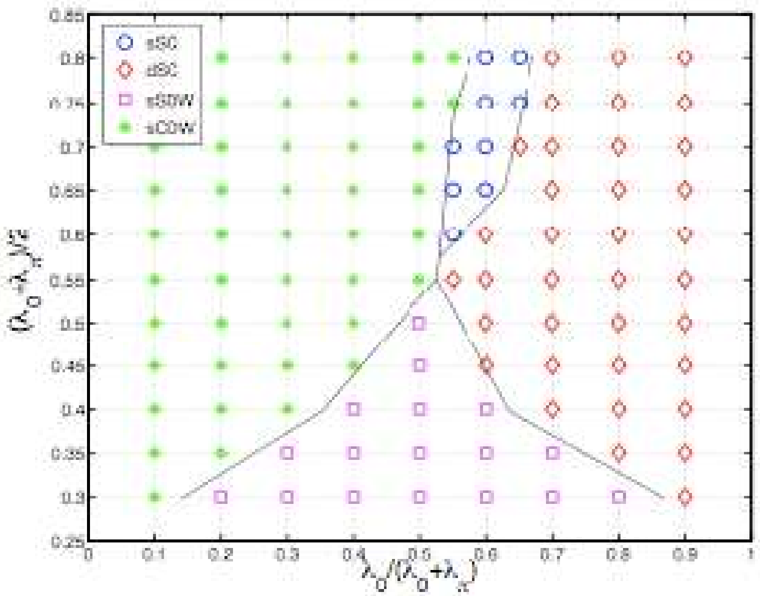


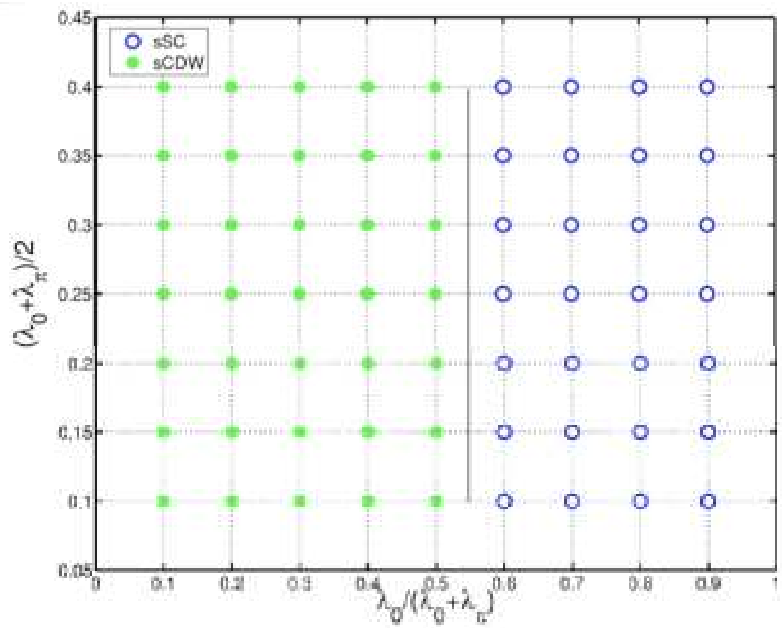
FIG. 1: (Color online) Phase diagram for Einstein phonons of frequency $\omega_E = 1.0$. Four phases involving antiferromagnetism (sSDW) (purple squares), charge density wave (sCDW) (green stars) and s-wave (sSC) (blue circles) and d-wave (dSC) (red rhombs) superconductivity compete in the vicinity where the average phononic strength $\bar{\lambda}$ approaches the bare on-site repulsion $u_0 = 0.5$. The lines distinguishing the different domains are guides to the eye.

$$g_{1,3}^{\ell=0}(\omega_1,\omega_2,\omega_3) = u_0 - \lambda_\pi \frac{\omega_E^2}{\omega_E^2 + (\omega_1-\omega_3)^2}$$

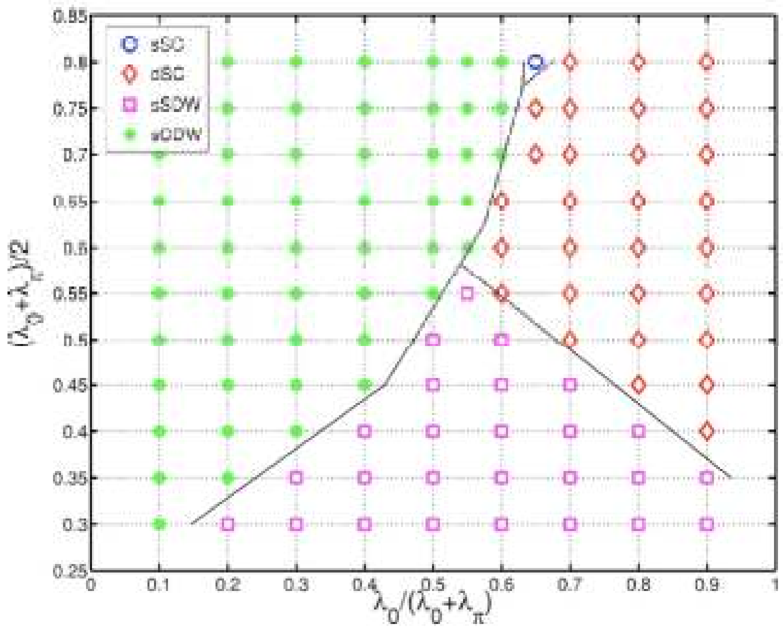
$$g_{2,4}^{\ell=0}(\omega_1,\omega_2,\omega_3) = u_0 - \lambda_0 \frac{\omega_E^2}{\omega_E^2 + (\omega_1-\omega_3)^2}$$

F. D. Klironomos and SWT, PRB **74**, 205109 (2006)

$u_0 = 0,\ \omega_E = 1.0$	$u_0 = 0.5,\ \omega_E = 0.1$



Need repulsive component for d-wave SC to develop.



Density-wave phases regions increase when ω_E is decreased.

F. D. Klironomos and SWT, PRB **74**, 205109 (2006)

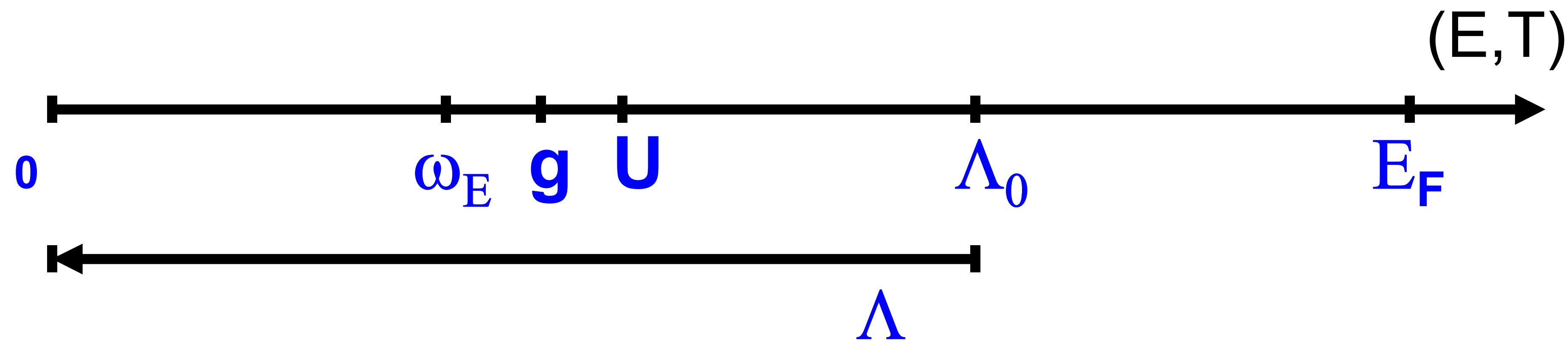
Eliashberg theory (‘60)

1) theory for strong-coupling superconductivity
2) mean-field theory
3) assumes a broken symmetry with SC gap $\Delta(\omega)$
4) provides self-consistent relations for the self-energy
5) assumes Migdal’s theorem (more on this later…)

→ this RG calculation is the first derivation of Eliashberg theory starting from the Fermi liquid state.

→ the RG is useful when the broken symmetry state is not known, or when there are several competing orders.

Electron-electron plus electron-phonon



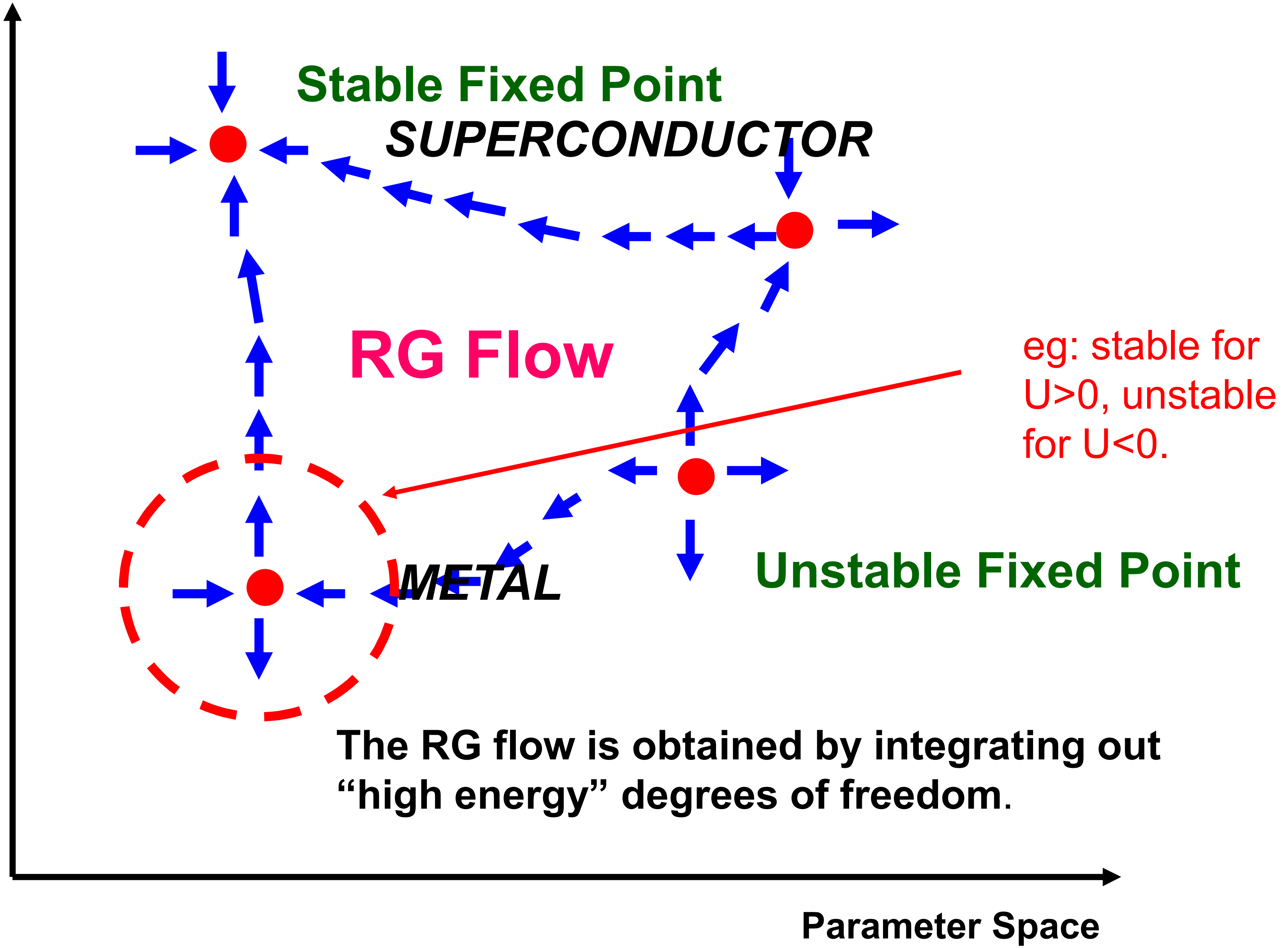
E_F: Fermi energy
g : electron-phonon
U : electron-electron
ω_E: phonon frequency

$$\lambda = 2N(0)g^2/\omega_E$$

$E_F >> \Lambda_0 >> \{g, U, \omega_E\}$

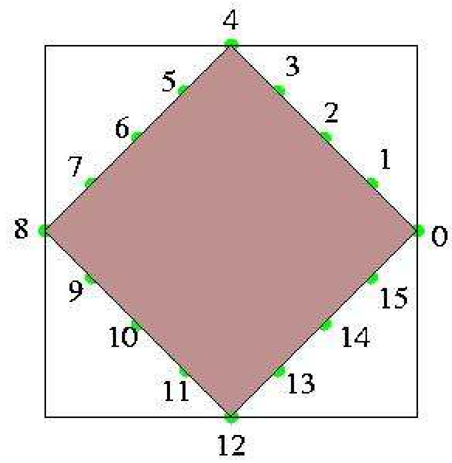
Large N: $N = \frac{E_F}{\Lambda}$

$N_0 = \frac{E_F}{\Lambda_0} >> 1$


Stable Fixed Point
SUPERCONDUCTOR
RG Flow
eg: stable for U>0, unstable for U<0.
METAL
Unstable Fixed Point
The RG flow is obtained by integrating out “high energy” degrees of freedom.
Parameter Space

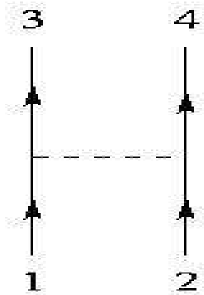
- discretization of the Fermi surface

For m=16 patches, there are m^3 = 4096 coupled differential equations (at most).



$$\frac{dU_l(i_1, i_2, i_3)}{dl} = \sum_{i=0}^{m} C_{pp}(i_1, i_2, i) U_{l_{pp}}(i_1, i_2, i) U_{l_{pp}}(i_3, i_4, i) + \ldots$$

- relate U's with CDW, AF, BCS, etc:

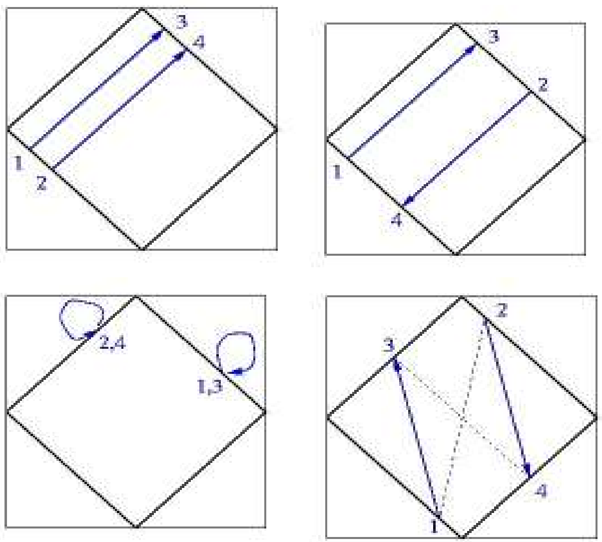


$V^{CDW}(i, j) = 4\, U_C(i, j, \underline{i}\square)$

$V^{AF}(i, j) = 4\, U_S(i, j, \underline{i}\square)$

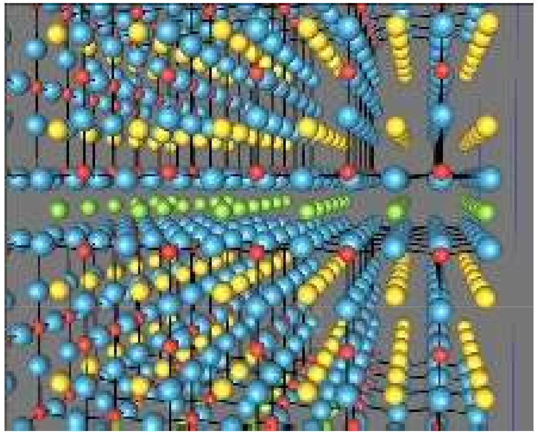
$F(i, j) = U_c(i, j, i)$

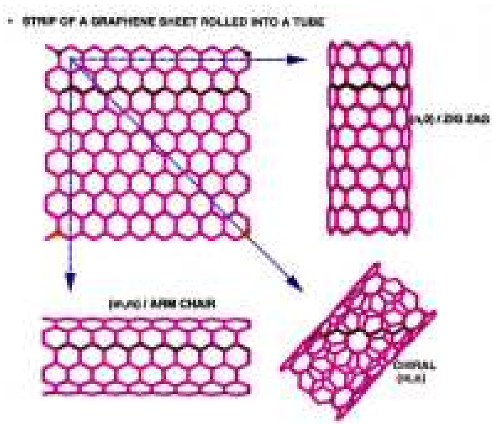
$V^{BCS}(i, j) = U_c(i, -i, j)$

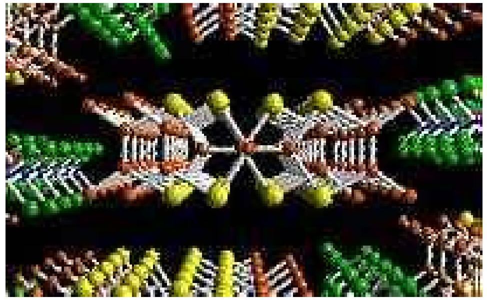


Fermions and bosonic modes in natural lattices:

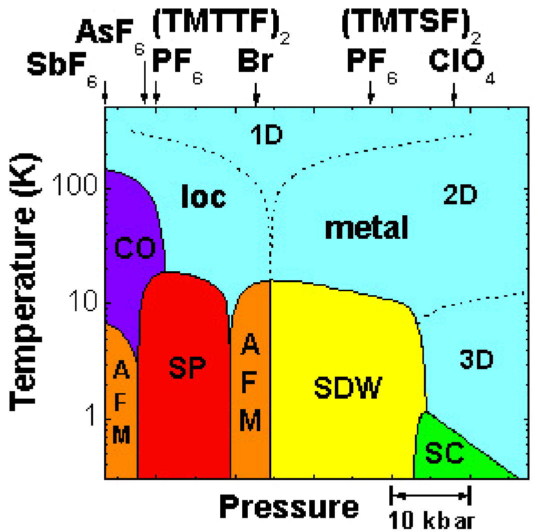
• Natural lattices: electrons, holes + phonons, magnons, … in crystals







• electrons and phonons in solid state materials: organic conductors, MgB_2, intercalated graphite superconductors, filled skutterudites …



J. Moser *et al.*, Eur. Phys. J **B1**, 39 (1998)

Artificial lattices: cold atoms in optical lattices

Proposed by: D. Jaksch *et al*., PRL **81**, 3108 (1998).

Expmt: M. T. de Pue *et al*., PRL **82**, 2262 (1999);
M. Greiner *et al*., PRL **87**, 160405 (2001);
M. Greiner *et al*., Nature **415**, 39 (2002);
T. Storferle *et al*., PRL **92**, 130403 (2004), etc.

QuickTime™ and a
TIFF (LZW) decompressor
are needed to see this picture.

This talk:
- neglect finite-size effects
- use simplified models
- consider fermions + boson BEC

$$V = V_0[\sin^2(kx) + \sin^2(ky) + \sin^2(kz)]$$ + magnetic trap

Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms

Markus Greiner*, Olaf Mandel*, Tilman Esslinger†, Theodor W. Hänsch* & Immanuel Bloch*

* *Sektion Physik, Ludwig-Maximilians-Universität, Schellingstrasse 4/III, D-80799 Munich, Germany, and Max-Planck-Institut für Quantenoptik, D-85748 Garching, Germany*
† *Quantenelektronik, ETH Zürich, 8093 Zurich, Switzerland*

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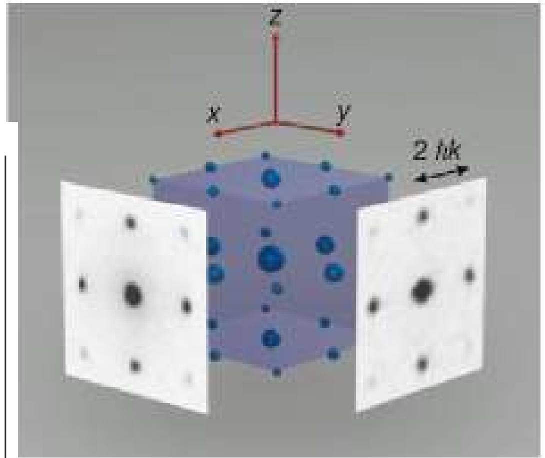


Figure 1 Schematic three-dimensional interference pattern with measured absorption images taken along two orthogonal directions. The absorption images were obtained after ballistic expansion from a lattice with a potential depth of $V_0 = 10E_r$ and a time of flight of 15 ms.

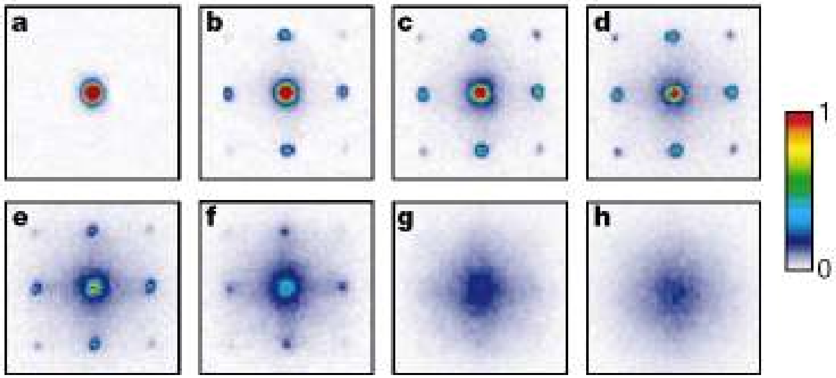
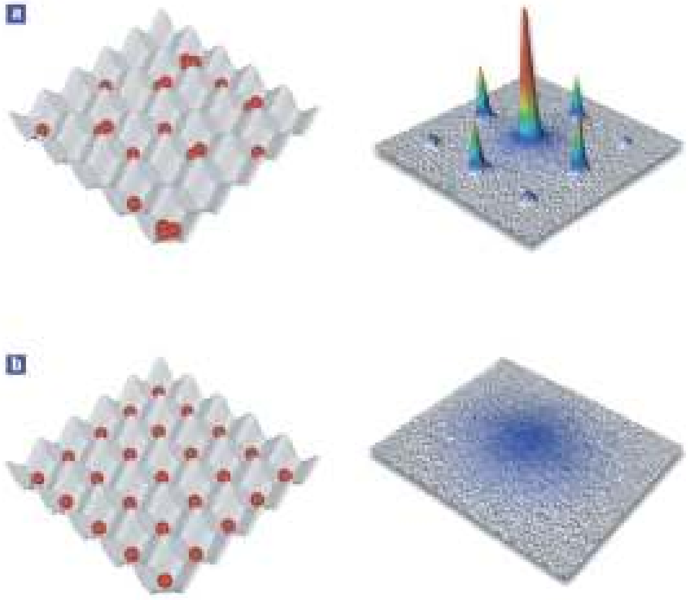


Figure 2 Absorption images of multiple matter wave interference patterns. These were obtained after suddenly releasing the atoms from an optical lattice potential with different potential depths V_0 after a time of flight of 15 ms. Values of V_0 were: **a**, 0 E_r; **b**, 3 E_r; **c**, 7 E_r; **d**, 10 E_r; **e**, 13 E_r; **f**, 14 E_r; **g**, 16 E_r; and **h**, 20 E_r.

Bosons in artificial lattices:



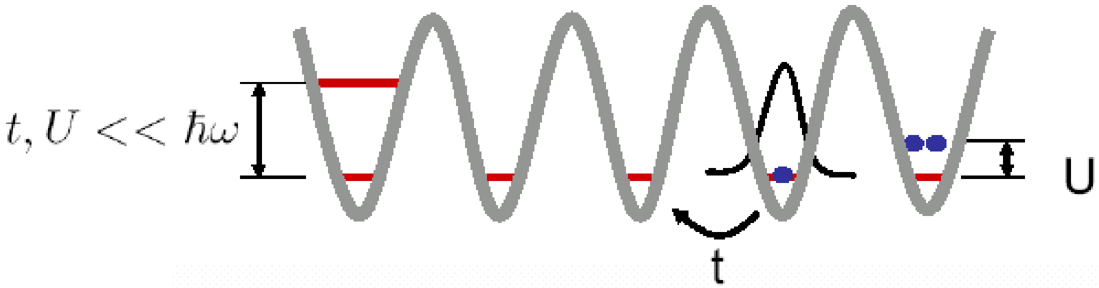
Rubidium-87 (^{87}Rb) = 37p + 37e + 50n = boson

• two-components: O. Mandel *et al*., Nature **425**, 937 (2003)

^{87}Rb: $|\uparrow\rangle$ = $|F = 1, m_F = -1\rangle$

$|\downarrow\rangle$ = $|F = 1, m_F = -2\rangle$

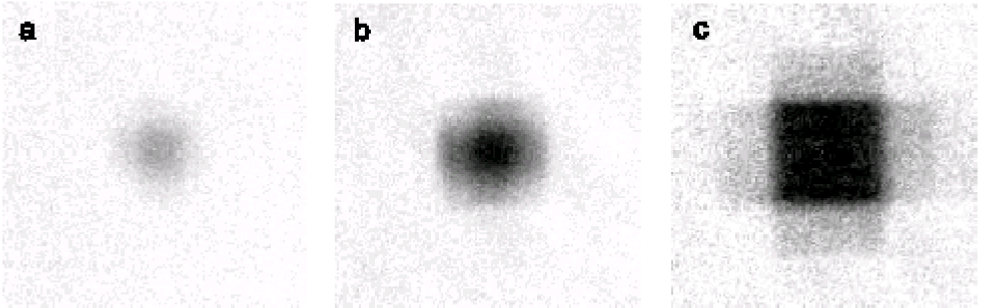
• Bosonic Hubbard model: J. Hubbard, Proc. R. Soc. Lond. **A276**, 238 (1963).



$$\mathcal{H} = -t \sum_{\langle ij \rangle} b_i^\dagger b_j + U \sum_i n_i^2 - \mu \sum_i n_i$$

Fermions in artificial lattices:

G. Modugno *et al*., PRA **68**, 011601 (2003);
M. Kohl, *et al*., cond-mat/0410389



"Snapshots of the Fermi surface"

Stoferle *et al*., cond-mat/0601045, ETH group

Bose-Fermi mixtures in artificial lattices:

PRL **97,** 120403 (2006) PHYSICAL REVIEW LETTERS week ending 22 SEPTEMBER 2006

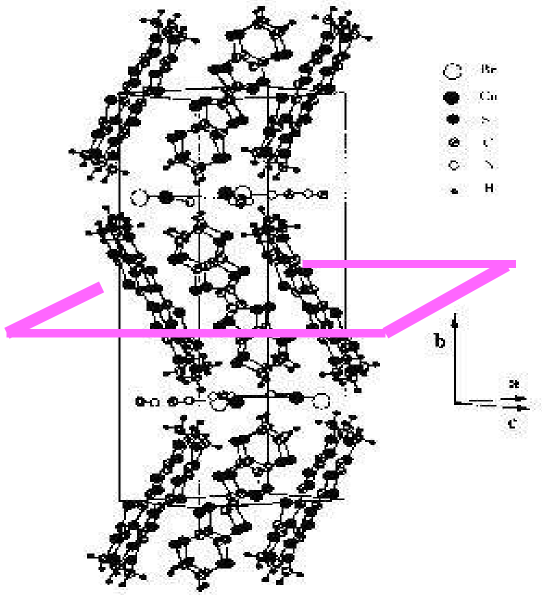
Tuning of Heteronuclear Interactions in a Degenerate Fermi-Bose Mixture

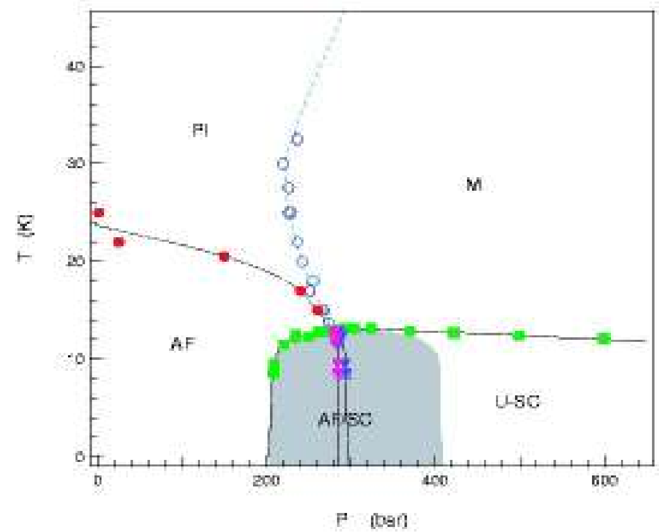
S. Ospelkaus, C. Ospelkaus, L. Humbert, K. Sengstock, and K. Bongs
Institut für Laserphysik, Luruper Chaussee 149, 22761 Hamburg, Germany

Modugno *et al*., Science **297**, 2240 (2002);
Goldwin *et al*., PRA **70**, 021601 (2004);
Ospelkaus *et al*.,PRL 96, 020401 (2006)

Organic superconductor $\kappa - (\text{BEDT-TTF})_2 X$

Ching *et al*., PRB **55**, 2780 (1997)

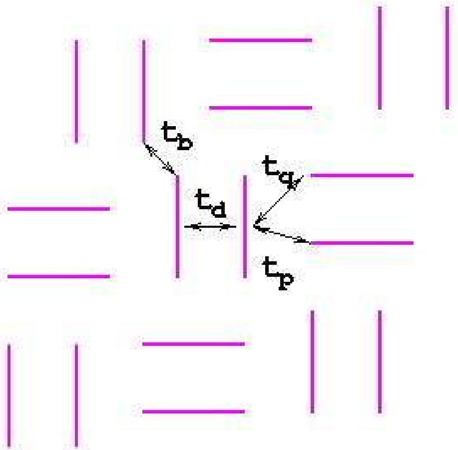




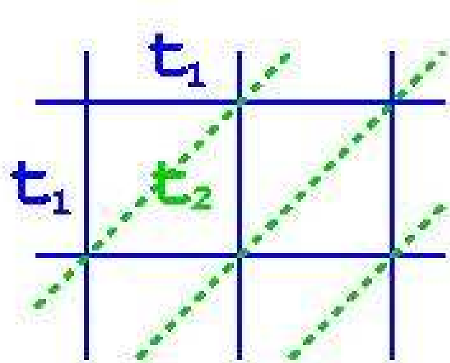
J. M. Williams et al. 1991
D. Jerome et al. 1991

from: S. Lefebvre *et al*., PRL **85**, 5420 (2000).

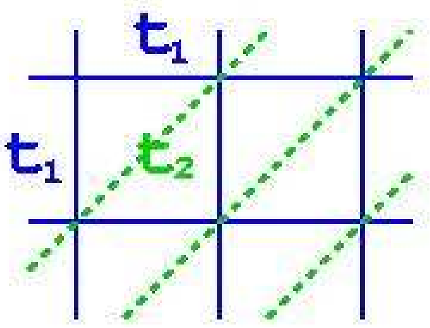
Kino and Fukuyama, J. Phys. Soc. Jpn. **65**, 2158 (1996)
R. H. McKenzie, Science **278**, 280 (1998)



$t_d \gg t_q \sim t_p > t_b$



Simplified model for κ-compounds:

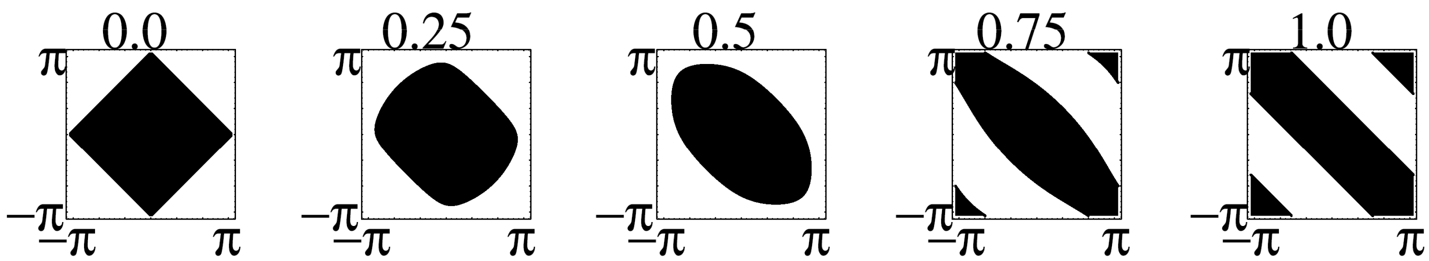


$$H = -t_1 \sum_{<\mathrm{ij}>} (c_\mathrm{i}^{\dagger\sigma} c_{\mathrm{j}\sigma} + h.c.) - t_2 \sum_{<<\mathrm{ij}>>} (c_\mathrm{i}^{\dagger\sigma} c_{\mathrm{j}\sigma} + h.c.) + U \sum_\mathrm{i} (n_{\mathrm{i}\uparrow} - \frac{1}{2})(n_{\mathrm{i}\downarrow} - \frac{1}{2}) + \mu \sum_\mathrm{i} n_\mathrm{i}$$

Fermi surface at half-filling for different values of t_2:

$t_1 + t_2 = 1$

t_2:



- square lattice ($t_1 = 1$, $t_2 = 0$)
- decoupled chains ($t_1 = 0$, $t_2 = 1$)
- intermediate region: isotropic triangular lattice ($t_1 = t_2$)

  this model ($t_1 > t_2$)

SWT, J. B. Marston, Can. J. Phys. **79**, 1463 (2000)

Einstein phonons

$$S_{ph} = \int_{\Omega\mathbf{q}} \phi_q^\dagger (i\Omega - \omega_E)\phi_q$$

Phonon propagator

$$D(q) = \omega_E/(\Omega^2 + \omega_E^2) \qquad q = (\Omega, \mathbf{q})$$

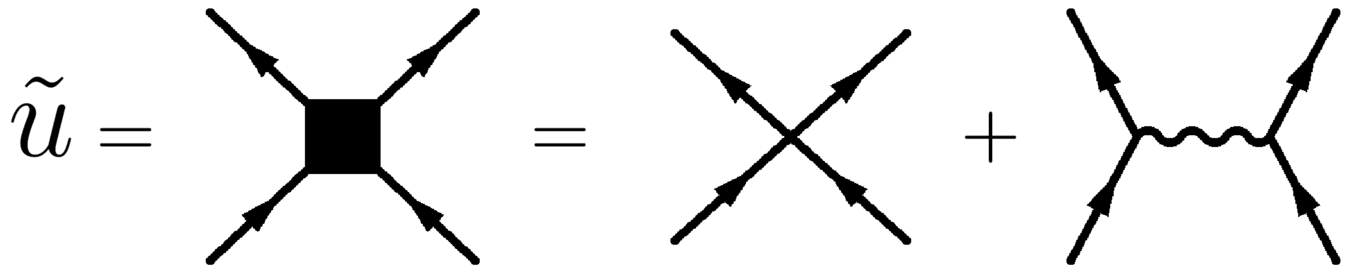
Electron-phonon interaction

$$S_{e-ph} = \int_{\omega\mathbf{k}} \int_{\Omega\mathbf{q}} g(q)\psi_{k+q}^\dagger \psi_k (\phi_q + \phi_{-q}^\dagger)$$

Retarded electron-electron interaction

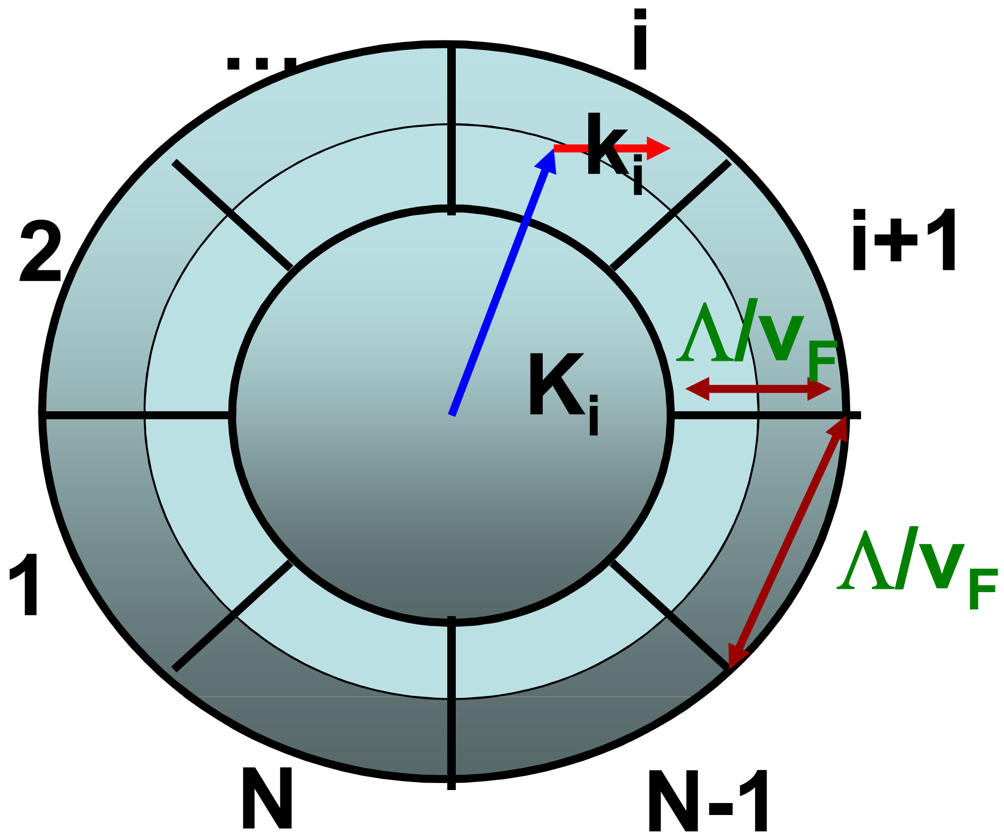
$$\begin{aligned} \tilde{u}(k_4, k_3, k_2, k_1) &= u(k_4, k_3, k_2, k_1) \\ &- 2g(k_1, k_3)g(k_2, k_4)D(k_1 - k_3) \end{aligned}$$

Feynman Diagrams

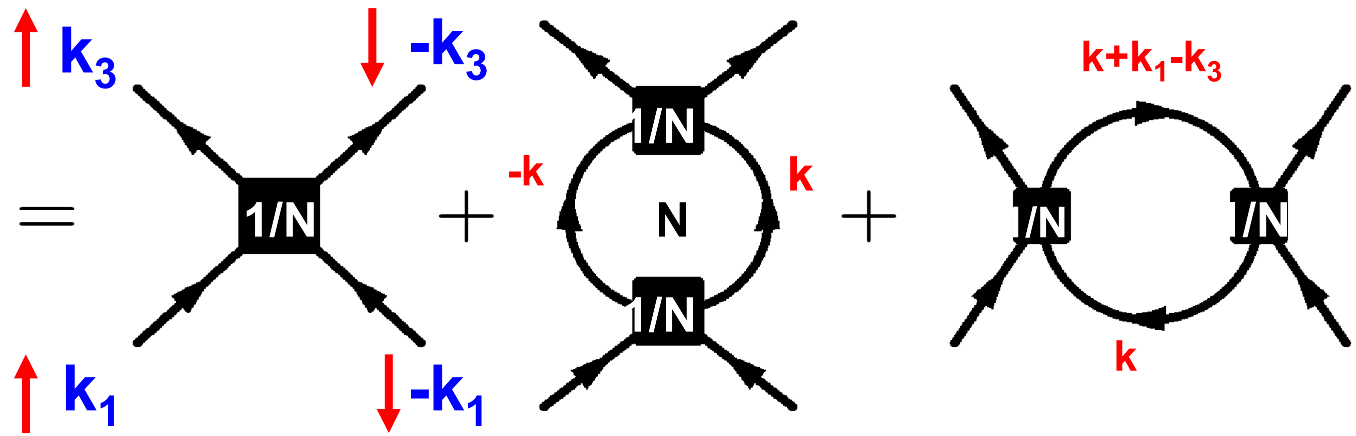


Large-N analysis:

$N = E_F / \Lambda$

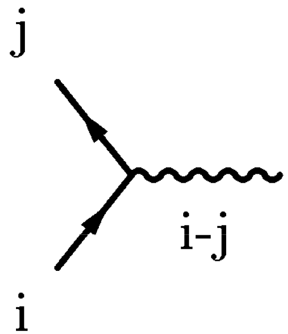


$$\Gamma^{(4)} =$$

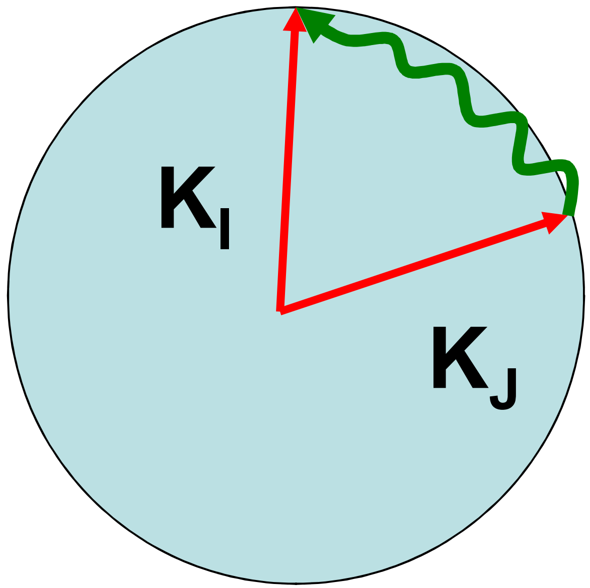


Migdal's theorem ('58)

u ~ g² ~ 1/N



$\propto \; g/\sqrt{N}$

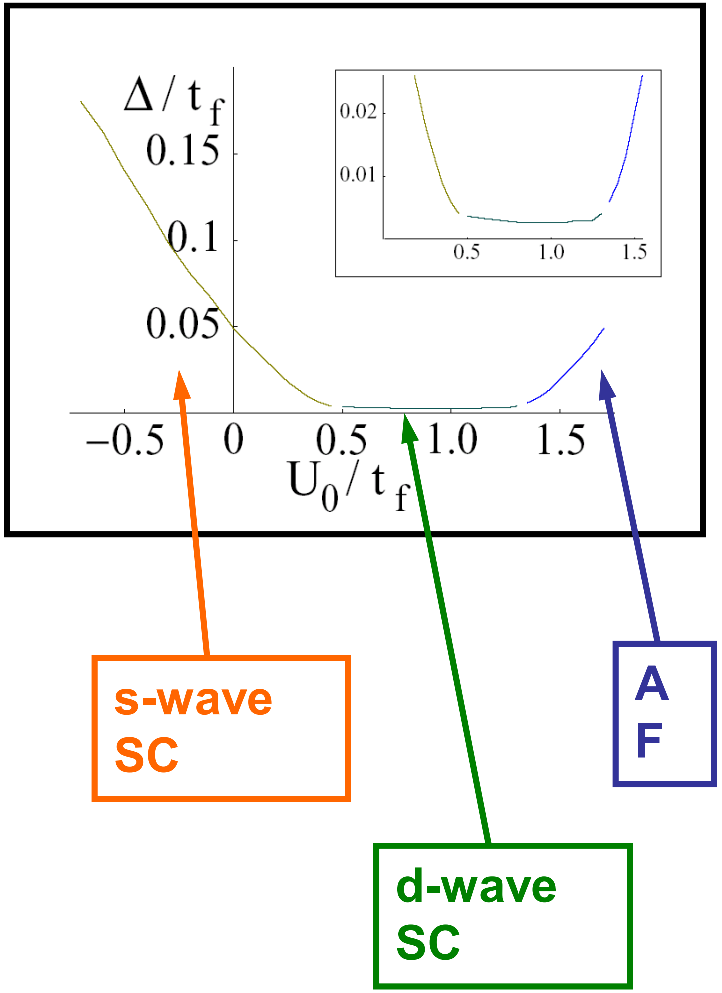


t'Hooft 1974

Self-energy

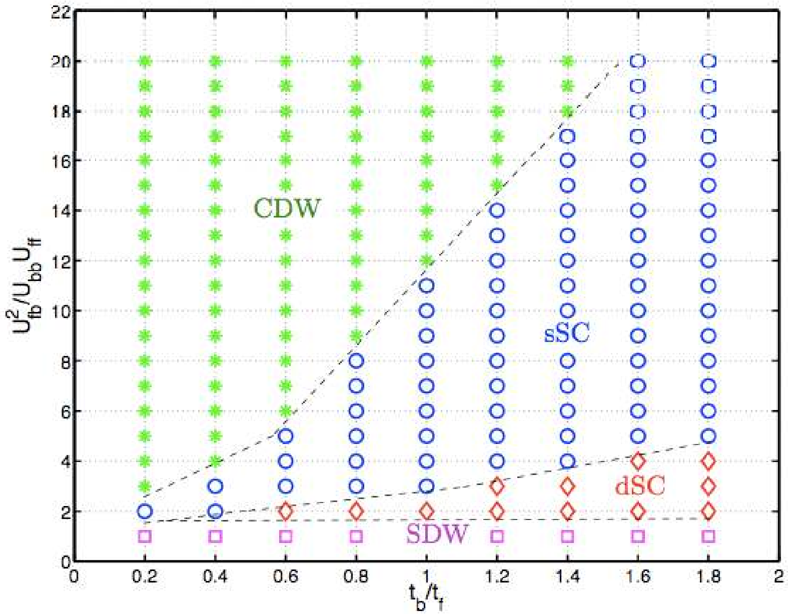
No e-ph vertex corrections!

- Instability gaps:



F. D. Klironomos and SWT, PRL **99**, 100401 (2007)

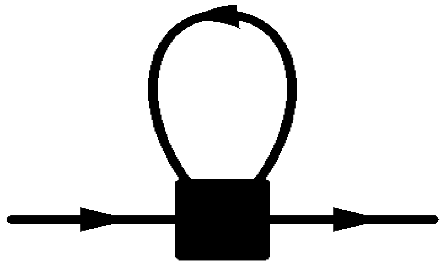
- with retardation (half-filling):



$$\tilde{U}_{ff}(k_1,k_2,k_3) = U_{ff} - \frac{U_{fb}^2/U_{bb}}{1+4\xi^2-2\xi^2\left(\cos(\boldsymbol{k}_1-\boldsymbol{k}_3)_x+\cos(\boldsymbol{k}_1-\boldsymbol{k}_3)_y\right)}\,\frac{\omega^2_{\boldsymbol{k}_1-\boldsymbol{k}_3}}{(\omega_1-\omega_3)^2+\omega^2_{\boldsymbol{k}_1-\boldsymbol{k}_3}},$$

Renormalization Group

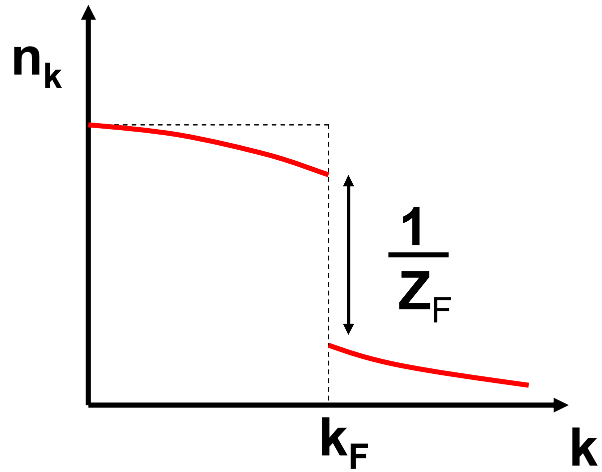
Electron self-energy

$\Sigma =$ 

$$\Sigma(\omega, \mathbf{k}) = \Sigma_0 + i(1 - Z(\omega, \mathbf{k}))\omega$$

Chemical potential

Wavefunction renormalization



$$\frac{\partial}{\partial \ell}\Sigma''_\ell(\omega) = -\int_{-\infty}^{+\infty}\frac{d\Omega}{\pi}\lambda\omega_E D(\Omega-\omega)\frac{\Lambda_\ell(\Omega-\Sigma''_\ell(\Omega))}{(\Omega-\Sigma''_\ell(\Omega))^2+\Lambda_\ell^2}$$

Finite temperature: find a temperature T * above which Fermi liquid is stable

$$\omega_n = \pi T^*(2n+1) \qquad \Lambda_c \to 0$$

Define: $$\phi(\omega_n) = f(\omega_n)/Z(\omega_n)$$

Interaction :

$$Z(\omega_n)\phi(\omega_n) = -\pi T^* \sum_m [u_0 - \lambda \omega_E D(\omega_n - \omega_m)] \frac{\phi(\omega_m)}{|\omega_m|}$$

Self-energy :

$$Z(\omega_n) = 1 + \lambda \omega_E \frac{\pi T^*}{\omega_n} \sum_m \mathrm{sgn}(\omega_m) D(\omega_n - \omega_m)$$

ELIASHBERG's equations at T=T_c !

Eliashberg theory ('60)

1) Assumes a broken symmetry with SC gap Δ(ω);
2) Assumes Migdal's theorem (more on this later…)

Self consistent relations for the self-energy.

Diagonal elements:

$$Z(\omega_n) = 1 + \frac{\pi T}{\omega_n} \sum_{m=-\infty}^{\infty} \frac{\lambda \omega_E^2}{(\omega_n - \omega_m)^2 + \omega_E^2} \frac{\omega_m}{\sqrt{\omega_m^2 + \Delta^2(\omega_m)}}$$

Off-diagonal elements:

$$Z(\omega_n)\Delta(\omega_n) = -\pi T \sum_{m=-\infty}^{+\infty} \left[u_0 - \frac{\lambda \omega_E^2}{(\omega_n - \omega_m)^2 + \omega_E^2} \right] \frac{\Delta(\omega_m)}{\sqrt{\omega_m^2 + \Delta^2(\omega_m)}}$$

At $T=T_c$: $\Delta \longrightarrow 0$ **Conclusion: $T^* = T_c$**

Nevertheless, $\Delta(\omega) \neq \phi(\omega)$

Evolution of the couplings in the BCS channel (λ = 0.3)

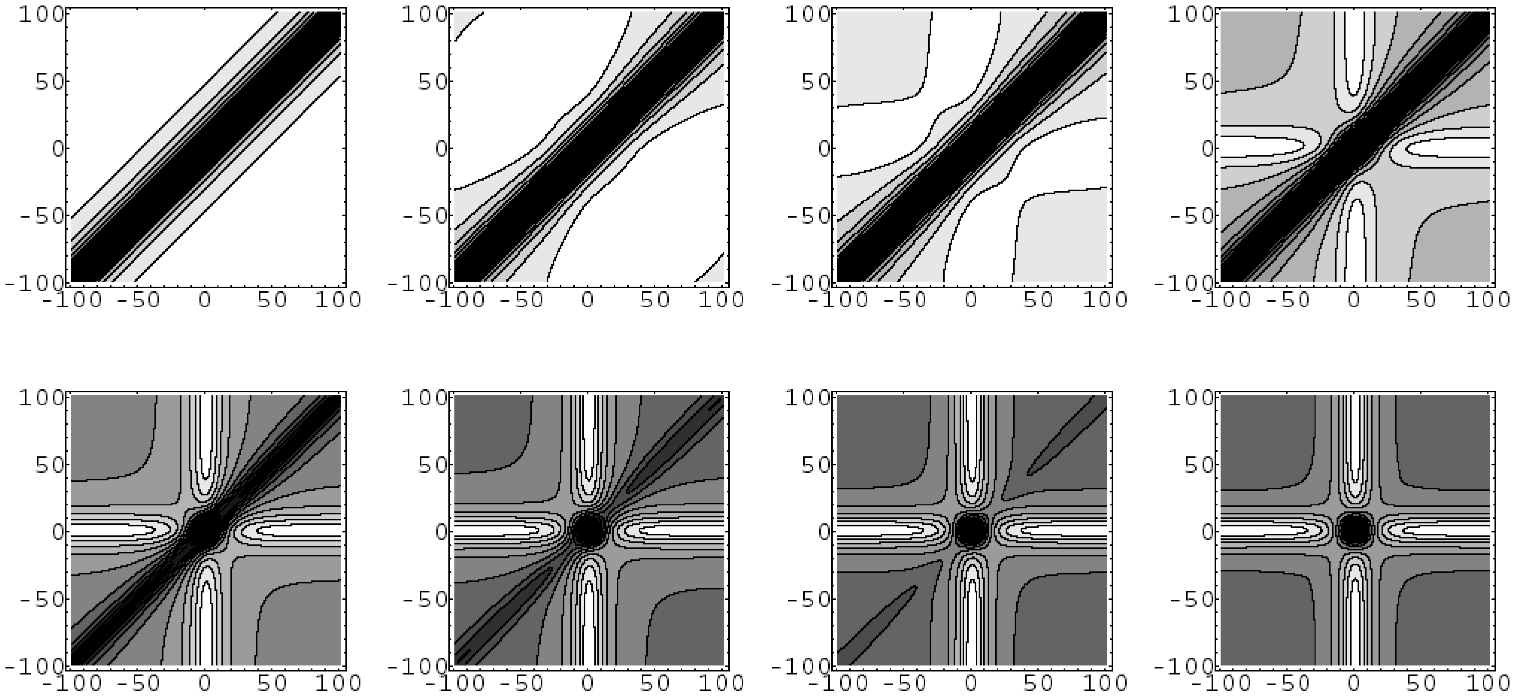


Figure 1: Plots of the NxN matrix U at different RG scales ℓ. Here the number of frequency divisions $N = 200$, and the value of the parameters used are $\lambda = 0.3$, $\Lambda_0 = 100$, $\omega_E = 10$, $u_0 = 0.1$. Panels correspond to ℓ = 0, 2.5, 3, 5, 6.5, 6.9, 7.1, and 7.19.

SWT, AH Castro Neto, R Shankar, DK Campbell, cond-mat/0505426

Evolution of the couplings in the BCS channel (λ = 4)

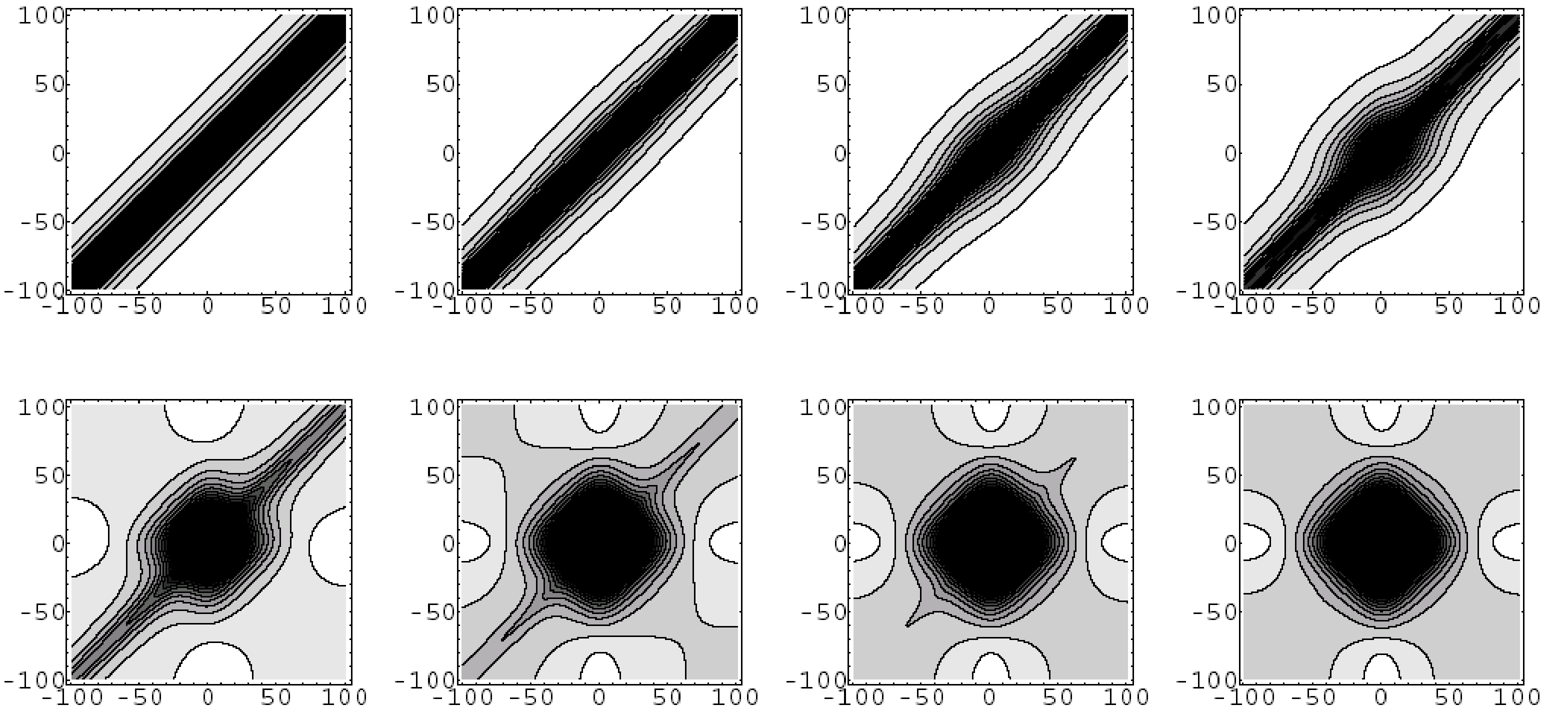


Figure 2: Plots of the NxN matrix U at different RG scales ℓ. Here the number of frequency divisions $N = 200$, and the value of the other parameters are $\lambda = 4$, $\Lambda_0 = 100$, $\omega_E = 10$, and $u_0 = 0.1$. Panels correspond to ℓ = 0, 1, 2, 2.5, 3, 3.13, 3.157, 3.172. The scale $2W_c \approx 40$ distinguishes the high and low frequencies close to ℓ_c.

SWT, AH Castro Neto, R Shankar, DK Campbell, cond-mat/0505426

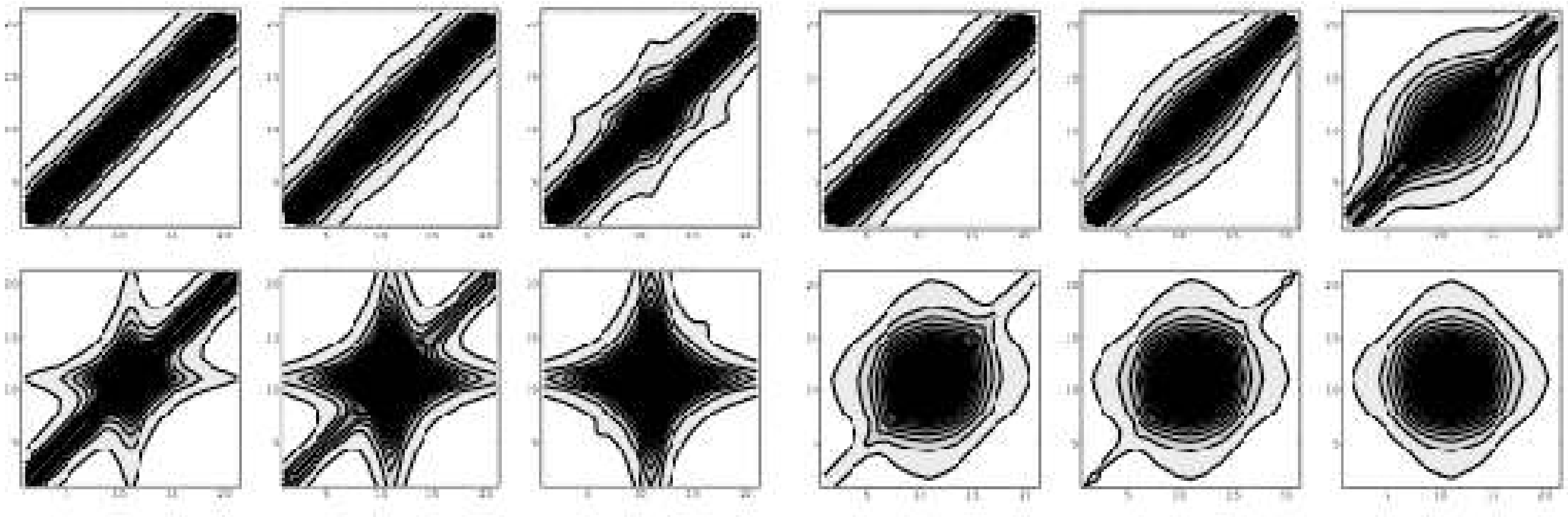
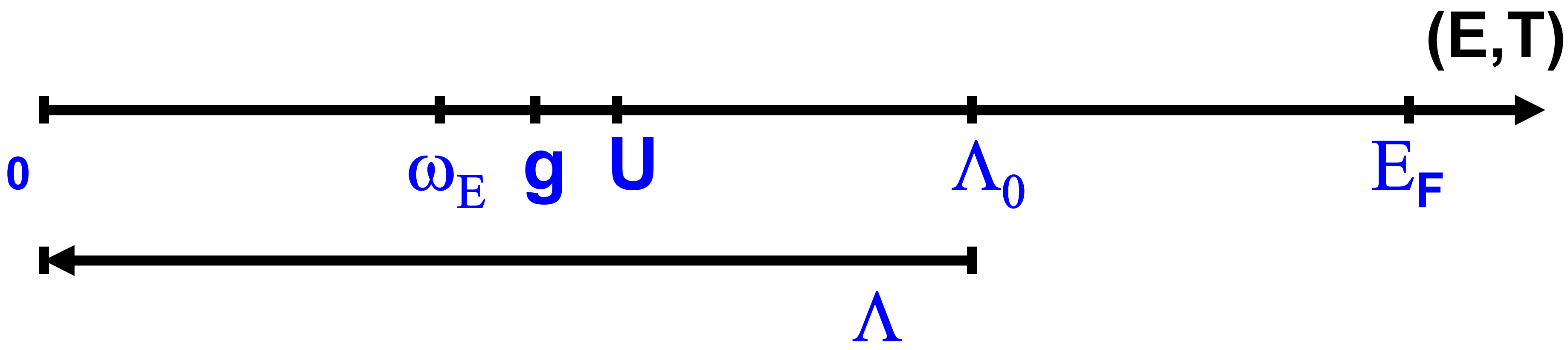


FIG. 2: Same as Fig.[1] but for the $d_{x^2-y^2}$-channel, $\gamma = 2$. The six panels on the left side are the evolution of the matrix elements at weak coupling, $\lambda = 0.4$ and the panels on the right side are the same but for the strong coupling regime, $\lambda = 4.0$

Electron-electron plus electron-phonon



E_F: Fermi energy
g : electron-phonon
U : electron-electron
ω_E: phonon frequency

$$\lambda = 2N(0)g^2/\omega_E$$

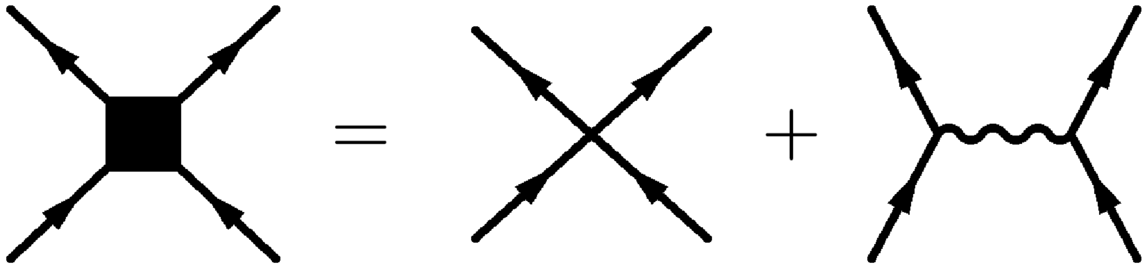
$E_F \gg \Lambda_0 \gg \{g, U, \omega_E\}$

Large N: $N = \dfrac{E_F}{\Lambda}$

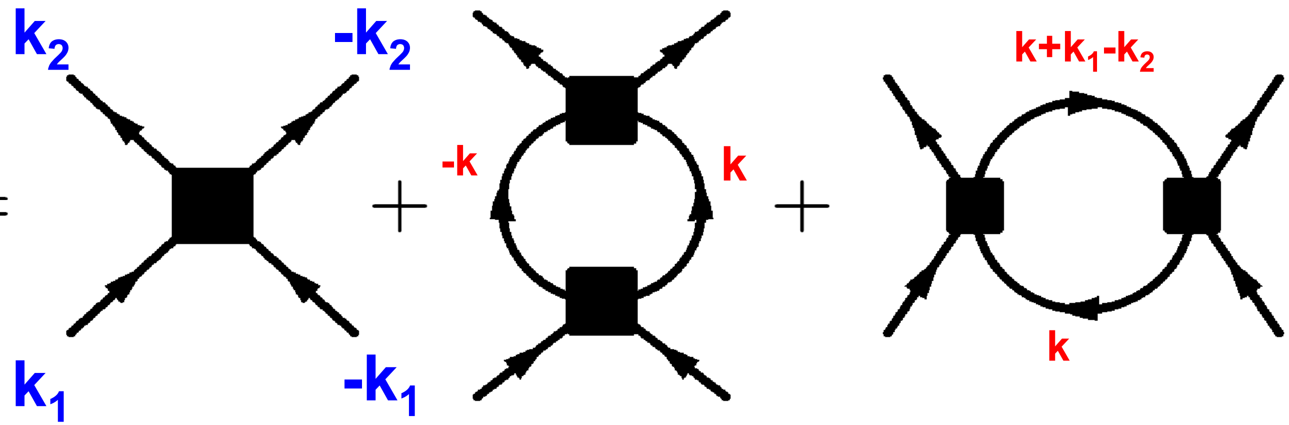
$N_0 = \dfrac{E_F}{\Lambda_0} \gg 1$

Feynman Diagrams

$\tilde{u} =$

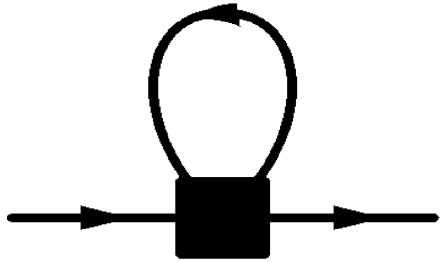


$\Gamma^{(4)} =$



Renormalization Group

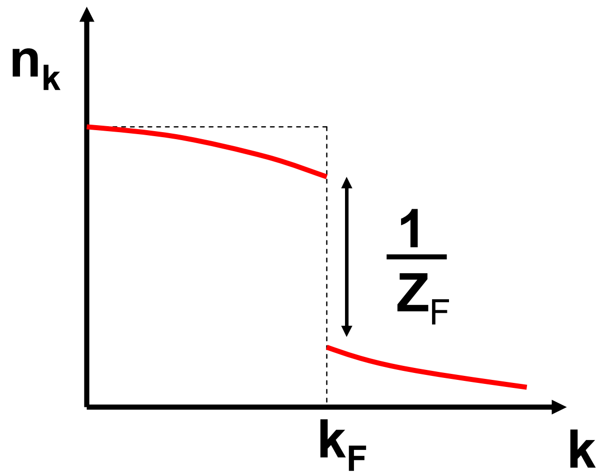
Electron self-energy

$\Sigma =$ 

$$\Sigma(\omega, \mathbf{k}) = \Sigma_0 + i(1 - Z(\omega, \mathbf{k}))\omega$$

Chemical potential

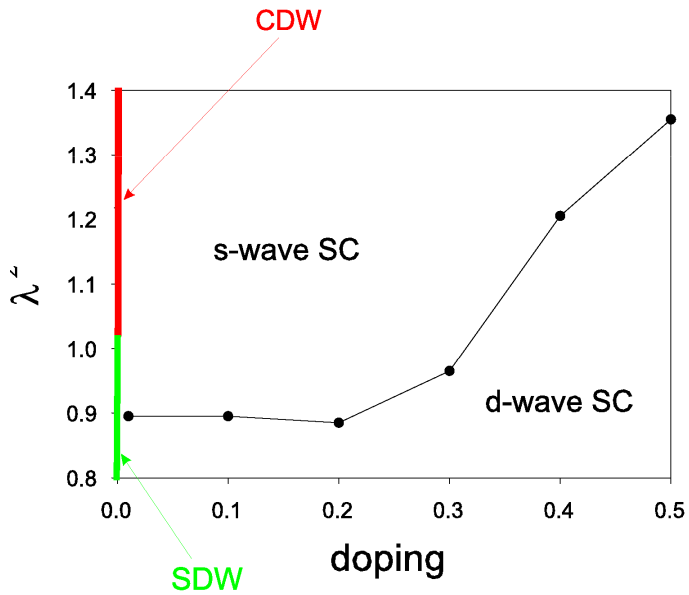
Wavefunction renormalization



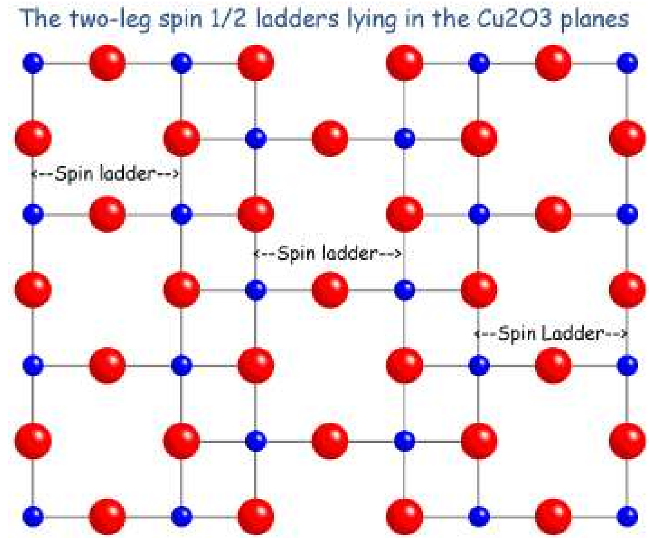
Holstein-Hubbard ladder:

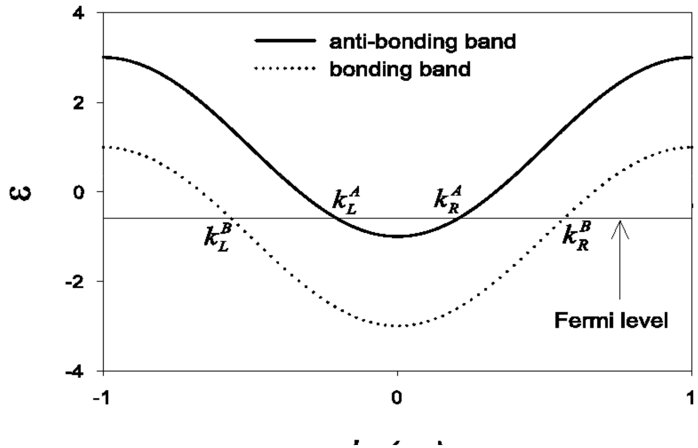
(Ka-Ming Tam, BU)

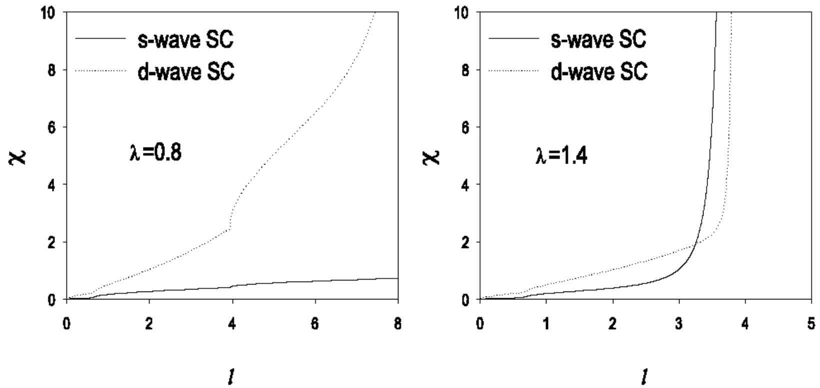
$$H = -t \sum_{\langle i,j\rangle,\sigma} (c^\dagger_{i,\sigma} c_{j,\sigma} + H.c.) + U \sum_i n_{i,\uparrow} n_{i,\downarrow}$$
$$+\lambda \sum_{i,\sigma} (a^\dagger_i + a_i) n_{i,\sigma} + \omega_0 \sum_i a^\dagger_i a_i \,,$$



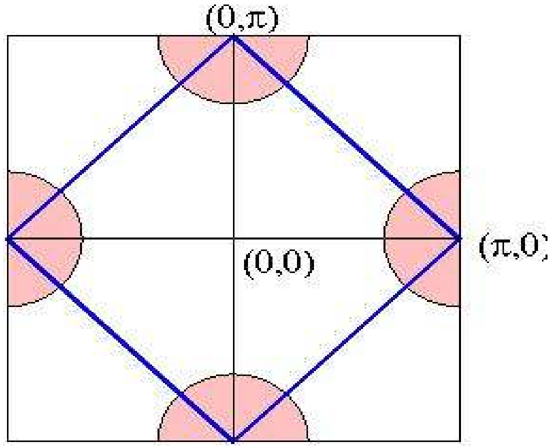
however: S.Seidel, H.-H.Lin, D.-H. Lee, PRB 2005





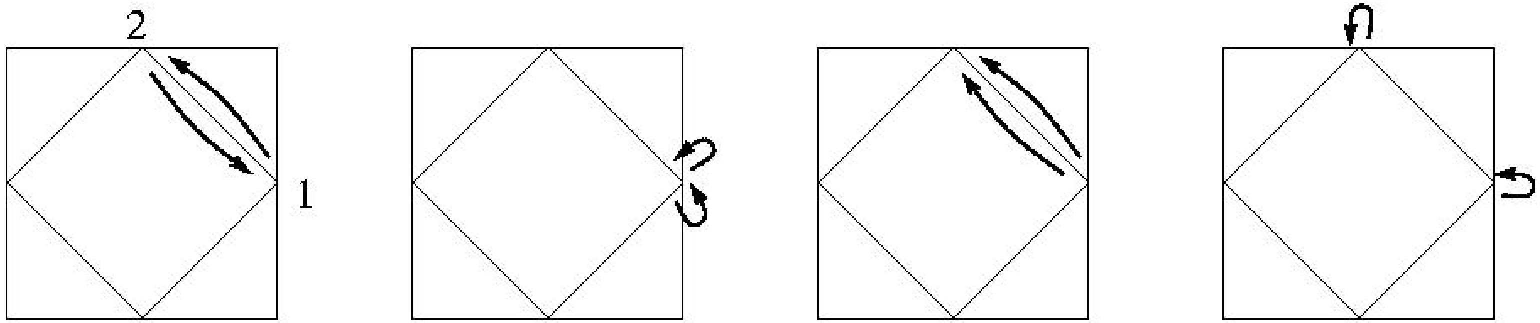


Two-patch model for van Hove problem:



H. Schulz, Europhys. Lett. (1987)

$$
\begin{aligned}
g_{SCs} &= g_2 + g_3 \\
g_{SCd} &= g_2 - g_3 \\
g_{SDW} &= -g_1 - g_3
\end{aligned}
$$



$g_1 = u(1221)$ $g_2 = u(1111)$ $g_3 = u(2211)$ $g_4 = u(1212)$

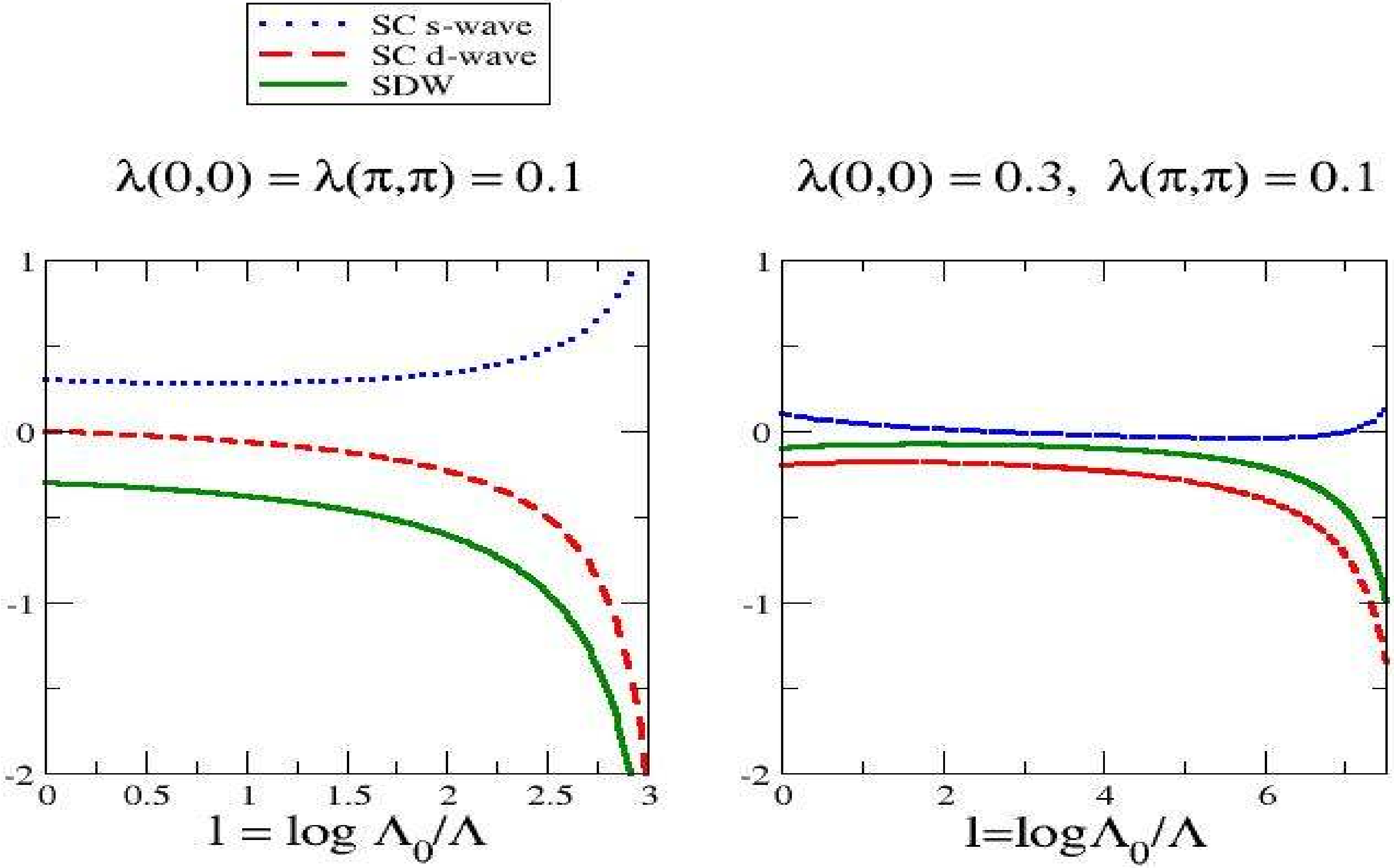
phonon coupling: $\lambda = 2N(0)g^2/\omega_E$

Fermions with spin:

van Hove problem without phonons has been extensively studied, *e. g.,* J. Gonzalez, F. Guinea, M. A. H. Vozmediano 1997, N. Furukawa, T. M. Rice and M. Salmhofer 1998, B. Binz, D. Baeriswyl, B. Doucot, 2002, …

- What is the interplay between effects of nesting and phonons?
- Are phonons always pair-breaking in the d-wave superconducting channel?
- Can phonons and AF fluctuations cooperate to enhance Tc for d-wave superconducivity ?

Isotropic x anisotropic phonons:



u_0=0.2

(Filippos Klironomos, UCR)

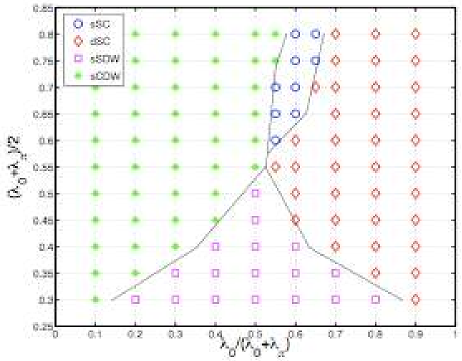


FIG. 1: (Color online) Phase diagram for Einstein phonons of frequency $\omega_E = 1.0$. Four phases involving antiferromagnetism (sSDW) (purple squares), charge density wave (sCDW) (green stars) and s-wave (sSC) (blue circles) and d-wave (dSC) (red rhombs) superconductivity compete in the vicinity where the average phononic strength $\bar{\lambda}$ approaches the bare on-site repulsion $u_0 = 0.5$. The lines distinguishing the different domains are guides to the eye.

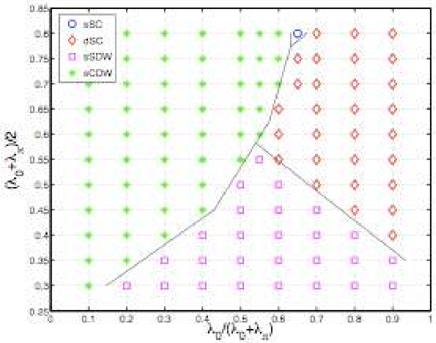


FIG. 4: (Color online) Phase diagram for the same on-site repulsion ($u_0 = 0.5$) as in Fig. (1), but for a smaller phonon frequency ($\omega_E = 0.1$). Slower phonons suppress superconductivity over the nesting-associated CDW and SDW phases. Color scheme is identical to Fig. (1).

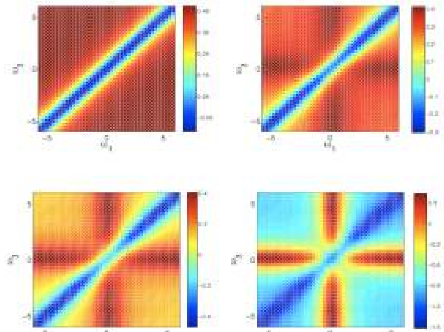
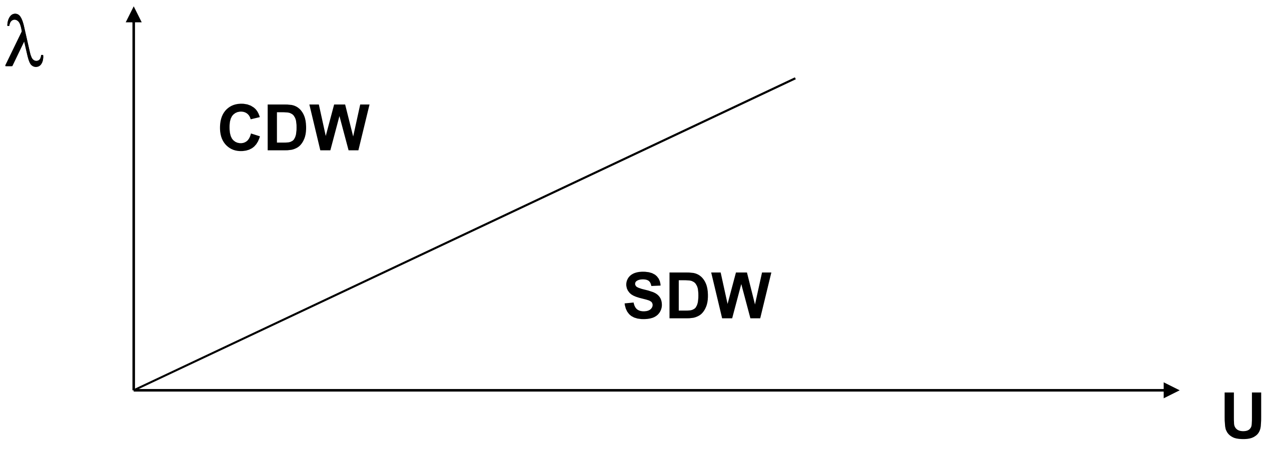


FIG. 5: (Color online) Evolution of $g_2(\omega_1, -\omega_1, \omega_3)$ coupling for $\lambda_0 = 0.6$, $\lambda_\pi = 0.4$, and $\omega_E = 1.0$. The RG steps chosen are $\ell = 0.4, 2.4, 3.4, 4.9$ (top left, top right, bottom left, bottom right). The color scheme is from lower dark blue (attractive) to higher dark red (repulsive) values.

1D Holstein-Hubbard model:

$$H = -t \sum_{i,\sigma} (c^\dagger_{i+1,\sigma} c_{i,\sigma} + H.c.) + U \sum_i n_{i,\uparrow} n_{i,\downarrow}$$
$$+ g_{ep} \sum_{i,\sigma} (a^\dagger_i + a_i) n_{i,\sigma} + \omega_0 \sum_i b^\dagger_i b_i,$$

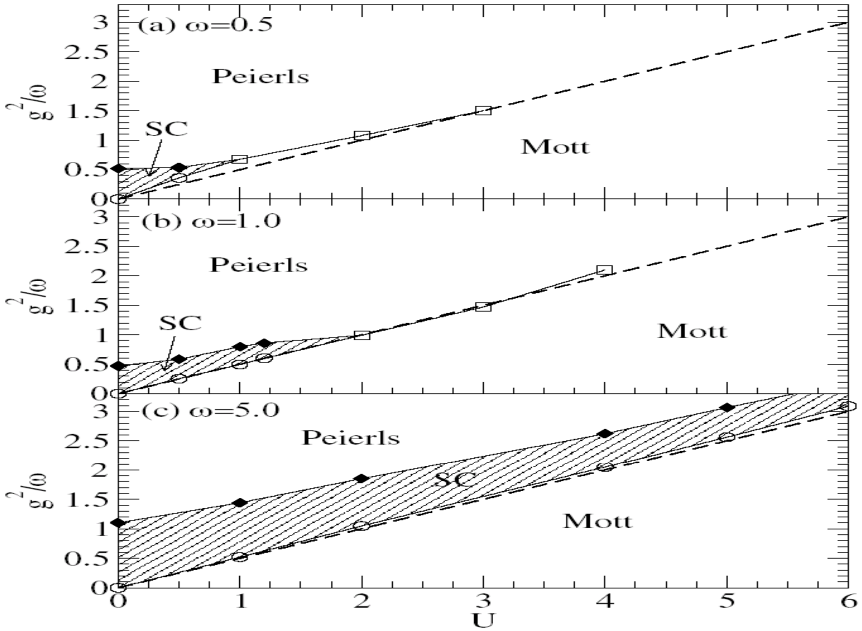


$\lambda = 2g_{ep}^2 / \omega_0$ $\quad\quad$ $U_{eff} \equiv U - \lambda$

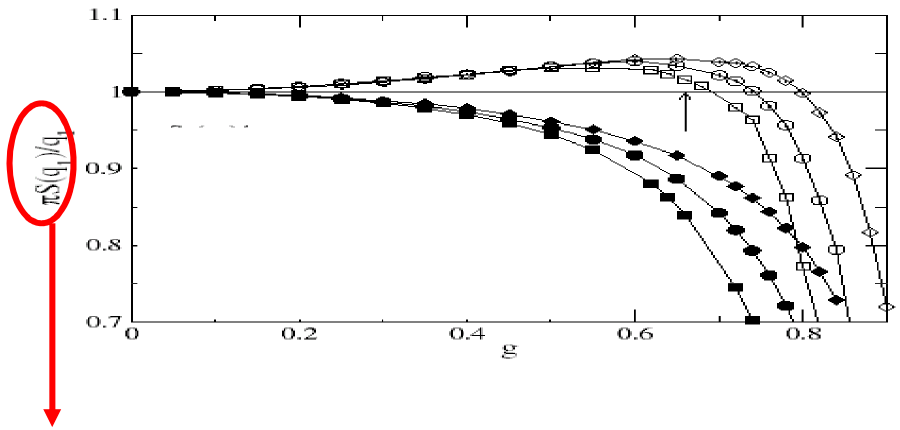
J. E. Hirsch and E. Fradkin, PRB 27, 4302 (1983)

also: H. Fehske, *et al.*, PRB 69, 165115 (2004);
I. P. Bindloss, PRB 71, 205113 (2005)

More recently a third phase has been proposed:



R. T. Clay and R. P. Hardikar, PRL 95, 096401 (2005)



Kρ

From Tomonaga-Luttinger liquid theory:

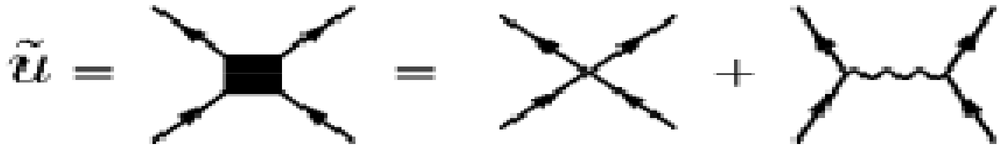
$O^{CDW} \sim x^{-K\rho}$ $\qquad$ $O^{SC} \sim x^{-1/K\rho}$

also: C. Wu, *et al.*, PRB 52, 15683 (1995)
E. Jeckelmann, *et al.*, PRB 60, 7950 (1999)
Y. Takada and A. Chatterjee, PRB 67, 0811102 (2003)
Y. Takada, J. Phys. Soc. Jpn., 65, 1544 (1996)

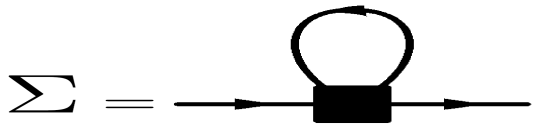
Multiscale RG for interacting fermions coupled to bosons:

- retardation effects
- no assumptions made on symmetry breaking
- f-f and f-b interactions treated simultaneously
- competition/cooperation between different instability channels

Interaction vertex corrections:



Self-energy corrections:

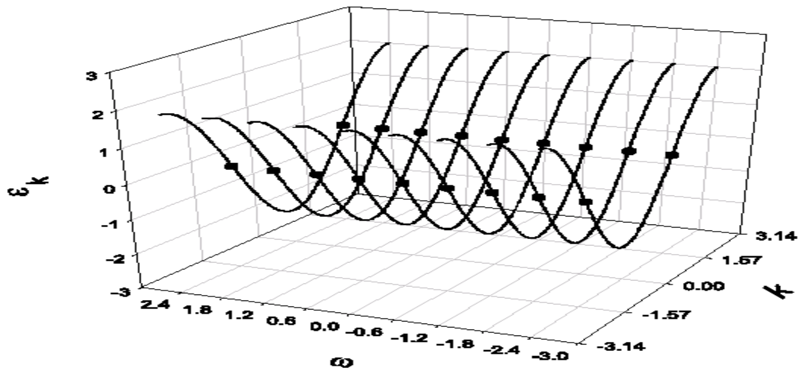


For a circular Fermi surface + phonons:

- Eliashberg's theory is obtained at Tc
- asymptotically exact due to large N ~ $\Lambda_0 / \Lambda \rightarrow \infty$
- generalized Migdal's theorem

SWT, A. H. Castro Neto, R. Shankar, D. K. Campbell, Phys. Rev. B, 72, 054531 (2005), cond-mat/0505426

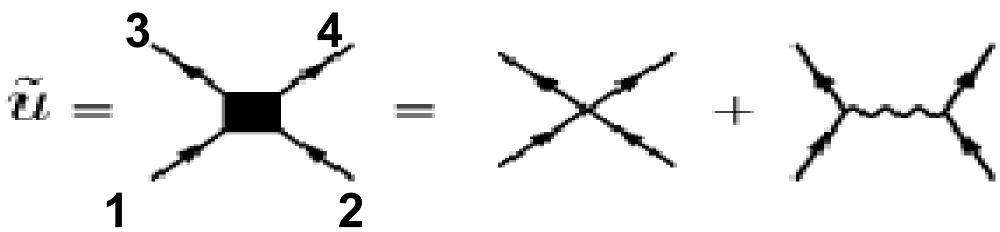
Multiscale RG analysis:



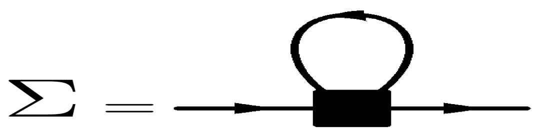
$$g_1 \longrightarrow g_1(\omega_1, \omega_2, \omega_3, \omega_4)$$
$$g_2 \longrightarrow g_2(\omega_1, \omega_2, \omega_3, \omega_4)$$
$$g_3 \longrightarrow g_3(\omega_1, \omega_2, \omega_3, \omega_4)$$
$$g_4 \longrightarrow g_4(\omega_1, \omega_2, \omega_3, \omega_4)$$

$$\omega_1 + \omega_2 = \omega_3 + \omega_4$$

Initial conditions:



$$g_i(\omega_1, \omega_2, \omega_3, \omega_4) = U - \frac{2g_{ep}^2}{\omega_0}\left(\frac{\omega_0^2}{\omega_0^2 + (\omega_1 - \omega_3)^2}\right)$$

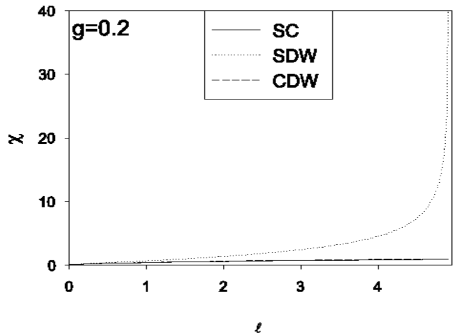


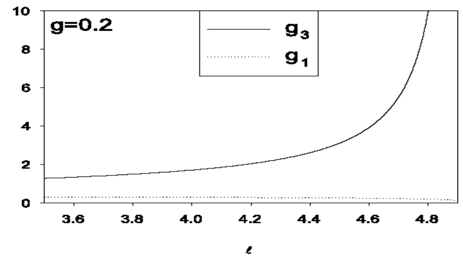
RG flows of susceptibilities and couplings ($\omega_0 = 1$):

SDW

$\Delta_C \neq 0$

$\Delta_S = 0$

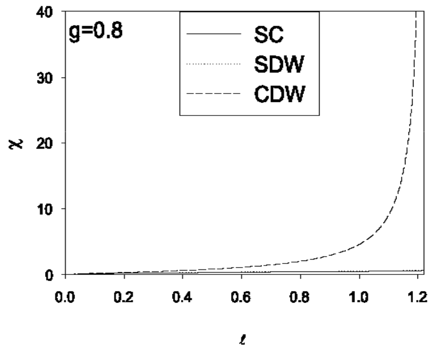


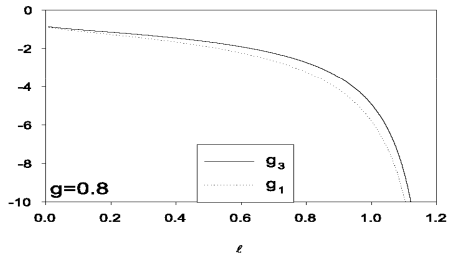


CDW

$\Delta_C \neq 0$

$\Delta_S \neq 0$



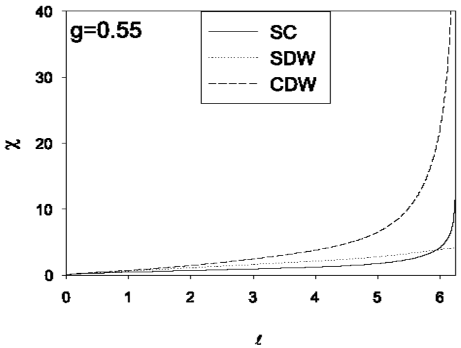


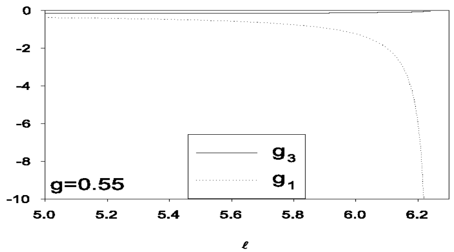
"Intermediate" region:

CDW

$\Delta_C = ?$

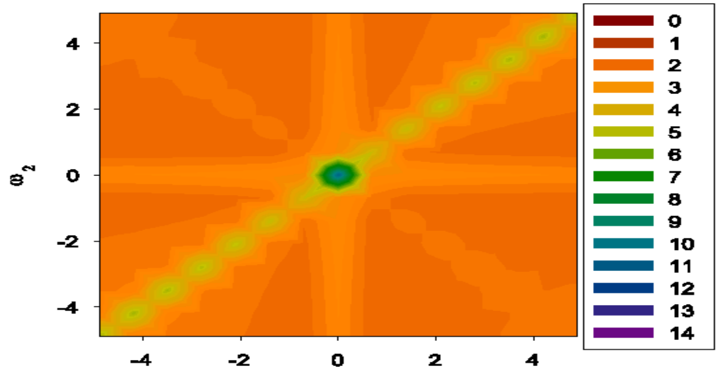
$\Delta_S \neq 0$



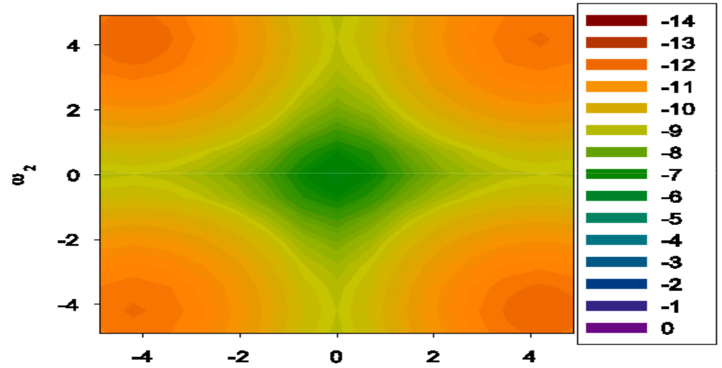


Frequency structure of $g_3(\omega_1,\omega_2,\omega_1,\omega_2)$

λ = 0.2 (SDW)

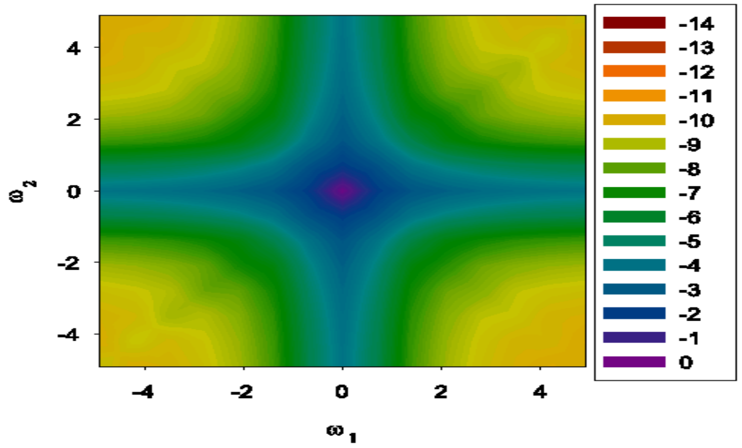


λ = 0.8 (CDW)



λ = 0.55 (CDW)

$U_{eff} \approx 0$



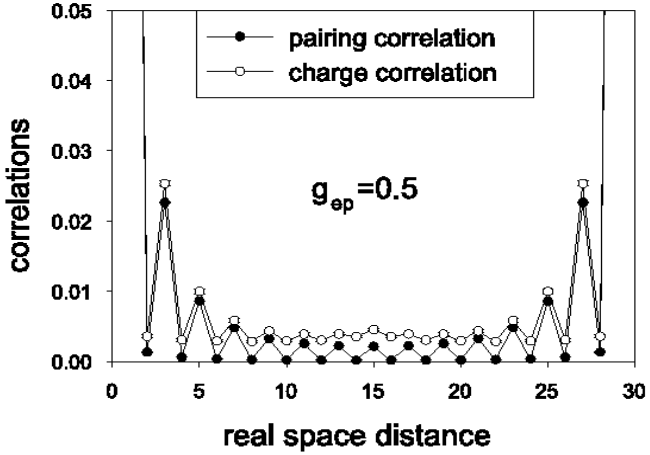
How to conciliate with Kρ > 1:

- Kρ > 1 does not mean SC is dominant!

$$O^{CDW}(x) \propto x^{-\alpha K_\rho} \equiv x^{-K_{CDW}}$$
$$O^{SC}(x) \propto x^{-\beta/K_\rho} \equiv x^{-K_{SC}}$$

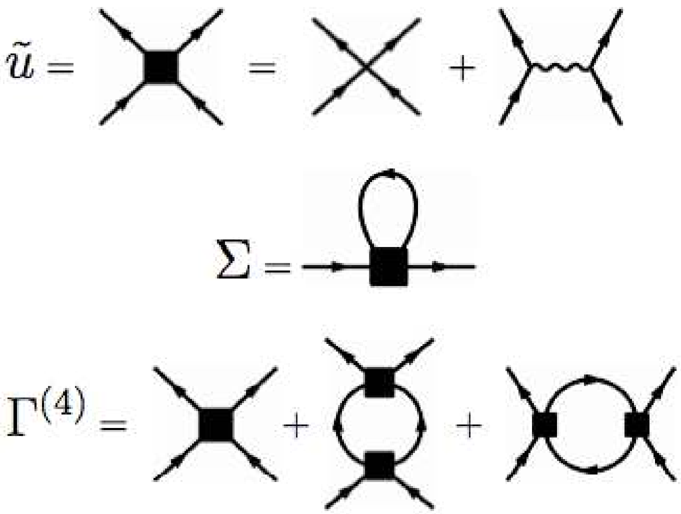
D. Loss and T. Martin, PRB 50, 12160 (1994)
M. Tezuka, *et al.*, PRL 96, 226401 (2005)

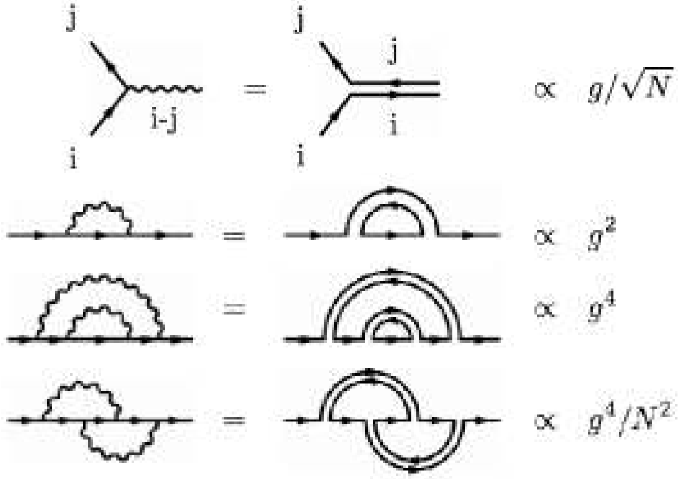
Direct calculation of susceptibilities (Quantum Monte-Carlo):



Renormalization-group approach to interacting fermions coupled to bosons

SWT, A. H. Castro Neto, R. Shankar and D. K. Campbell, PRB 72, 054531 (2005)





For a circular Fermi surface + phonons:

- **Eliashberg's theory is obtained at Tc**
- **asymptotically exact due to large N ~ $\Lambda_0 / \Lambda \rightarrow \infty$**
- **generalized Migdal's theorem**