Scattering amplitudes, Wilson loops and correlation functions in $\mathcal{N} = 4$ super-Yang-Mills theory

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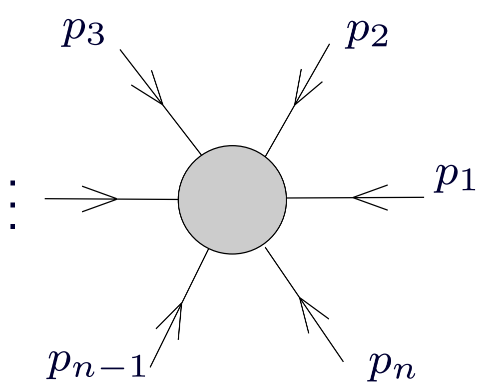
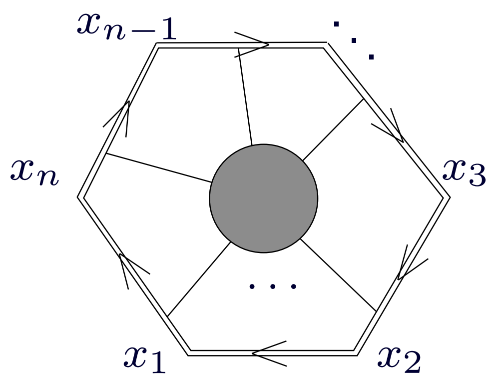
Outline

- ✔ On-shell gluon scattering amplitudes
- ✔ Iterative structure at weak/strong coupling in $\mathcal{N}=4$ SYM
- ✔ Dual conformal invariance – hidden symmetry of planar amplitudes
- ✔ Maximally helicity violating (MHV) scattering amplitude/Wilson loop duality in $\mathcal{N}=4$ SYM



$\Longleftrightarrow$



$\langle 0|S|1^- 2^- 3^+ \dots n^+\rangle$

$\langle 0|\,\mathrm{tr}\,\mathrm{P}\exp\left(i\oint_C dx\cdot A(x)\right)|0\rangle$

- ✔ Correlation function/Scattering amplitude/Wilson loop triality in $\mathcal{N}=4$ SYM

Why is $\mathcal{N}=4$ super Yang-Mills theory interesting?

- ✔ Four-dimensional gauge theory with extended spectrum of physical states/symmetries

  2 gluons with helicity ± 1, *6 scalars with helicity* 0, *8 gauginos with helicity* $\pm \frac{1}{2}$

  all in the adjoint of the $SU(N_c)$ gauge group

- ✔ Classical symmetries survive at the quantum level:
  - ✗ β-function vanishes to all loops $\Longrightarrow$ the theory is (super)conformal
  - ✗ Only two free parameters: 't Hooft coupling $\lambda = g^2_{\rm YM} N_c$ and number of colors N_c

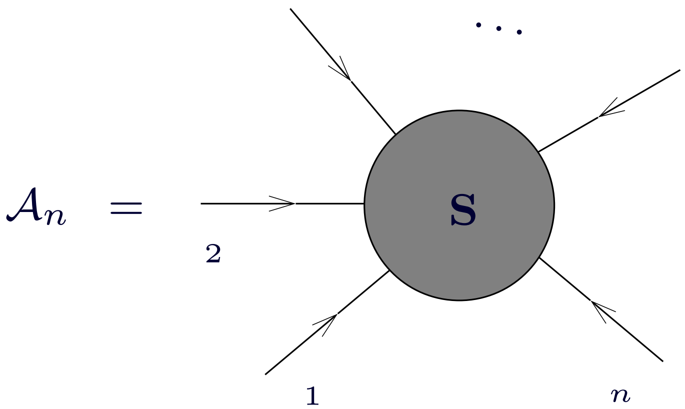
- ✔ Why is $\mathcal{N}=4$ SYM fascinating?
  - ✗ *At weak coupling*, $\mathcal{L}_{\mathcal{N}=4}$ is more complicated than $\mathcal{L}_{QCD}$, the number of Feynman integrals contributing to amplitudes is *MUCH* bigger compared to QCD ... but the final answer is *MUCH* simpler (examples to follow)
  - ✗ *At strong coupling*, the conjectured AdS/CFT correspondence [Maldacena],[Gubser,Klebanov,Polyakov],[Witten]

    Strongly coupled planar $\mathcal{N}=4$ *SYM* $\Longleftrightarrow$ *Weakly coupled string theory on* ${\rm AdS}_5 \times {\rm S}^5$

  - ✗ Final goal (dream or reality?):

$\mathcal{N}=4$ SYM is the unique example of a four-dimensional gauge theory that can be/ should be/ will be solved exactly for arbitrary values of the coupling !!!

Why scattering amplitudes?

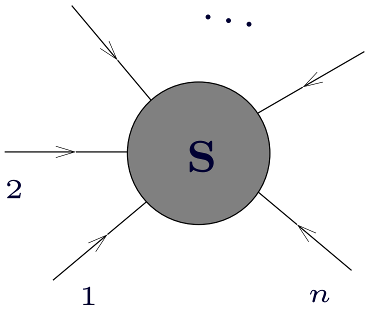


- On-shell matrix elements of $S-$matrix:
  - Probe (hidden) symmetries of gauge theory
  - Are independent of gauge choice
  - Nontrivial functions of Mandelstam's variables $s_{ij} = (p_i + p_j)^2$
- Simpler than QCD amplitudes but they share many properties
- In planar $\mathcal{N} = 4$ SYM they have a remarkable structure
- All-order conjectures and a proposal for strong coupling via AdS/CFT
- New dynamical symmetry: dual superconformal invariance $\Rightarrow$ Yangian $\Rightarrow$ integrability?
- Recently discovered recursive structure of the loop intgerlas

On-shell gluon scattering amplitudes in $\mathcal{N}=4$ SYM

- ✔ Gluon scattering amplitudes in $\mathcal{N}=4$ SYM

$\mathcal{A}_n =$



- ✗ Quantum numbers of on-shell gluons $|i\rangle = |p_i, h_i, a_i\rangle$:
  momentum ($(p_i^\mu)^2 = 0$), helicity ($h = \pm 1$), color (a)
- ✗ Suffer from IR divergences $\mapsto$ require IR regularization
- ✗ Close cousins of QCD gluon amplitudes

- ✔ Color-ordered planar partial amplitudes

$$\mathcal{A}_n = \operatorname{tr}\left[T^{a_1} T^{a_2} \ldots T^{a_n}\right] A_n^{h_1,h_2,\ldots,h_n}(p_1, p_2, \ldots, p_n) + \text{[Bose symmetry]}$$

  - ✗ Color-ordered amplitudes are classified according to their helicity content $h_i = \pm 1$
  - ✗ Supersymmetry relations:

$$A^{++\ldots+} = A^{-+\ldots+} = 0\,, \qquad A^{(\mathrm{MHV})} = A^{--+\ldots+}\,, \qquad A^{(\mathrm{next-to-MHV})} = A^{---+\ldots+}\,, \quad \ldots$$

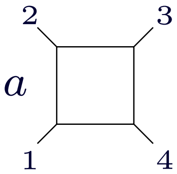
  - ✗ The $n=4$ and $n=5$ planar gluon amplitudes are all MHV

$$\{A_4^{++--}\,,\quad A_4^{+-+-}\,,\quad \ldots\}\,,\quad \{A_5^{+++--}\,,\quad A_5^{+-+--}\,,\quad \ldots\}$$

  - ✗ *Weak/strong coupling corrections to all MHV amplitudes are described by a single function of the 't Hooft coupling and kinematical invariants!* [Parke,Taylor]

Four-gluon amplitude in $\mathcal{N}=4$ SYM at weak coupling

$$M_4(s,t) \equiv \mathcal{A}_4/\mathcal{A}_4^{(\text{tree})} = 1 + a$$



$$+ O(a^2)\,, \quad a = \frac{g_{\text{YM}}^2 N_c}{8\pi^2}\,, \quad s = (p_1+p_2)^2\,,\ t = (p_3+p_4)^2$$

All-order planar amplitudes can be split into (universal) IR divergent and (nontrivial) finite part

$$M_4(s,t) = \text{Div}(s,t,\epsilon_{\text{IR}})\,\text{Fin}(s/t)$$

- ✔ IR divergences appear at all loops as poles in ϵ_{IR} (in dimreg with $D = 4 - 2\epsilon_{\text{IR}}$)

- ✔ IR divergences exponentiate (in any gauge theory!)

$$\text{Div}(s,t,\epsilon_{\text{IR}}) = \exp\left\{ -\frac{1}{2} \sum_{l=1}^{\infty} a^l \left(\frac{\Gamma_{\text{cusp}}^{(l)}}{(l\epsilon_{\text{IR}})^2} + \frac{G^{(l)}}{l\epsilon_{\text{IR}}} \right) \left[(-s/\mu^2)^{l\epsilon_{\text{IR}}} + (-t/\mu^2)^{l\epsilon_{\text{IR}}} \right] \right\}$$

- ✔ *IR divergences* are in one-to-one correspondence with *UV divergences* of cusped Wilson loops

  $\Gamma_{\text{cusp}}(a) = \sum_l a^l \Gamma_{\text{cusp}}^{(l)} =$ *cusp anomalous dimension of Wilson loops*

  $G(a) = \sum_l a^l G_{\text{cusp}}^{(l)} =$ *collinear anomalous dimension*

- ✔ *What about the finite part of the amplitude* $\text{Fin}(s/t)$*? Does it have a simple structure?*

$$\text{Fin}_{\text{QCD}}(s/t) = [\text{4 pages long mess}]\,, \qquad \text{Fin}_{\mathcal{N}=4}(s/t) = \text{BDS conjecture}$$

Finite part of four-gluon amplitude in QCD at two loops

$$\text{Fin}_{\text{QCD}}{}^{(2)}(s,t,u) = A(x,y,z) + O(1/N_c^2, n_f/N_c)$$

[Glover,Oleari,Tejeda-Yeomans'01]

with notations $x = -\frac{t}{s}$, $y = -\frac{u}{s}$, $z = -\frac{u}{t}$, $X = \log x$, $Y = \log y$, $S = \log z$

$$\begin{aligned}
A \;=\; & \Big\{ \Big(48\,\text{Li}_4(x) - 48\,\text{Li}_4(y) - 128\,\text{Li}_4(z) + 40\,\text{Li}_3(x)\,X - 64\,\text{Li}_3(x)\,Y - \tfrac{98}{3}\,\text{Li}_3(x) + 64\,\text{Li}_3(y)\,X - 40\,\text{Li}_3(y)\,Y + 18\,\text{Li}_3(y) \\
& + \tfrac{98}{3}\,\text{Li}_2(x)\,X - \tfrac{16}{3}\,\text{Li}_2(x)\,\pi^2 - 18\,\text{Li}_2(y)\,Y - \tfrac{37}{6}\,X^4 + 28\,X^3\,Y - \tfrac{23}{3}\,X^3 - 16\,X^2\,Y^2 + \tfrac{49}{3}\,X^2\,Y - \tfrac{35}{3}\,X^2\,\pi^2 - \tfrac{38}{3}\,X^2 \\
& - \tfrac{22}{3}\,S\,X^2 - \tfrac{20}{3}\,X\,Y^3 - 9\,X\,Y^2 + 8\,X\,Y\,\pi^2 + 10\,X\,Y - \tfrac{31}{12}\,X\,\pi^2 - 22\,\zeta_3\,X + \tfrac{22}{3}\,S\,X + \tfrac{37}{27}\,X + \tfrac{11}{6}\,Y^4 - \tfrac{41}{9}\,Y^3 - \tfrac{11}{3}\,Y^2\,\pi^2 \\
& - \tfrac{22}{3}\,S\,Y^2 + \tfrac{266}{9}\,Y^2 - \tfrac{35}{12}\,Y\,\pi^2 + \tfrac{418}{9}\,S\,Y + \tfrac{257}{9}\,Y + 18\,\zeta_3\,Y - \tfrac{31}{30}\,\pi^4 - \tfrac{11}{9}\,S\,\pi^2 + \tfrac{31}{9}\,\pi^2 + \tfrac{242}{9}\,S^2 + \tfrac{418}{9}\,\zeta_3 + \tfrac{2156}{27}\,S \\
& - \tfrac{11093}{81} - 8\,S\,\zeta_3 \Big)\,\frac{t^2}{s^2} + \Big(-256\,\text{Li}_4(x) - 96\,\text{Li}_4(y) + 96\,\text{Li}_4(z) + 80\,\text{Li}_3(x)\,X + 48\,\text{Li}_3(x)\,Y - \tfrac{64}{3}\,\text{Li}_3(x) - 48\,\text{Li}_3(y)\,X \\
& + 96\,\text{Li}_3(y)\,Y - \tfrac{304}{3}\,\text{Li}_3(y) + \tfrac{64}{3}\,\text{Li}_2(x)\,X - \tfrac{32}{3}\,\text{Li}_2(x)\,\pi^2 + \tfrac{304}{3}\,\text{Li}_2(y)\,Y + \tfrac{26}{3}\,X^4 - \tfrac{64}{3}\,X^3\,Y - \tfrac{64}{3}\,X^3 + 20\,X^2\,Y^2 \\
& + \tfrac{136}{3}\,X^2\,Y + 24\,X^2\,\pi^2 + 76\,X^2 - \tfrac{88}{3}\,S\,X^2 + \tfrac{8}{3}\,X\,Y^3 + \tfrac{104}{3}\,X\,Y^2 - \tfrac{16}{3}\,X\,Y\,\pi^2 + \tfrac{176}{3}\,S\,X\,Y - \tfrac{136}{3}\,X\,Y - \tfrac{50}{3}\,X\,\pi^2 \\
& - 48\,\zeta_3\,X + \tfrac{2350}{27}\,X + \tfrac{440}{3}\,S\,X + 4\,Y^4 - \tfrac{176}{9}\,Y^3 + \tfrac{4}{3}\,Y^2\,\pi^2 - \tfrac{176}{3}\,S\,Y^2 - \tfrac{494}{9}\,Y\,\pi^2 + \tfrac{5392}{27}\,Y - 64\,\zeta_3\,Y + \tfrac{496}{45}\,\pi^4 \\
& - \tfrac{308}{9}\,S\,\pi^2 + \tfrac{200}{9}\,\pi^2 + \tfrac{968}{9}\,S^2 + \tfrac{8624}{27}\,S - \tfrac{44372}{81} + \tfrac{1864}{9}\,\zeta_3 - 32\,S\,\zeta_3 \Big)\,\frac{t}{u} + \Big(\tfrac{88}{3}\,\text{Li}_3(x) - \tfrac{88}{3}\,\text{Li}_2(x)\,X + 2\,X^4 - 8\,X^3\,Y \\
& - \tfrac{220}{9}\,X^3 + 12\,X^2\,Y^2 + \tfrac{88}{3}\,X^2\,Y + \tfrac{8}{3}\,X^2\,\pi^2 - \tfrac{88}{3}\,S\,X^2 + \tfrac{304}{9}\,X^2 - 8\,X\,Y^3 - \tfrac{16}{3}\,X\,Y\,\pi^2 + \tfrac{176}{3}\,S\,X\,Y - \tfrac{77}{3}\,X\,\pi^2 \\
& + \tfrac{1616}{27}\,X + \tfrac{968}{9}\,S\,X - 8\,\zeta_3\,X + 4\,Y^4 - \tfrac{176}{9}\,Y^3 - \tfrac{20}{3}\,Y^2\,\pi^2 - \tfrac{176}{3}\,S\,Y^2 - \tfrac{638}{9}\,Y\,\pi^2 - 16\,\zeta_3\,Y + \tfrac{5392}{27}\,Y - \tfrac{4}{15}\,\pi^4 - \tfrac{308}{9}\,S\,\pi^2 \\
& - 20\,\pi^2 - 32\,S\,\zeta_3 + \tfrac{1408}{9}\,\zeta_3 + \tfrac{968}{9}\,S^2 - \tfrac{44372}{81} + \tfrac{8624}{27}\,S \Big)\,\frac{t^2}{u^2} + \Big(\tfrac{44}{3}\,\text{Li}_3(x) - \tfrac{44}{3}\,\text{Li}_2(x)\,X - X^4 + \tfrac{110}{9}\,X^3 - \tfrac{22}{3}\,X^2\,Y \\
& + \tfrac{14}{3}\,X^2\,\pi^2 + \tfrac{44}{3}\,S\,X^2 - \tfrac{152}{9}\,X^2 - 10\,X\,Y + \tfrac{11}{2}\,X\,\pi^2 + 4\,\zeta_3\,X - \tfrac{484}{9}\,S\,X - \tfrac{808}{27}\,X + \tfrac{7}{30}\,\pi^4 - \tfrac{31}{9}\,\pi^2 \\
& + \tfrac{11}{9}\,S\,\pi^2 - \tfrac{418}{9}\,\zeta_3 - \tfrac{242}{9}\,S^2 - \tfrac{2156}{27}\,S + 8\,S\,\zeta_3 + \tfrac{11093}{81} \Big)\,\frac{ut}{s^2} + \Big(-176\,\text{Li}_4(x) + 88\,\text{Li}_3(x)\,X - 168\,\text{Li}_3(x)\,Y - \ldots
\end{aligned}$$

Four-gluon amplitude in $\mathcal{N}=4$ SYM at weak coupling II

- Bern-Dixon-Smirnov (BDS) conjecture:

$$\text{Fin}_4(s/t) = 1 + \frac{a}{2}\ln^2(s/t) + O(a^2) \quad \overset{\text{all loops}}{\Longrightarrow} \quad \exp\left[\tfrac{1}{4}\Gamma_{\text{cusp}}(a)\ln^2(s/t)\right]$$

  - Compared to QCD,

    (i) the complicated functions of s/t are replaced by the elementary function $\ln^2(s/t)$;

    (ii) the coefficient of $\ln^2(s/t)$ is determined by the cusp anomalous dimension $\Gamma_{\text{cusp}}(a)$ just like the coefficient of the double IR pole.

  - The conjecture has been verified up to three loops
  - A similar conjecture exists for n-gluon MHV amplitudes
  - It has been confirmed for $n=5$ at two loops
  - Agrees with the strong coupling prediction from the AdS/CFT correspondence [Alday,Maldacena]

- Surprising features of the finite part of the MHV amplitudes in planar $\mathcal{N}=4$ SYM:

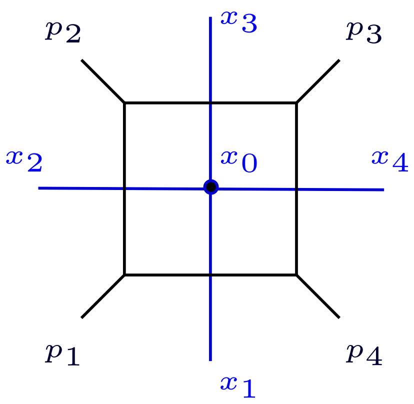
  Why should finite corrections exponentiate? And be related to the cusp anomaly of Wilson loops?

Dual conformal symmetry

Examine one-loop 'scalar box' diagram

- ✔ Change variables to go to a *dual 'coordinate space'* picture (not a Fourier transform!)

$$p_1 = x_1 - x_2 \equiv x_{12}\,, \quad p_2 = x_{23}\,, \quad p_3 = x_{34}\,, \quad p_4 = x_{41}\,, \quad k = x_{10}$$



$$= \int \frac{d^4k\,(p_1+p_2)^2(p_2+p_3)^2}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2} = \int \frac{d^4x_0\; x_{13}^2 x_{24}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}$$

Check conformal invariance by inversion $x_i^\mu \to x_i^\mu / x_i^2$

[Broadhurst],[Drummond,Henn,Smirnov,ES]

- ✔ The integral is invariant under $SO(2,4)$ conformal transformations in dual space!
- ✔ This symmetry *is not related* to the $SO(2,4)$ conformal symmetry of $\mathcal{N}=4$ SYM
- ✔ All scalar integrals contributing to A_4 up to 4 loops are dual conformal! [Bern,Czakon,Dixon,Kosower,Smirnov]
- ✔ The dual conformal symmetry allows us to determine four- and five-gluon planar scattering amplitudes to all loops! [Drummond,Henn,Korchemsky,ES],[Alday,Maldacena]
- ✔ Dual conformality is "slightly" broken by the infrared regulator: $d^4x \;\Rightarrow\; d^{4-2\epsilon}x$
- ✔ For *planar* integrals only!

From gluon amplitudes to Wilson loops

Properties of gluon scattering amplitudes in $\mathcal{N}=4$ SYM:

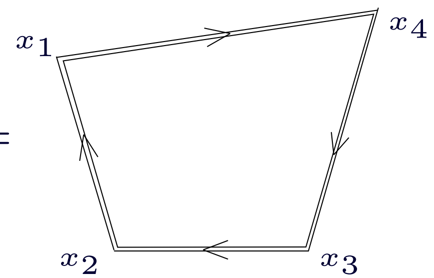
(1) IR divergences of M_4 exactly match UV divergences of *cusped Wilson loops*

(2) Perturbative corrections to M_4 possess a hidden *dual conformal symmetry*

☞ *Is it possible to find an $\mathcal{N}=4$ SYM object for which both properties are manifest ?*

Yes! The expectation value of a light-like Wilson loop in $\mathcal{N}=4$ SYM [Alday,Maldacena], [DHKS]

$$W(C_4) = \frac{1}{N_c}\langle 0|\mathrm{Tr\,P}\exp\left(ig\oint_{C_4} dx^\mu A_\mu(x)\right)|0\rangle\,, \qquad C_4 =$$



✔ Gauge invariant functional of the integration contour C_4 in Minkowski space-time

✔ The contour is made out of 4 light-like segments $C_4 = \ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4$ joining the cusp points x_i^μ

$$x_i^\mu - x_{i+1}^\mu = p_i^\mu = \text{on-shell gluon momenta}$$

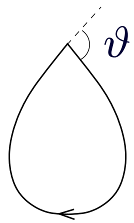
✔ The contour C_4 has four light-like cusps $\mapsto$ $W(C_4)$ has UV divergences

✔ Conformal symmetry of $\mathcal{N}=4$ SYM $\mapsto$ conformal invariance of $W(C_4)$ in dual coordinates x^μ

Cusp anomalous dimension

- Cusp anomaly is a very 'unfortunate' feature of Wilson loops evaluated on a *Euclidean* closed contour with a cusp – generates an anomalous dimension [Polyakov'80]

$$\langle \mathrm{tr}\, \mathsf{P} \exp \left(i \oint_C dx \cdot A(x) \right) \rangle \sim (\Lambda_{\mathrm{UV}})^{\Gamma_{\mathrm{cusp}}(g,\vartheta)} ,$$

$C =$ 

- A very 'fortunate' property of Wilson loops – the cusp anomaly controls the *infrared* asymptotics of scattering amplitudes in gauge theories [Korchemsky, Radyushkin'86]
  - The integration contour C is defined by the particle momenta
  - The cusp angle ϑ is related to the scattering angles in *Minkowski* space-time, $|\vartheta| \gg 1$

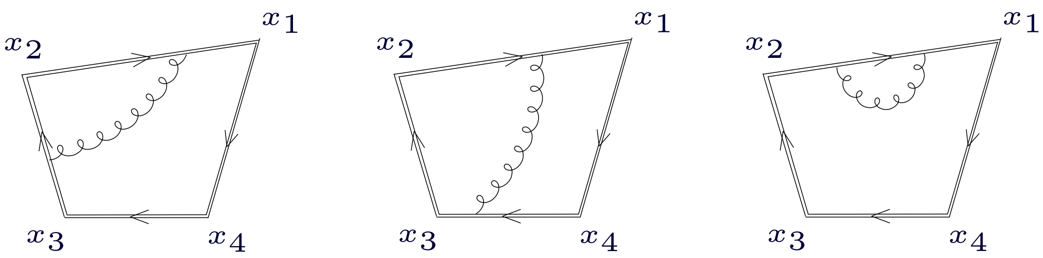
$$\Gamma_{\mathrm{cusp}}(g,\vartheta) = \vartheta\, \Gamma_{\mathrm{cusp}}(g) + O(\vartheta^0) ,$$

- *The cusp anomalous dimension* $\Gamma_{\mathrm{cusp}}(g)$ is an observable in gauge theories appearing in many contexts:
  - Logarithmic scaling of anomalous dimensions of high-spin Wilson operators;
  - IR singularities of on-shell gluon scattering amplitudes;
  - Gluon Regge trajectory;
  - Sudakov asymptotics of elastic form factors;
  - ...

MHV scattering amplitudes/Wilson loop duality I

The one-loop expression for the light-like Wilson loop (with $x_{jk}^2 = (x_j - x_k)^2$)

$\ln W(C_4) =$



$$= \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{\epsilon_{\rm UV}{}^2} \left[(-x_{13}^2\mu^2)^{\epsilon_{\rm UV}} + (-x_{24}^2\mu^2)^{\epsilon_{\rm UV}} \right] + \frac{1}{2} \ln^2 \left(\frac{x_{13}^2}{x_{24}^2} \right) + \text{const} \right\} + O(g^4)$$

The one-loop expression for the gluon scattering amplitude

$$\ln M_4(s,t) = \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{\epsilon_{\rm IR}{}^2} \left[(-s/\mu_{\rm IR}^2)^{\epsilon_{\rm IR}} + (-t/\mu_{\rm IR}^2)^{\epsilon_{\rm IR}} \right] + \frac{1}{2} \ln^2 \left(\frac{s}{t} \right) + \text{const} \right\} + O(g^4)$$

✔ Identify the light-like segments with the on-shell gluon momenta $x_{i,i+1}^\mu \equiv x_i^\mu - x_{i+1}^\mu := p_i^\mu$:

$$x_{13}^2\,\mu^2 := s/\mu_{\rm IR}^2\,, \qquad x_{24}^2\,\mu^2 := t/\mu_{\rm IR}^2\,, \qquad x_{13}^2/x_{24}^2 := s/t$$

☞ UV divergences of the light-like Wilson loop match IR divergences of the gluon amplitude

☞ the finite $\sim \ln^2(s/t)$ corrections coincide at one loop!

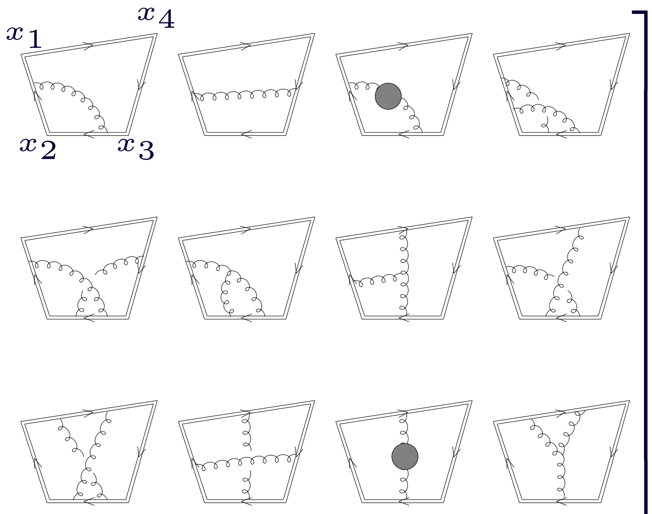
MHV scattering amplitudes/Wilson loop duality II

Conjecture: *MHV gluon amplitudes are dual to light-like Wilson loops*

$$\ln \mathcal{A}_4 = \ln W(C_4) + O(1/N_c^2, \epsilon_{\rm IR})\,.$$

- ✔ At strong coupling, the relation holds to leading order in $1/\sqrt{\lambda}$ [Alday,Maldacena]
- ✔ At weak coupling, the relation was verified at two loops [Drummond,Henn,Korchemsky,ES]

$$\ln \mathcal{A}_4 = \ln W(C_4) = \left[\ \right]$$



$$= \frac{1}{4}\Gamma_{\rm cusp}(g)\ln^2(s/t) + \text{Div}$$

- ✔ Generalization to $n \geq 5$ gluon MHV amplitudes

$$\ln \mathcal{A}_n^{(\rm MHV)} = \ln W(C_n) + O(1/N_c^2)\,, \quad C_n = \text{light-like } n\text{−(poly)gon}$$

  - ✗ At weak coupling, matches the n-gluon amplitude at one loop [Brandhuber,Heslop,Travaglini]
  - ✗ The duality relation for $n = 5$ (pentagon) was verified at two loops [DHKS]

Conformal Ward identities for light-like Wilson loops

Main idea: *Make use of the conformal invariance of light-like Wilson loops in $\mathcal{N}=4$ SYM + duality relation to constrain the finite part of $n-$gluon amplitudes*

- ✔ Conformal transformations map the light-like polygon C_n into another light-like polygon C'_n
- ✔ If the Wilson loop $W(C_n)$ were well defined (=finite) in $D=4$ dimensions, we would have

$$W(C_n)=W(C'_n)$$

- ✔ ... but $W(C_n)$ has cusp UV singularities $\mapsto$ dimreg breaks conformal invariance

$$W(C_n)=W(C'_n)\times \text{[cusp anomaly]}$$

- ✔ *All-loop* anomalous conformal Ward identities for the *finite part* of the Wilson loop

$$W(C_n)=\exp(F_n)\times \text{[UV divergences]}$$

Under dilatations, $\mathbb{D}$, and special conformal transformations, $\mathbb{K}^\mu$, [DHKS]

$$\mathbb{D}\,F_n \equiv \sum_{i=1}^{n}(x_i\cdot\partial_{x_i})F_n = 0$$

$$\mathbb{K}^\mu\,F_n \equiv \sum_{i=1}^{n}\left[2x_i^\mu(x_i\cdot\partial_{x_i})-x_i^2\partial_{x_i}^\mu\right]F_n = \frac{1}{2}\Gamma_{\rm cusp}(a)\sum_{i=1}^{n}x^\mu_{i,i+1}\ln\Big(\frac{x^2_{i,i+2}}{x^2_{i-1,i+1}}\Big)$$

The same relations also hold at strong coupling [Alday,Maldacena],[Komargodski]

Finite part of MHV amplitudes

Corollaries of the conformal WI for the finite part of the Wilson loop/ MHV scattering amplitudes:

- ✔ $n = 4, 5$ are special: there are no conformal invariants (too few distances due to $x^2_{i,i+1} = 0$)
  $\Longrightarrow$ the Ward identity has a *unique all-loop solution* (up to an additive constant)

$$F_4 = \frac{1}{4}\Gamma_{\text{cusp}}(a)\ln^2\left(\frac{x^2_{13}}{x^2_{24}}\right) + \text{ const } ,$$

$$F_5 = -\frac{1}{8}\Gamma_{\text{cusp}}(a)\sum_{i=1}^{5}\ln\left(\frac{x^2_{i,i+2}}{x^2_{i,i+3}}\right)\ln\left(\frac{x^2_{i+1,i+3}}{x^2_{i+2,i+4}}\right) + \text{ const}$$

  Exactly the functional forms of the BDS ansatz for the 4- and 5-point MHV amplitudes!

- ✔ Starting from $n = 6$ there are conformal invariants in the form of cross-ratios, e.g.

$$u_1 = \frac{x^2_{13}x^2_{46}}{x^2_{14}x^2_{36}}, \qquad u_2 = \frac{x^2_{24}x^2_{15}}{x^2_{25}x^2_{14}}, \qquad u_3 = \frac{x^2_{35}x^2_{26}}{x^2_{36}x^2_{25}}$$

  Hence the general solution of the Ward identity for $W(C_n)$ with $n \geq 6$ contains *an arbitrary function* of the conformal cross-ratios.

- ✔ The BDS ansatz is a solution of the conformal Ward identity for *arbitrary* n but does it actually work for $n \geq 6$? [Alday, Maldacena] [Bartels, Lipatov, Sabio Vera]

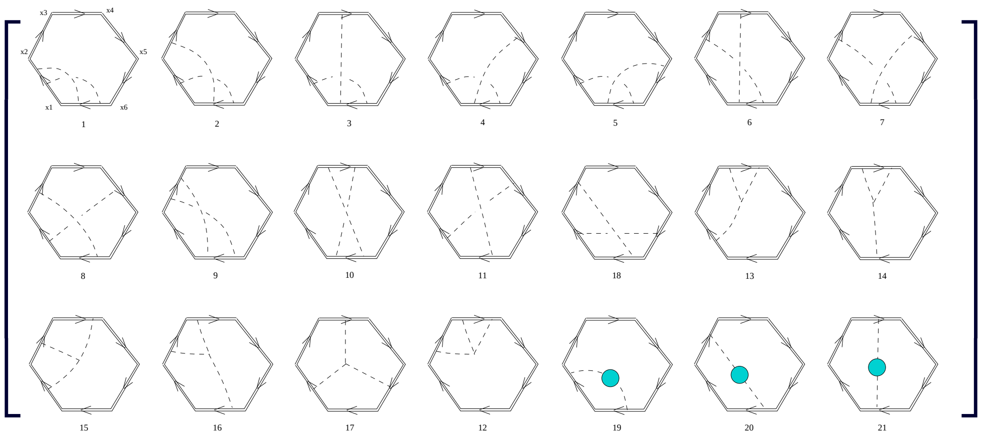
- ✔ If not, what is the “remainder" function

$$R(u_1, u_2, u_3) = \ln\mathcal{M}_6^{(\text{MHV})} - \ln\mathcal{M}_6^{(\text{BDS})}$$

Remainder function

✔ We computed the two-loop hexagon Wilson loop $W(C_6)$ ... [DHKS]

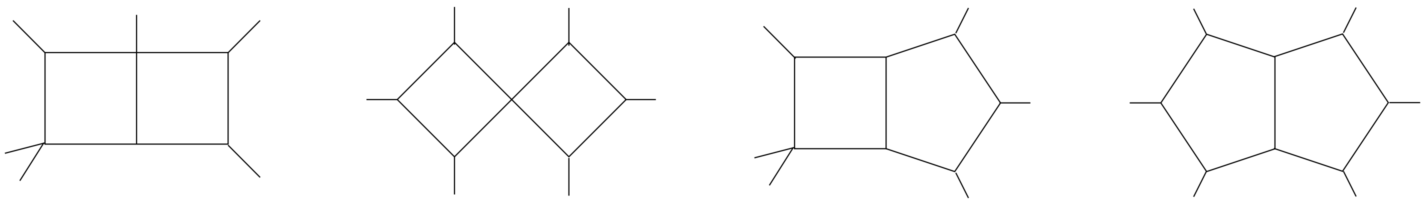
$$\ln W(C_6) =$$



... and found a **discrepancy** $\ln W(C_6) \neq \ln \mathcal{M}_6^{(\mathrm{BDS})}$

✔ Bern-Dixon-Kosower-Roiban-Spradlin-Vergu-Volovich computed the 6-gluon 2-loop amplitude

$$\mathcal{M}_6^{(\mathrm{MHV})} =$$



$$+ \ldots$$

... and found a **discrepancy** $\ln \mathcal{M}_6^{(\mathrm{MHV})} \neq \ln \mathcal{M}_6^{(\mathrm{BDS})}$

☞ The BDS ansatz **fails** for $n = 6$ starting from two loops.

☞ ... but the Wilson loop/MHV amplitude duality still holds $\ln \mathcal{M}_6^{(\mathrm{MHV})} = \ln W(C_6)$

All-order MHV superamplitude

✔ All MHV amplitudes can be combined into a single superamplitude

$$\mathcal{A}_n^{\rm MHV}(p_1,\eta_1;\ldots;p_n,\eta_n) = i(2\pi)^4 \frac{\delta^{(4)}\left(\sum_{i=1}^n p_i\right)\,\delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A\right)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle} M_n^{(\rm MHV)}\,,$$

Here $p_i^{\alpha\dot\alpha} = \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha}$ solves $p_i^2=0$, and η_i^A ($A=1\ldots4$) are Grassmann variables.
Helicity: $h[\lambda]=1/2$, $h[\tilde\lambda]=h[\eta]=-1/2$

✗ Perturbative corrections to all MHV amplitudes are factorized into a universal factor $M_n^{(\rm MHV)}$

✗ The all-loop MHV amplitudes are the coefficients in the expansion of $\mathcal{A}_n^{\rm MHV}$ in powers of η's

$$\mathcal{A}_n^{\rm MHV} = (2\pi)^4\delta^{(4)}\Big(\sum_{i=1}^n p_i\Big)\sum_{1\le j<k\le n}(\eta_j)^4(\eta_k)^4 A_n^{(\rm MHV)}(1^+\ldots j^-\ldots k^-\ldots n^+) + \ldots\,,$$

✗ The function $M_n^{(\rm MHV)}$ is dual to a light-like $n-$gon Wilson loop

$$\ln M_n^{(\rm MHV)} = \ln W_n + O(\epsilon, 1/N^2)$$

✔ The MHV superamplitude possesses a bigger, dual superconformal symmetry which acts on the dual coordinates x_i^μ and their superpartners $\theta_{i\,\alpha}^A$ [DHKS]

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu\,,\qquad \lambda_i^\alpha\,\eta_i = \theta_i^\alpha - \theta_{i+1}^\alpha$$

Dual superconformal invariance

- Tree-level MHV superamplitude (in the spinor formalism $\langle ij\rangle = \lambda_i^\alpha \lambda_{j\,a}$)

$$\mathcal{A}_n^{\mathrm{MHV;tree}} = i(2\pi)^4 \frac{\delta^{(4)}\left(\sum_{i=1}^n p_i\right)\ \delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A\right)}{\langle 12\rangle\langle 23\rangle \dots \langle n1\rangle}$$

- The same amplitude in the dual superspace $\quad p_i^\mu = x_i^\mu - x_{i+1}^\mu\,, \quad \lambda_i^\alpha\,\eta_i = \theta_i^\alpha - \theta_{i+1}^\alpha$

$$\mathcal{A}_n^{\mathrm{MHV;tree}} = i(2\pi)^4 \frac{\delta^{(4)}\left(x_1 - x_{n+1}\right)\ \delta^{(8)}(\theta_1 - \theta_{n+1})}{\langle 12\rangle\langle 23\rangle \dots \langle n1\rangle}$$

- Define inversions in the dual superspace

$$I[\lambda_i^\alpha] = (x_i^{-1})^{\dot\alpha\beta}\lambda_{i\beta}\,, \qquad I[\theta_i^{\alpha\,A}] = (x_i^{-1})^{\dot\alpha\beta}\theta_{i\,\beta}^A$$

  Neighboring contractions are dual conformal covariant

$$I[\langle i\, i+1\rangle] = (x_i^2)^{-1}\langle i\, i+1\rangle$$

- The tree-level MHV amplitude is covariant under dual conformal inversions

$$I\left[\mathcal{A}_n^{\mathrm{MHV;tree}}\right] = \left(x_1^2 x_2^2 \dots x_n^2\right) \times \mathcal{A}_n^{\mathrm{MHV;tree}}$$

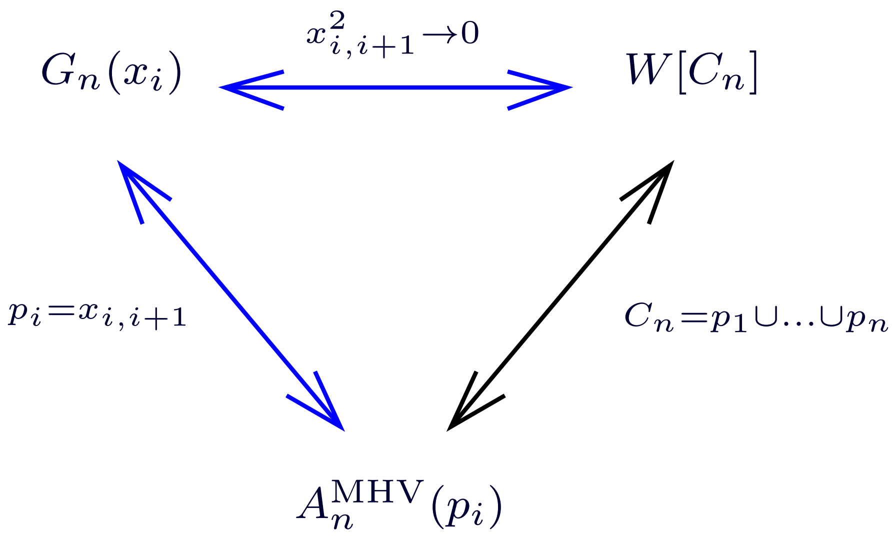
- Generalization: dual superconformal covariance is a property of all tree-level superamplitudes (MHV, NMHV, N^{2}MHV, ...) in $\mathcal{N}=4$ SYM theory

Triality correlators/Wilson loops/amplitudes in planar $\mathcal{N}=4$ SYM

✔ Three natural observables in a conformal gauge theory:

- ✗ Correlators of gauge inv. operators: $G_n(x_i) = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\ldots\mathcal{O}(x_n)\rangle$
- ✗ Light-like Wilson loops: $W[C_n] = \frac{1}{N_c}\langle 0|\,\mathrm{tr}\,\mathrm{P}\exp\left(i\oint_{C_n} dx\cdot A(x)\right)|0\rangle$
- ✗ Scattering amplitudes: $A_n(p_i) = \langle p_1, p_2, \ldots, p_n|S|0\rangle$

✔ They seem to be related to each other in planar $\mathcal{N}=4$ SYM:



- ✗ The duality $A_n^{\mathrm{MHV}}(p_i) \leftrightarrow W[C_n]$ is well studied (but still not understood)
- ✗ The dualities $A_n^{\mathrm{MHV}} \leftrightarrow G_n$ and $G_n \leftrightarrow W[C_n]$ are new [Alday,Eden,Korchemsky,Maldacena,ES'10]
- ✗ Triality relation in planar $\mathcal{N}=4$ SYM

Correlation functions

- Protected superconformal operators made from the 6 scalars $\phi_{AB} = \frac{1}{2}\epsilon_{ABCD}\bar{\phi}^{CD}$

$$\mathcal{O}(x) = \mathrm{Tr}(\phi_{12}\phi_{12})\,, \qquad \tilde{\mathcal{O}}(x) = \mathrm{Tr}(\bar{\phi}^{12}\bar{\phi}^{12}) : \qquad \mathbf{20'} \text{ of } SU(4)$$

  Quantum conformal dimension = tree-level canonical dimension

  Two- and three-point correlation functions do not receive quantum corrections

- Simplest non-trivial correlation function

$$G_4 = \langle \mathcal{O}(x_1)\tilde{\mathcal{O}}(x_2)\mathcal{O}(x_3)\tilde{\mathcal{O}}(x_4)\rangle = \frac{N_c^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2}\mathcal{F}(u,v;a)$$

  Conformal cross-ratios

$$u = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$$

  The conformally invariant (coupling dependent) function $\mathcal{F}(u,v;a)$ is finite as long as $x_i \neq x_j$

  The limit $x_i \to x_j$ corresponds to the standard OPE

- Novel limit: all neighboring points simultaneously become light-like separated

$$x_{i,i+1}^2 \to 0\,, \qquad x_i \neq x_{i+1}\,, \qquad (i = 1, \dots, n)$$

Correlation functions on the light-cone

The light-cone limit of G_4 is singular:

(i) For $x^2_{i,i+1} \to 0$ the correlator develops pole singularities already at tree level

$$G_4^{(\text{tree})} \sim \frac{N_c^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2} + \text{subleading terms}$$

The way out – consider the ratio

$$\mathcal{F}_4 \equiv \lim_{x^2_{i,i+1}\to 0} G_4(x_i)/G_4^{(\text{tree})}(x_i)$$

(ii) Loop integrals develop additional light-cone singularities (cross-ratios vanish $u, v \to 0$)

$$\mathcal{F}_4 = 1 + a\,\frac{i}{\pi^2}\int \frac{d^4x_0\, x_{13}^2 x_{24}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} + \ldots = 1 - a \ln u \ln v + \ldots\,.$$

Leading divergent terms can be resummed *to all loops*

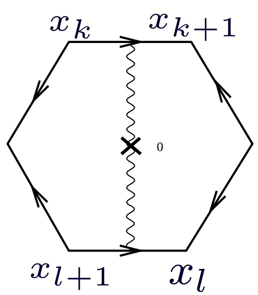
$$\mathcal{F}_4 \sim \exp\left(-\frac{1}{2}\Gamma_{\text{cusp}}(a)\ln u \ln v\right)$$

✔ Light-cone singularities need to be regularized

Correlation functions on the light-cone II

- ✔ Two choices of regularization procedure:
  - ✗ Use the small distances $\delta = x^2_{i,i+1}$ as a cutoff in $D = 4$ dimensions;
  - ✗ Employ dimensional regularization with $D = 4 - 2\epsilon$ and set $x^2_{i,i+1} = 0$ from the start
- ✔ One-loop calculation of the correlation function in dimreg for $x^2_{i,i+1} = 0$

$$\left[G_n/G_n^{(\mathrm{tree})}\right]_{\mathrm{light-cone}} = \sum_{k>l=1}^{n}$$



- ✔ Result ($x_{s_k} = x_k - s_k x_{k,k+1}$; $D_{\mu\nu}$ – gluon propagator in Landau gauge)

$$\ln\left[G_n/G_n^{(\mathrm{tree})}\right]_{\mathrm{l.c.}} = (ig)^2 N_c \sum_{k>l}\int_0^1 ds_k \int_0^1 ds_l\, x^\mu_{k,k+1} x^\nu_{l,l+1} D_{\mu\nu}(x_{s_k,s_l}) + \ldots = 2\ln W[C_n]$$

Coincides with the one-loop expression for the light-like polygonal Wilson loop!

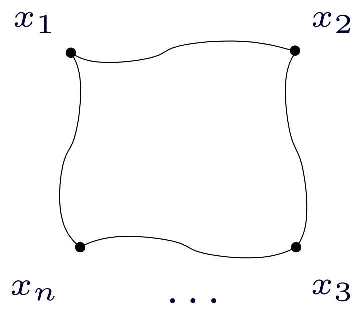
$$\left[G_n/G_n^{(\mathrm{tree})}\right]_{\mathrm{l.c.}} \propto (W[C_n])^2$$

The square comes from $\mathrm{adjoint} = (\mathrm{fundamental})^2$ of the gauge group

From correlation functions to Wilson loops

- ✔ Correlation function on the light-cone

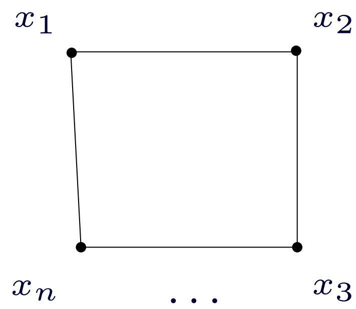
$$G_n \to \sum_C \mathrm{e}^{-iL(C)}\, \langle 0|\mathrm{Tr}_{\mathrm{adj}}\, \mathrm{P}\, \mathrm{e}^{i\oint_C dx^\mu A_\mu(x)}|0\rangle\,, \qquad C =$$



Infinitely fast particle interacting with a slowly varying gauge field (for $x_{i,i+1}^2\mu^2 \ll 1$ only!)

- ✔ The path-integral is dominated by the saddle point $C_n =$ classical trajectory of a particle

$$G_n \to G_n^{(\mathrm{tree})} \times \langle 0|\mathrm{Tr}_{\mathrm{adj}}\, \mathrm{P}\, \mathrm{e}^{i\oint_{C_n} dx^\mu A_\mu(x)}|0\rangle\,, \qquad C_n =$$



- ✔ All-loop result, valid in any gauge theory

$$\lim_{x_{i,i+1}^2 \to 0} \left(G_n / G_n^{(\mathrm{tree})} \right) = W_{\mathrm{adj}}[C_n] = \left(W[C_n] \right)^2 + O(1/N_c^2)$$

$W[C_n]$ satisfies anomalous conformal Ward identity $\longrightarrow$ new results for light-cone asymptotics of the correlation functions G_n!

From correlation functions to amplitudes

New duality between integrands of correlators and amplitudes

- Correlation functions as path integrals

$$G_n = \int \mathcal{D}\phi \exp\left\{\frac{i}{g^2}\int d^4x_0 L_{\mathcal{N}=4}(x_0)\right\} \mathcal{O}(x_1)\tilde{\mathcal{O}}(x_2)\dots\mathcal{O}(x_{n-1})\tilde{\mathcal{O}}(x_n)$$

- Compute one-loop correction via Lagrangian insertion:

$$g^2\frac{\partial}{\partial g^2}G_n = -i\int d^4x_0 \langle L(x_0)\mathcal{O}(x_1)\tilde{\mathcal{O}}(x_2)\dots\mathcal{O}(x_{n-1})\tilde{\mathcal{O}}(x_n)\rangle^{\text{tree}} + O(g^4)$$

- Multiloop corrections $\Leftrightarrow$ multiple Lagrangian insertions

- Example: four points at one loop

$$\lim_{x^2_{i,i+1}\to 0}\frac{\langle L(x_0)\mathcal{O}(x_1)\tilde{\mathcal{O}}(x_2)\mathcal{O}(x_3)\tilde{\mathcal{O}}(x_4)\rangle^{\text{tree}}}{\langle\mathcal{O}(x_1)\tilde{\mathcal{O}}(x_2)\mathcal{O}(x_3)\tilde{\mathcal{O}}(x_4)\rangle^{\text{tree}}} = \frac{x^2_{13}x^2_{24}}{x^2_{10}x^2_{20}x^2_{30}x^2_{40}} \Leftrightarrow \frac{(p_1+p_2)^2(p_2+p_3)^2}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2}$$

  This is the integrand of the one-loop amplitude!

- New duality between conformal objects in four dimensions (no regularization!)

- Has been verified at one loop for arbitrary n and at two loops for $n = 4,\ 5,\ 6$

- New result: Complete agreement with the twistor construction of Arkani-Hamed et al

Conclusions and recent developments

- ✔ MHV amplitudes in $\mathcal{N} = 4$ theory
  - ✗ possess dual conformal symmetry both at weak and at strong coupling
  - ✗ Dual to light-like Wilson loops

  ... but what about NMHV, NNMHV, *etc.* amplitudes?

- ✔ This symmetry is part of a bigger dual superconformal symmetry of all planar tree-level superamplitudes in $\mathcal{N} = 4$ SYM [DHKS], [Brandhuber,Heslop,Travaglini]
  - ✗ Relates various particle amplitudes with different helicity configurations (MHV, NMHV,...)
  - ✗ Interesting twistor space structure [Witten'03], [Arkani-Hamed et al], [Hodges], [Mason,Skinner], [Korcemsky,ES]
  - ✗ Broken by loop corrections, but how?

- ✔ Dual superconformal symmetry is now explained better through the AdS/CFT correspondence by a combined bosonic [Kallosh,Tseytlin] and fermionic T-duality symmetry [Berkovits,Maldacena], [Beisert,Ricci,Tseytlin,Wolf]

- ✔ What is the generalization of the Wilson loop/amplitude duality beyond MHV?
- ✔ What is the role of ordinary superconformal symmetry?
  - ✗ Exact symmetry at tree level, closure [ordinary, dual] = Yangian [Drummond,Henn,Plefka]
  - ✗ Not sufficient to fix the tree, need analytic properties [Korchemsky,ES], [Beisert et al]
  - ✗ At loop level broken by IR divergences, hard to control
- ✔ Is the theory integrable (in some sense)?