[assumption]Theorem [assumption]Corollary [assumption]Lemma [assumption]Definition

κ-Minkowski star product and its symmetries

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THE LANDSCAPE:

- κ-Minkowski was introduced by Lukierski, Nowicki, Ruegg:

$$[x_0, x_i] = \frac{i}{\kappa} x_i, \quad [x_i, x_j] = 0, \quad i,j = 1,\ldots,d-1, \tag{1}$$

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- So far: most of the approaches were formal or not manageable.
- Earlier attempts: D'Andrea, Agostini, Dabrowski-Piacitelli.

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- We fix κ:

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- the corresponding connected and simply connected Lie group is the group G_2 of 2×2-matrices of the form:

$$S(a, b) = \begin{pmatrix} e^{-a} & b \\ 0 & 1 \end{pmatrix}, \quad a, b \in \mathbb{R}, \tag{4}$$

THE CONVOLUTION ALGEBRA

- Let $\mathcal{A}$ be the the convolution algebra of G_2 with respect to the right invariant measure. Identifying functions on G_2 with functions on $\mathbb{R}^2$:

$$(f \hat{\star} g)(a,b) = \int da' db' f(a-a', b-e^{a'-a}b')g(a',b'), \tag{5}$$

$$f^\dagger(a,b) = e^a \bar{f}(-a, -e^a b), \tag{6}$$

where $f, g \in \mathcal{A}$ and $\bar{f}$ is the complex conjugate of f.

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where $f, g \in \mathcal{A}$ and $\bar{f}$ is the complex conjugate of f.

- If π is a unitary representation (always assumed to be strongly continuous) of G_2 it is well known that π gives rise to a $\star$-representation, also denoted by π, of $\mathcal{A}$ by setting

$$\pi(f) = \int dadb\, f(a,b) \pi(S(a,b)). \tag{7}$$

Thus, we have

$$\pi(f \hat{*} g) = \pi(f)\pi(g) \quad \text{and} \quad \pi(f^\dagger) = \pi(f)^*. \tag{8}$$

WEYL QUANTIZATION

Following the same procedure as in the case of the Weyl quantisation we define the Weyl map W_π associated with the representation π by

$$W_\pi(f) = \pi(\mathcal{F}f) \text{ for } f \in \mathrm{L}_1(\mathbb{R}^2) \cap \mathcal{F}^{-1}(\mathrm{L}_1(\mathbb{R}^2)),$$

where $\mathcal{F}$ denotes the Fourier transformation.

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where $\mathcal{F}$ denotes the Fourier transformation. Finally we define the star-product:

DEFINITION

$$f * g = \frac{1}{2\pi}\mathcal{F}^{-1}\left((\mathcal{F}f)\hat{*}(\mathcal{F}g)\right). \tag{9}$$

and

$$f^* = \mathcal{F}^{-1}(\mathcal{F}(f)^\dagger), \tag{10}$$

κ-$*$-PRODUCT: THE DOMAIN AND THE FORMULA.

As in the case of the standard Moyal product, one needs to exercise some care about the domain of definition for the righthand sides of (9) and (10). As a first result in this direction we note the following.

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PROPOSITION

Let $\mathcal{C}_c^\infty$ denote the space of smooth functions on $\mathbb{R}^2$ with compact support. If $f, g \in \mathcal{F}^{-1}(\mathcal{C}_c^\infty)$ then f^* and $f * g$ also belong to $\mathcal{F}^{-1}(\mathcal{C}_c^\infty)$ and are given by

$$f * g(\alpha, \beta) = \frac{1}{2\pi} \int dv \int du\, f(\alpha + u, \beta) g(\alpha, e^{-v}\beta) e^{-iuv} \tag{11}$$

and

$$f^*(\alpha, \beta) = \frac{1}{2\pi} \int dv \int du\, \bar{f}(\alpha + u, e^{-v}\beta) e^{-iuv}. \tag{12}$$

respectively.

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Clearly, the $*$-product defined earlier is well defined as a function on $\mathbb{R}^2$ for a larger class of functions than those discussed above. We note the following:

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- If $g(\alpha, \beta) = g(\alpha)$ is any function depending only on α and f is, say, a Schwartz function of α for fixed value of β, the $f * g$ is well defined and $f * g(\alpha, \beta) = f(\alpha, \beta)g(\alpha)$.
- If $f(\alpha, \beta) = f(\beta)$ is any function depending only on β and g is e.g. smooth with compact support as a function of β for fixed value of α, then the integral in (11) can be interpreted in a distributional sense and yields $f * g(\alpha, \beta) = f(\beta)g(\alpha, \beta)$.

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- If $f(\alpha,\beta) = \alpha$ and $g(\alpha,\beta) = g(\beta)$ is a smooth function of β of compact support, a distributional interpretation yields

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- In particular,

$$\alpha * g(\beta) - g(\beta) * \alpha = i\beta g'(\beta) \, .$$

  Note that, formally, setting $g(\beta) = \beta$ in this relation yields a representation of the defining relation (2) in terms of a $*$-commutator with t, x corresponding to α, β. Note also that, $\alpha^* = \alpha$ and $\beta^* = \beta$ by (12), suitably interpreted.

THE COMPLETION OF THE ALGEBRA $\mathcal{B}$

Let $\mathcal{B}'$ denote the subspace of $\mathcal{B}$ consisting of Fourier transforms of derivatives w.r.t. the second variable of functions in $\mathcal{C}_0^\infty$. $\mathcal{B}'_2$ is dense in $\mathrm{L}_2(\mathbb{R}^2, d\mu)$, where the measure $d\mu = |\beta|^{-1} d\alpha d\beta$.

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$$\begin{aligned}
\langle\varphi, W_\pm(f)\psi\rangle &= \int dadbds\, \mathcal{F}f(a,b)\,\overline{\varphi}(s)\, e^{\pm ibe^{-s}}\psi(s+a) \\
&= \int dsdudb\, \overline{\varphi}(s)\mathcal{F}f(u-s,b)\, e^{\pm ibe^{-s}}\psi(u) \\
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The map

$$W : f \to W_+(f) \oplus W_-(f)$$

is injective from $\mathrm{L}_2(\mathbb{R}^2, d\mu)$ into $\mathcal{H} \oplus \mathcal{H}$, where $\mathcal{H}$ denotes the space of Hilbert-Schmidt operators on $\mathrm{L}_2(\mathbb{R})$.

THE COMPLETION OF THE ALGEBRA $\mathcal{B}$

THEOREM

Let $\mathcal{B}'$ and W be as defined above and set $\bar{\mathcal{B}} = \mathrm{L}_2(\mathbb{R}^2, d\mu)$. Then the $*$-product (11) *and involution* (12) *have unique extensions from $\mathcal{B}'$ to $\bar{\mathcal{B}}$, such that $\mathcal{B}$ becomes an involutive algebra and W an isomorphism,*

$$W(f * g) = W(f)W(g) \qquad W(f^*) = W(f)^* .$$

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COROLLARY

The integrals w.r.t. $d\mu$ over $\mathbb{R} \times \mathbb{R}_\pm$ are positive traces on $\bar{\mathcal{B}}$ in the sense that

$$\int duds(f * f^*)(u, \pm e^{-s}) \geq 0,$$

$$\int duds(f * g)(u, \pm e^{-s}) = \int duds(g * f)(u, \pm e^{-s}).$$

THE LEFT-INVARIANT STAR-PRODUCT

- We can apply the same procedure as above using instead the left invariant Haar measure on G_2. It is then convenient to use the parametrisation

$$R(a,c) = S(a, e^{-a}c)\,, \quad a, c \in \mathbb{R}\,, \tag{13}$$

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- The corresponding star-product $*_\phi$ and involution $^{*_\phi}$ are given by:

$$f \star g(\alpha,\beta) = \frac{1}{2\pi}\int dv \int du\, f(\alpha, e^v\beta) g(\alpha+u,\beta) e^{-iuv}\,, \tag{14}$$

and the involution * is

$$f^\star(\alpha,\beta) = \frac{1}{2\pi}\int dv \int du\, \bar{f}(\alpha+u, e^v\beta) e^{-iuv}\,. \tag{15}$$

All star product...

Proposition

Assume ϕ is positive and smooth. For $f \in \mathrm{L}_1(\mathbb{R}^2) \cap \mathcal{F}^{-1}(\mathrm{L}_1(\mathbb{R}^2))$ the operators $W_{\pm}^{\phi}(f)$ are integral operators on $\mathrm{L}_2(\mathbb{R})$ with kernels given by

$$K_f^{\pm}(s,u) = \sqrt{2\pi}\tilde{f}(u-s, \pm\phi(u-s)e^{-s}) = \int dv f(v, \pm\phi(u-s)e^{-s})e^{-iv(u-s)}.$$

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It can now be seen that the norm and trace formulas hold independently of the choice of ϕ.

THEOREM

The involutive algebras $\mathcal{B}_\phi$, resp. $\bar{\mathcal{B}}_\phi$, where ϕ is positive and smooth are isomorphic.

COPRODUCT, COUNIT AND ANTIPODE

The κ-Minkowski, as originally defined by LNR has a natural Hopf algebra structure, which arises by dualisation from the momenta subalgebra of the κ-Poincaré. The coalgebra structure alone is underformed when compared to the classical case:

$$\triangle x_0 = x_0 \otimes 1 + 1 \otimes x_0, \quad x_i = x_i \otimes 1 + 1 \otimes x_i. \tag{16}$$

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PROPOSITION

The usual cocommutative coproduct on the space of functions $\mathcal{B}$, $\triangle : \mathcal{B} \to \mathcal{B}_2$

$$\triangle f(\alpha, \beta; \alpha', \beta') = f(\alpha + \alpha'; \beta + \beta').$$

together with the counit map: $\varepsilon : \mathcal{B} \to \mathbb{C}$, and the antipode $S : \mathcal{B} \to \mathcal{B}$:

$$\varepsilon : \mathcal{B} \ni f \mapsto f(0,0) \in \mathbb{C}, \quad S(f)(\alpha, \beta) = \int dpds\, f(-\alpha - s, -e^p \beta) e^{-ips},$$

equip $\mathcal{B}$ with a Hopf algebra structure.

THE LORENTZ SYMMETRY

$$[P, E] = 0, \qquad [N, E] = P,$$

$$\triangle P = P \otimes 1 + e^{-\frac{E}{\kappa}} \otimes P, \quad \triangle E = E \otimes 1 + 1 \otimes E,$$

and for the relations involving the boost,

$$[N, P] = -\frac{\kappa}{2}(1 - e^{\frac{-2E}{\kappa}}) + \frac{1}{2\kappa}P^2, \quad \triangle N = N \otimes 1 + e^{-\frac{E}{\kappa}} \otimes N.$$

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DEFINITION

Let $f \in \mathcal{B}$ so that it is a Fourier transform of $\hat{f} \in C_0(\mathbb{R}^2)$. We define a one parameter group of linear operations on $\mathcal{B}$ in the following way. For any $\gamma \in \mathbb{R}$ let:

$$T_\gamma(f)(\alpha, \beta) = \frac{1}{2\pi} \int dudv\, \hat{f}(u, v) e^{\gamma u} e^{-i(\alpha u + \beta v)}. \qquad (17)$$

THE ACTION OF TRANSLATION

The explicit formula for the map $T_\gamma : f \to T_g f$ is:

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PROPOSITION

The map $T_\gamma : f \to T_\gamma f$ is an algebra automorphism.

The T_γ automorphism does not preserve the involution in the algebra $\mathcal{B}$, however we have:

$$T_\gamma(f^*) = (T_{-\gamma} f)^*.$$

THE LORENTZ SYMMETRY

PROPOSITION

The algebra $\mathcal{B}$ is a Hopf module algebra with respect to the following action of the momentum algebra, with the generators $E, P, \mathcal{E}$ represented as linear operators on $\mathcal{B}$.

$$(E \triangleright f)(\alpha, \beta) = \frac{\partial}{\partial \alpha}, \quad (P \triangleright f)(\alpha, \beta) = \frac{\partial}{\partial \beta}, \quad (\mathcal{E} \triangleright f)(\alpha, \beta) = (T_1 f)(\alpha, \beta).$$

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PROPOSITION

A linear operator N:

$$N = -L_\alpha P - \frac{i}{2}(1 - \mathcal{E}^2) L_\beta + \frac{i}{2} L_\beta P^2,$$

acts on $\mathcal{B}$ as boost.

THE HAAR INTEGRAL

REMARK

The usual Lebesgue measure on $\mathbb{R}^2$ gives rise to a left (and right) integrals on the Hopf algebra $\mathcal{B}$:

$$\Omega_H(f) := \int dadb\, f(a,b),$$

$$\Omega_H(f) = \int da' db'\, f(a+a', b+b') = \int dadb\, f(a+a', b+b').$$

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The Haar integral Ω_H has the following properties:

- Ω_H is compatible with the conjugation in $\mathcal{B}$, for every $f \in \mathcal{B}$:

$$\Omega_H(f) = \overline{\Omega(f^*)},$$

THE HAAR INTEGRAL - PROPERTIES

- Ω_H is a twisted trace, that is for all $f, g \in \mathcal{B}$:

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- For every $f \in \mathcal{B}$:

$$\Omega_H(f * f^*) = ||f||^2_{L_2},$$

REINTRODUCING κ:

We can write the product with κ present:

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The formal powers series expansion of the product:

$$(f *_\kappa g)(\alpha,\beta) = \sum_n \frac{i^n}{\kappa^n n!} (\partial_\alpha^n f(\alpha,\beta)) \left(\sum_{k=1}^{n} B_k^n \beta^k (\partial_\beta^k g)(\alpha,\beta) \right).$$

where B_k are integer coeeficients of the expansion:

$$(x\partial_x)^n = \sum_{k=1}^{n} B_k x^k (\partial_x)^k.$$

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- The symmetries are well defined (in terms of Hopf algebras)
- The twisted trace provides an equivariant representation on $L^2(\mathbb{R}^2)$
- The geometry *a la Connes* - Dirac operator, spectral triple - no problem.
- *Field theory - like in the Moyal case* - is within reach !!!

OUTLOOK

- We have a well-defined algebra with a nice C^* completion.
- The symmetries are well defined (in terms of Hopf algebras)
- The twisted trace provides an equivariant representation on $L^2(\mathbb{R}^2)$
- The geometry *a la Connes* - Dirac operator, spectral triple - no problem.
- *Field theory - like in the Moyal case* - is within reach !!!

THANK YOU!