High spin limits and non-abelian T-duality

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Motivation

Field equations in curved stringy backgrounds

In string theory want to go beyond (super)gravity.

- $\exists$ exact theories realizing this, particularly, coset (G/H) CFTs .
- When groups are non-abelian there are no isometries (generic).
- Solving the field equations is an impossible task with traditional methods, i.e. separation of variables.
- In physical applications this is precisely what is needed, i.e. propagating fluctuations, semiclassical quantization etc.

Understanding Non-abelian T-duality

Unlike abelian T-duality:

- Not well understood.
- Not likely to be an exact symmetry.
- Yet, what is it good for? Maybe for some effective description?

Outline

- Non-abelian T-duality in WZW actions:
  - How to obtain the σ-model action.
  - Relation to gauged WZW actions.

- Field equations and the large spin limit:
  - Solving field equations, with group theoretical techniques.
  - How the large spin limit gets involved.

- Example with $SU(2)$'s.

- Concluding remarks.

Non-abelian T-duality in WZW actions

Let a group G, a subgroup $H \in G$. Introduce an element $g \in G$ and gauge fields $A_\pm \in \mathcal{L}(H)$.

- The gauged WZW action is

$$S_{\mathrm{gWZW}}(g, A_\pm) = k \overbrace{I_0(g)}^{\mathrm{WZW}} + \frac{k}{\pi} \int \mathrm{Tr}\Big[A_- \partial_+ g g^{-1} - A_+ g^{-1} \partial_- g + A_- g A_+ g^{-1} - A_- A_+\Big] .$$

- Non-abelian T-duality action is

$$S_{\mathrm{NonAb}}(g, v, A_\pm) = S_{\mathrm{gWZW}}(g, A_\pm) - i \frac{k}{\pi} \int \underbrace{\mathrm{Tr}(v F_{+-})}_{\text{Lagrange mult.}} ,$$

  with field strength $F_{+-} = \partial_+ A_- - \partial_- A_+ - [A_+, A_-]$.

- Gauge invariance

$$g \to \Lambda^{-1} g \Lambda , \quad v \to \Lambda^{-1} v \Lambda , \quad A_\pm \to \Lambda^{-1}(A_\pm - \partial_\pm)\Lambda .$$

Obtaining the dual σ-model

- Gauge fix $\dim(H)$ parameters in g and in v, leaving $\dim(G)$ variables X^μ.
- Integrate out the $A_\pm$'s, substitute back into the action and get

$$S = \frac{k}{\pi}\int (G_{\mu\nu} + B_{\mu\nu})\partial_+ X^\mu \partial_- X^\nu \ , \quad \text{also } \Phi = \cdots$$

- Properties:
  - Isometries $G_L \times G_R$ of the WZW broken.
  - Even if G is compact, (some) variables of its non-abelian dual appear non-compact.
  - Transformation is not invertible at the action level.
  - Drastically different than abelian T-duality.
- Is it useful? What does it describe?

Relating non-abelian T-duality to gauged WZW models

- Start with the gauged WZW action for $\frac{G_k \times H_\ell}{H_{k+\ell}}$, that is

$$\overbrace{kI_0(g) + \ell I_0(h)}^{\text{WZW}} + \frac{1}{\pi}\int \mathrm{Tr}\Big[kA_-\partial_+ gg^{-1} + \ell A_-\partial_+ hh^{-1} - kA_+ g^{-1}\partial_- g$$
$$-\ell A_+ h^{-1}\partial_- h + kA_- gA_+ g^{-1} + \ell A_- hA_+ h^{-1} - (k+\ell)A_-A_+\Big] \,,$$

  Invariant under $(g,h) \to \Lambda^{-1}(g,h)\Lambda$ etc, $\Lambda(\sigma^+,\sigma^-) \in H$.

- Expand infinitesimally around the identity

$$h = \mathbb{I} + i\,\frac{k}{\ell}\,v + \mathcal{O}\left(\frac{1}{\ell^2}\right)$$

  and take the limit $\ell \to \infty$.

- We get the non-abelian T-duality action, i.e. classically [KS 94]

$$\boxed{\left.\frac{G_k \times H_\ell}{H_{k+\ell}}\right|_{\ell\to\infty} \quad = \quad \text{dual of } G_k \text{ with respect to } H}\ .$$

The T-dual as an effective geometry

- We explore the area around the identity element of the group.
- A well defined limit can be taken on the geometric background. Some variables become non-compact.
- Can this be the effective background describing a consistent sector of states of the parent theory?
- This will put the equivalence to a firmer quantum mechanical footing.

Field equations and the large spin limit

Solving field equations

Consider the scalar equation for the coset background.

- We will obtain its general solution from that of the scalar equation for the WZW model for $G \times H$.
- Start with Reps of $G \times H$. The eigenstates are

$$R_{\alpha\beta}(g)\ r_{\mu\nu}(h) \ ,$$

with eigenvalues (semiclassically, for $k, \ell \gg 1$)

$$E(R, r) = \frac{C_2(R)}{k} + \frac{C_2(r)}{\ell} \ ,$$

where the C_2's are the Casimirs.

- Under the vector H-transf they transform as

$$(R \times r) \times (\bar{R} \times \bar{r}) = (r_1 \oplus r_2 \oplus \cdots) \otimes (\bar{r}_1 \oplus \bar{r}_2 \oplus \cdots) \ .$$

- We decompose $R \times r$ and its conjugate into Reps r_i of H.
- We get a singlet from all products of the form $r_i \times \bar{r}_i$.
- The coset eigenstates are

$$\boxed{\psi_{R,r;r_i}(g,h) = \sum_{a;\alpha,\beta,\mu,\nu} \underbrace{C^a_{\alpha\mu}(R,r;r_i)\, C^a_{\beta\nu}(R,r;r_i)}_{\text{Clebsch-Gordan}} \underbrace{\overbrace{R_{\alpha\beta}(g)\, r_{\mu\nu}(h)}^{G\times H}}_{\text{gauged fixed}}} \ .$$

  Combinations of states of the $G \times H$ WZW.
- The eigenvalues get shifted as

$$E(R,r;r_i) = \frac{C_2(R)}{k} + \frac{C_2(r)}{\ell} - \frac{C_2(r_i)}{k+\ell} \ .$$

- The coset background receives $1/k$ corrections.
  It becomes simple for $k \gg 1$.
- Remarkably, the eigenstates do not depend on $\alpha' \sim 1/k$, only the eigenvalues do (indicated expressions are for $k \gg 1$).

The large spin limit

What is the effect of this limit on the states of the theory?

- Consider Reps for $\mathcal{L}(H)$ with highest weight (spin) $j \gg 1$. The Reps in its tensor product with Reps in $\mathcal{L}(G)$ have also large spin.
- We may expand as

$$C_2(r) = a(r)j^2 + b(r)j + \mathcal{O}(1) \ .$$

- Similarly for $C_2(r_i)$, with j replaced by $j + n$ ($n =$finite).
- Keeping the eigenenergies finite requires the correlated limit

$$\ell = \frac{k}{\delta}\, j \to \infty \ , \qquad \delta = \text{positive real} \ .$$

- The limit of the eigenfunction is delicate. It involves the limiting behaviour of the Clebsch–Gordan coefficients.

Example with SU(2)'s

The background fields

- $SU(2) \times SU(2)/SU(2)$ coset model:
  - Has a metric and a dilaton [Will not be displayed].
  - It is a bit complicated, with no isometries. Manifold compact.
- Non-abelian T-dual of the $SU(2)$ WZW model w.r.t. $SU(2)$

$$ds^2 = d\psi^2 + \frac{\cos^2\psi}{x_3^2}dx_1^2 + \frac{\left(x_3 dx_3 + (\sin\psi\cos\psi + x_1 + \psi)dx_1\right)^2}{x_3^2\cos^2\psi} ,$$

plus a dilaton.

- Also quite complicated. No isometries.
- ψ is periodic and x_1, x_3 are non-compact.

What do the eigenfunctions and eigenenergies look like?
They should effectively describe the large spin sector.

Solving the eigenvalue problem

$SU(2) \times SU(2)/SU(2)$: The general state is

$$\Psi^{j}_{j_1,j_2} = \sum_m \sum_{m_2,n_2=-j_2}^{j_2} \underbrace{C^{j,m}_{j_1,m-m_2,j_2,m_2} \; C^{j,m}_{j_1,m-n_2,j_2,n_2}}_{Clebsch-Gordan} \; \underbrace{\overbrace{R^{j_1}_{m-m_2,m-n_2}(g_1) R^{j_2}_{m_2,n_2}(g_2)}^{SU(2)\times SU(2)}}_{\text{gauged fixed}} \, .$$

- The $SU(2)$ Rep matrices

$$R^{j}_{m_1,m_2}(\phi,\theta,\psi) = e^{-i(m_1\phi+m_2\psi)} d^{j}_{m_1,m_2}(\theta) \; .$$

  Wigner's d-functions in terms of the Jacobi's polynomials.

- The eigenvalues are

$$E^{j}_{j_1,j_2} = \frac{j_1(j_1+1)}{k_1} + \frac{j_2(j_2+1)}{k_2} - \frac{j(j+1)}{k_1+k_2} \; .$$

- Expressions solve the scalar equation for the coset background.

Non-abelian T-dual of the $SU(2)$ WZW model w.r.t. $SU(2)$:

- Consider the high spin-level limit

$$j_1 = j - n \ , \quad k_1 = \frac{k_2}{\delta}\, j \ , \quad j_2, n = \text{finite} \ , \quad j \gg 1 \ .$$

- In this limit, the energy eigenvalues remain finite

$$E_{j_2,n,\delta} = \lim_{j\to\infty} E^{j}_{j_1,j_2} = \frac{j_2(j_2+1)}{k_2} + \frac{\delta - 2n}{k_2}\,\delta \ .$$

- In the high spin limit the Clebsch–Gordan coefficients

$$\lim_{j\to\infty} C^{j,m}_{j-n,m-m_2,j_2,m_2} = d^{j_2}_{m_2,n}(\zeta) \ , \quad \cos\zeta = \frac{m}{j} \ .$$

  - They get associated with an auxiliary $SU(2)$ rotation.
  - Expected for a classical body given extra angular momentum.

- At the end we obtain the finite sum

$$\Psi_{j_2,n,\delta}(x_1,x_3,\psi) = \lim_{j\to\infty} \Psi^{j}_{j-n,j_2} = \sum_{m_2=-j_2}^{j_2} \Gamma_{j_2,m_2,n,\delta}(x_3)\ \underbrace{R^{j_2}_{m_2,m_2}(g_2)}_{\text{gauged fixed}},$$

where

$$\Gamma_{j_2,m_2,n,\delta}(x_3) = \int_0^{\pi} d\zeta \sin\zeta \left(d^{j_2}_{m_2,n}(\zeta)\right)^2 e^{-2i\delta v_3 \cos\zeta} .$$

- Explicit expressions become complicated fast, as j_2 increases.
- Fair to say:
  Solution would have never been found without using this method.

Examples of states:

- Define

$$v_3 = \sqrt{(x_1+\psi)^2 + x_3^2}\ , \qquad \beta_0 = \sin\psi\ ,$$
$$\beta_1 = \frac{x_3\cos\psi}{\sqrt{(x_1+\psi)^2 + x_3^2}}\ , \qquad \beta_3 = \frac{(x_1+\psi)\cos\psi}{\sqrt{(x_1+\psi)^2 + x_3^2}}\ .$$

- For instance, for $j_2 = 1$ (and $\delta = 1$):

$$\Psi_{1,\pm 1} = \frac{\beta_1^2 - 2\beta_3(\beta_3 \mp 2\beta_0 v_3)}{2v_3^2}\ \cos 2v_3$$
$$+ \frac{2\beta_3^2 - \beta_1^2 + \mp 4\beta_0\beta_3 v_3 + 4(\beta_0^2 - \beta_3^2)v_3^2}{4v_3^3}\ \sin 2v_3\ ,$$
$$\Psi_{1,0} = \frac{2\beta_3^2 - \beta_1^2}{v_3^2}\ \cos 2v_3 + \frac{\beta_1^2 - 2\beta_3^2 + 2(1-2\beta_1^2)v_3^2}{2v_3^3}\ \sin 2v_3\ .$$

Concluding remarks

- Using group theoretical methods one may solve field equations for general G/H, for which:
  - Generically, there are no isometries.
  - Conventional techniques are not applicable.
- Non-abelian duality generates solutions that:
  - effectively describe high spin sectors.
  - Taking the limit is a delicate procedure, but nevertheless the only way to solve field equation of the T-dual background.
- Method works in other occasions with non-abelian isometries. For instance, when the symmetry group acts from on side, i.e. in Principal Chiral Models.
- Use non-compact groups leading to Minkowski signature spacetimes, i.e. $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/SL(2,\mathbb{R})$. Explore physical applications, i.e. in cosmology.