Evaluating Renormalization Group Equations via off-diagonal heat-kernel methods

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work in progress

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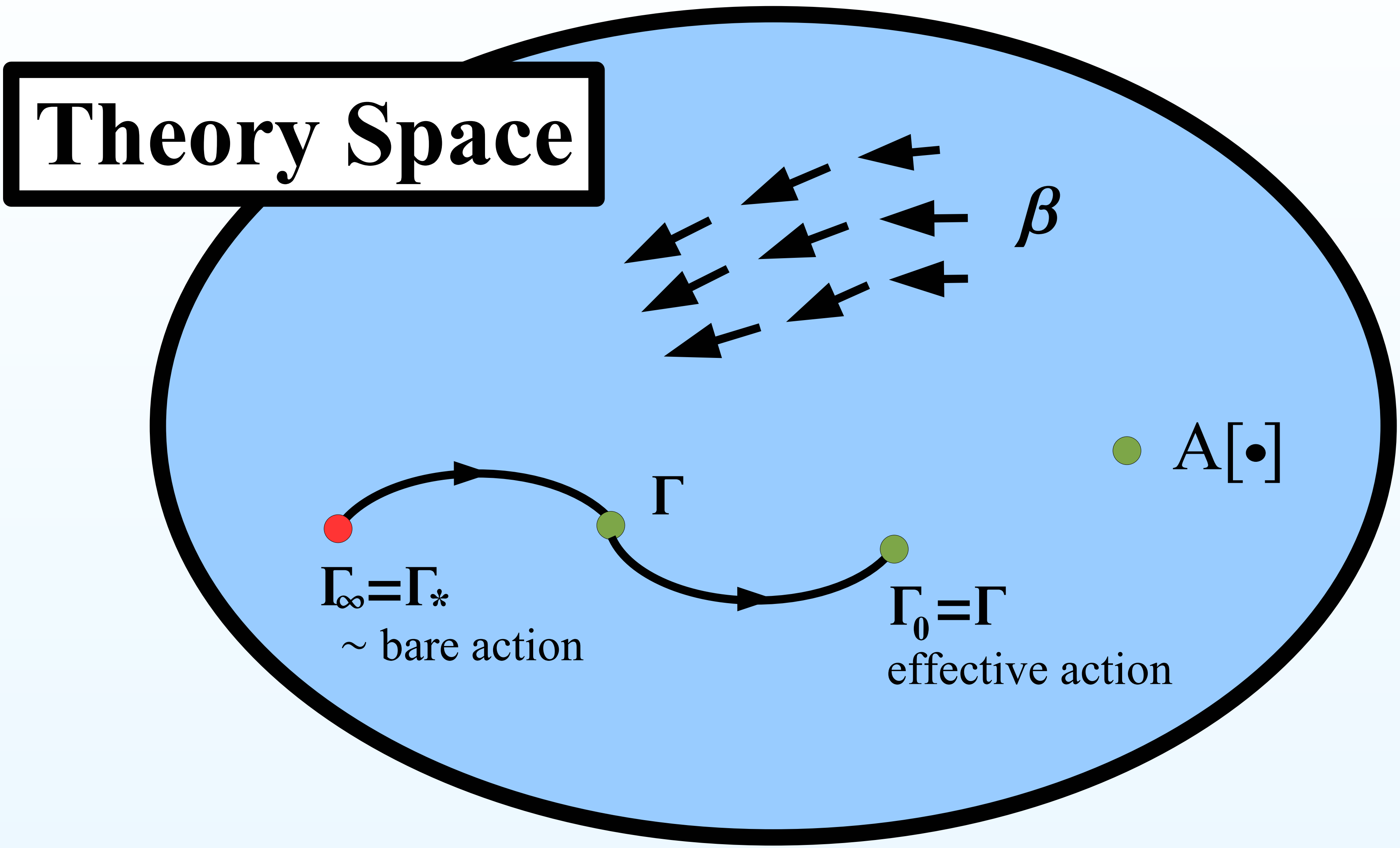
Theory space underlying the Functional Renormalization Group



Functional Renormalization Group Equation for gravity

Flow equation for effective average action Γ_k

(C. Wetterich, Phys. Lett. $\mathbf{B301}$ (1993) 90)

generalized to gravity

(M. Reuter, Phys. Rev. D $\mathbf{57}$ (1998) 971, hep-th/9605030)

$$\partial_t \Gamma_k[g, C, \bar{C}; \bar{g}, c, \bar{c}] = \frac{1}{2}\mathrm{STr}\left[\left[\Gamma_k^{(2)} + \mathcal{R}_k\right]^{-1} \partial_t \mathcal{R}_k\right]$$

- background-covariance via background gauge formalism

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,, \qquad \bar{C}_\mu = \bar{c}_\mu + \bar{f}_\mu\,, \qquad C^\mu = c^\mu + f^\mu$$

  - averaged fields: $g_{\mu\nu}, \bar{C}_\mu, C^\mu$
  - background fields: $\bar{g}_{\mu\nu}, \bar{c}_\mu, c^\mu$
  - fluctuation fields: $h_{\mu\nu}, \bar{f}_\mu, f^\mu$

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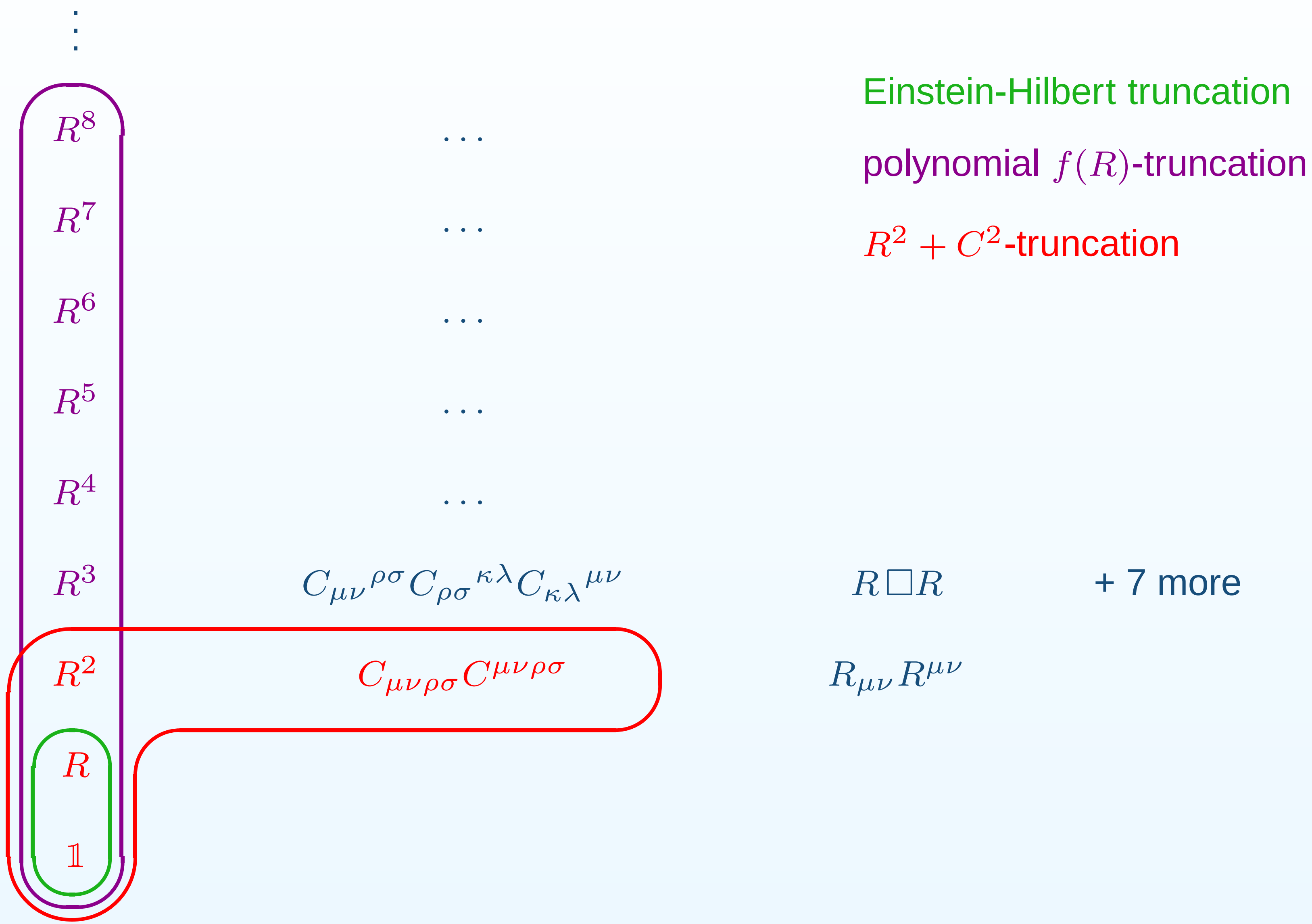
Recipe for extracting physics:

Make ansatz for Γ_k and let it flow!

Charting the theory space of gravity



Exploring the gravitational theory space

key results:

- tremendous evidence for non-Gaussian fixed point (NGFP)
  - $\Longrightarrow$ non-perturbative UV completion of gravity
- finite dimensional UV-critical surface
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Open Questions:

- Properties of NGFP in extended truncations?
- Dimension of UV-critical surface?
- What about . . . the Goroff-Sagnotti Counterterm?
- Physics associated with the fundamental theory (unitarity, . . .)?

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!!! Wanted !!!

systematic algorithm for implementing the derivative expansion for gravity letting things flow

The universal RG machine

The universal RG machine: blueprint

goal: systematic derivative expansion of $\mathrm{STr}\left[[\Gamma_k^{(2)} + \mathcal{R}_k]^{-1}\partial_t \mathcal{R}_k\right]$:

- curved space-times $\Longleftrightarrow$ heat-kernel methods:
  - works well for Laplace-Type Operators $\mathcal{D} = \Delta + V\,,\ \Delta = -g^{\mu\nu} D_\mu D_\nu$
  - higher-derivative truncations: limited applicability

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central idea:

convert operator structure in Str to "standard" heat-kernel form

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Implementation in 3 steps:

1. simplify operator structure of $\Gamma_k^{(2)}$
2. perturbative inversion of $[\Gamma_k^{(2)} + \mathcal{R}_k]$
3. evaluation of operator traces via off-diagonal heat-kernels

Step 1: simplify operator structure of $\Gamma_k^{(2)}$

generically, $\Gamma_k^{(2)}$ has non-Laplacian part:

$$\left[\Gamma_k^{(2)}\right]^{ij} = \underbrace{\mathbb{K}(\Delta)\,\delta^{ij}\,\mathbb{1}_i}_{\text{kin. terms}} + \underbrace{\mathbb{D}(D_\mu)}_{\text{uncontracted derivatives}} + \underbrace{\mathbb{M}(R, D_\mu)}_{\text{background curvature}}$$

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Examples:

$$\mathbb{D} = (1-\alpha)D_\mu D^\nu \quad , \quad \mathbb{D} = D^\mu D^\nu D_\alpha D_\beta$$

$$\mathbb{M} = R^{\mu\nu} D_\mu D_\nu \quad , \quad \mathbb{M} = D^\mu c_\nu$$

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Solution I: choose background gauge-fixing such that $\mathbb{D} = 0$

- gauge-freedom may be insufficient
- limited to very particular gauge-choice (e.g. $\alpha = 1$)

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generic solution: Transverse decomposition of fluctuation fields

(York, ’75)

- vector: $f_\mu = f_\mu^{\mathrm{T}} + D_\mu \eta \quad , \qquad D^\mu f_\mu^{\mathrm{T}} = 0$
- graviton: $h_{\mu\nu} = h_{\mu\nu}^{\mathrm{T}} + D_\mu \xi_\nu + D_\nu \xi_\mu - \frac{1}{2} g_{\mu\nu} D^\alpha \xi_\alpha + \frac{1}{4} g_{\mu\nu} h$

$$D^\mu h_{\mu\nu}^{\mathrm{T}} = 0\,, \qquad g^{\mu\nu} h_{\mu\nu}^{\mathrm{T}} = 0\,, \qquad g^{\mu\nu} h_{\mu\nu} = h$$

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Removes $\mathbb{D}$-part from $[\Gamma_k^{(2)}]^{ij}$

Step 2: Perturbative inversion of $\left[\Gamma^{(2)} + \mathcal{R}_k\right]^{ij}$

Implement type I cutoff:

$$\mathbb{K}(\Delta) \mapsto \mathbb{P}(\Delta)\,, \quad \text{following} \qquad \Delta \mapsto P_k = \Delta + R_k$$

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$\left[\Gamma^{(2)} + \mathcal{R}_k\right]^{ij}$ operator-valued matrix in field space:

$$\left[\Gamma^{(2)} + \mathcal{R}_k\right]^{ij} = \begin{bmatrix} \mathbb{P}_1 \mathbb{1}_1 + \mathbb{M}_1 & \mathbb{M}_\times \\ \tilde{\mathbb{M}}_\times & \mathbb{P}_2 \mathbb{1}_2 + \mathbb{M}_2 \end{bmatrix}$$

Formal inversion following inversion formulas for block matrices

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} \left(A - BD^{-1}C\right)^{-1} & -A^{-1}B\left(D - CA^{-1}B\right)^{-1} \\ -D^{-1}C\left(A - BD^{-1}C\right)^{-1} & \left(D - CA^{-1}B\right)^{-1} \end{bmatrix}$$

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Perturbative expansion of inverse matrix elements in $\mathbb{M}$: terminates at finite order:

$$\left[\Gamma^{(2)} + \mathcal{R}_k\right]^{-1}_{11} = \frac{1}{\mathbb{P}_1} - \frac{1}{\mathbb{P}_1}\mathbb{M}_1\frac{1}{\mathbb{P}_1} + \frac{1}{\mathbb{P}_1}\mathbb{M}_1\frac{1}{\mathbb{P}_1}\mathbb{M}_1\frac{1}{\mathbb{P}_1} + \frac{1}{\mathbb{P}_1}\mathbb{M}_\times\frac{1}{\mathbb{P}_2}\tilde{\mathbb{M}}_\times\frac{1}{\mathbb{P}_1} + \mathcal{O}(\mathbb{M}^3)$$

Step 3: Evaluate the operator traces including insertions $\mathbb{M}$

1. use commutators to bring trace argument into standard form:
   - contracted cov. derivatives: $\Longrightarrow$ collected into a single function $W(\Delta)$
   - remainder: $\Longrightarrow$ matrixvalued insertion $\mathcal{O}$

2. Laplace transform $W(\Delta) \to \tilde{W}(s)$

$$\mathrm{Tr}\,[\,W(\Delta)\,\mathcal{O}\,] = \int_0^\infty ds\,\tilde{W}(s)\,\langle x|\,\mathrm{e}^{-s\Delta}\,\mathcal{O}\,|x\rangle$$

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3. Evaluate trace using off-diagonal Heat-kernel (act $\mathcal{O}$ on H)

$$\langle x|\,\mathcal{O}\,\mathrm{e}^{-s\Delta}|x\rangle = \langle x|\,\mathcal{O}\,|x'\rangle\langle x'|\mathrm{e}^{-s\Delta}|x\rangle = \int d^4x\sqrt{g}\,\mathrm{tr}_i\,\left[\mathcal{O}H(s,x,x')\right]_{x=x'}$$

$$H(s,x,x') := \langle x'|\mathrm{e}^{-s\Delta}|x\rangle = \frac{1}{(4\pi s)^2}\,\mathrm{e}^{-\frac{\sigma(x,x')}{2s}}\sum_{n=0}^{\infty} s^n A_{2n}(x,x')$$

   - $A_{2n}(x,x')$: heat-coefficients at non-coincident point
   - $2\sigma(x,x')$: geodesic distance between x, x'

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Properties of $H(s,x,x')$ in the coincidence limit:

- $A_{2n}(x,x)$ $\longrightarrow$ standard heat-kernel coefficients
- derivatives of A_{2n} $\longrightarrow$ additional powers of curvatures
- $\sigma(x,x) = 0\,,\ \sigma_{;\mu} = 0$ $\longrightarrow$ vanish in coincidence limit
- $\sigma_{;\mu\nu}(x,x) = g_{\mu\nu}(x)$ $\longrightarrow$ non-vanishing

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Example: $\mathcal{O} = R^{\mu\nu}D_\mu D_\nu$

$$\mathrm{Tr}\,[\,W(\Delta)\,\mathcal{O}\,] = -\frac{1}{32\pi^2}\int_0^\infty ds\,\frac{1}{s^3}\tilde{W}(s)\int d^4x\sqrt{g}R + \mathcal{O}(R^2)$$

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Features:

- applications: gravity, gauge-theory $\ldots$
- method is algebraic $\Longleftrightarrow$ no numerical integrations
- easily implemented on your laptop

The universal RG machine at work:

Ghost wave-function renormalization for gravity

Ghost-improved Einstein-Hilbert truncation: setup

$$\Gamma_k^{\mathrm{GI-EH}}[g,C,\bar{C};\bar{g}] = \Gamma_k^{\mathrm{grav}}[g] + S^{\mathrm{gf}} + \Gamma_k^{\mathrm{ghost}}[g,C,\bar{C};\bar{g}]$$

- gravitational sector: same as Einstein-Hilbert

$$\Gamma_k^{\mathrm{grav}}[g] = \frac{1}{16\pi G_k} \int d^4x \sqrt{g}\, \{-R + 2\Lambda_k\}$$

- Harmonic gauge-fixing:

$$S^{\mathrm{gf}} = \frac{1}{32\pi G_k} \int d^4x \sqrt{\bar{g}} \bar{g}^{\mu\nu} F_\mu F_\nu \,, \quad F_\mu = \bar{D}^\nu h_{\mu\nu} - \tfrac{1}{2} \bar{D}_\mu h$$

- Ghost sector including wave-function renormalization Z_k^c:

$$\Gamma_k^{\mathrm{ghost}}[g,C,\bar{C};\bar{g},c,\bar{c}] = -\sqrt{2} Z_k^c \int d^4x \sqrt{\bar{g}}\, \bar{C}_\mu \, \mathcal{M}^\mu{}_\nu \, C^\nu$$

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- β-functions: track prefactors of:

$$I_1 = \int d^4x \sqrt{\bar{g}} \,, \qquad I_2 = \int d^4x \sqrt{\bar{g}} \bar{R} \,, \qquad I_3 = \int d^4x \sqrt{\bar{g}} \, \bar{c}^\mu \, \bar{D}^2 c_\mu$$

Ghost-improved EH truncation: Constructing the operator traces

background: non-trivial background ghost field:

$$g_{\mu\nu} = \underbrace{\bar{g}_{\mu\nu}}_{\text{sphere}} + h_{\mu\nu} \,, \qquad \bar{C}_\mu = \underbrace{\bar{c}_\mu}_{\neq 0, \bar{D}_\mu \bar{c}^\mu = 0} + \bar{h}_\mu \,, \quad C_\mu = \underbrace{c_\mu}_{\neq 0, \bar{D}_\mu c^\mu = 0} + h_\mu$$

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Decompose fluctuation fields: traceless-decomposition of metric suffices:

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \tfrac{1}{4}\bar{g}_{\mu\nu} h\,, \qquad h = \bar{g}^{\mu\nu} h_{\mu\nu}\,, \quad \bar{g}^{\mu\nu}\hat{h}_{\mu\nu} = 0$$

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$[\Gamma_k^{(2)} + \mathcal{R}_k]^{ij}$: Block-matrix in field space

$$\left[\Gamma_k^{(2)} + \mathcal{R}_k\right]^{ij} = \begin{bmatrix} \mathbb{P}_{\text{grav}} + \mathbb{M}_{\text{grav}} & \mathbb{M}_\times \\ \tilde{\mathbb{M}}_\times & \mathbb{P}_{\text{ghost}} + \mathbb{M}_{\text{ghost}} \end{bmatrix}$$

- $\mathbb{P}_{\text{grav}}, \mathbb{M}_{\text{grav}}, \mathbb{P}_{\text{ghost}}, \mathbb{M}_{\text{ghost}} \Longleftrightarrow$ standard Einstein-Hilbert
- $\mathbb{M}_\times, \tilde{\mathbb{M}}_\times \Longleftrightarrow$ vertices including one background ghost-field

Improved Einstein-Hilbert truncation: Evaluating operator traces

perturbative inversion of $\left[\Gamma_k^{(2)} + \mathcal{R}_k\right]^{ij}$ in powers of background ghost field c_μ:

$$\partial_t \Gamma_k = \underbrace{\mathcal{S}_{2\mathrm{T}} + \mathcal{S}_0 + \mathcal{S}_1}_{\text{no background ghosts}} + \underbrace{\mathcal{G}_{2\mathrm{T}} + \mathcal{G}_0 + \mathcal{G}_1}_{\text{one pair of background ghosts}}$$

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$\mathcal{S}_1$: feedback: ghost sector $\Longrightarrow$ gravitational couplings

$$\mathcal{S}_1 = -\,\mathrm{Tr}_1\left[\frac{\mathbb{1}_1}{Z_k^c(P_k - \frac{1}{4}\bar{R})}\,\partial_t(Z_k^c R_k)\right].$$

Improved Einstein-Hilbert truncation: Evaluating operator traces

perturbative inversion of $\left[\Gamma_k^{(2)} + \mathcal{R}_k\right]^{ij}$ in powers of background ghost field c_μ:

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running Z_k^c: captured by $\mathcal{G}_i$:

- Example: contribution of scalar trace

$$\mathcal{G}_0 \propto Z_k^c G_k^2 \operatorname{Tr}_0\left[\frac{1}{\mathbb{P}_0 - 2\Lambda_k}\Pi_0 \cdot \mathbb{M}_\times \frac{1}{\mathbb{P}_{\text{ghost}}} \tilde{\mathbb{M}}_\times \cdot \Pi_0 \frac{1}{\mathbb{P}_0 - 2\Lambda_k}\partial_t(G_k^{-1} R_k)\right]$$

- typical form of vertex insertion: $\mathcal{O} = (\bar{D}_\mu c^\alpha)(\bar{D}_\nu \bar{c}_\alpha)\bar{D}^\mu \bar{D}^\nu$

Improved Einstein-Hilbert truncation: Evaluating operator traces

1. use commutation rules to collect all Laplace-operators:
   - ghost-curvature terms outside truncation $\Longrightarrow$ all derivatives commute

$$\mathcal{G}_0 \propto \mathrm{Tr}_0 \left[W(\Delta) \, \mathcal{O} \right] \, , \qquad \mathcal{O} = (D_\mu c^\alpha)(D_\nu \bar{c}_\alpha) D^\mu D^\nu$$

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3. insert complete set of states $|x'\rangle\langle x'| \Longrightarrow$ off-diagonal heat kernel

$$H(s,x,x') := \langle x' | \mathrm{e}^{-s\Delta} | x \rangle = \frac{1}{(4\pi s)^2}\, \mathrm{e}^{-\frac{\sigma(x,x')}{2s}} \sum_{n=0}^{\infty} s^n A_{2n}(x,x') \,,$$

   act $\mathcal{O}$ on $H(s,x,x')$ and take coincidence limit:
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4. Result:

$$\mathrm{Tr}\left[W(\Delta)\,\mathcal{O}\right] = -\frac{1}{32\pi^2} \int_0^\infty \frac{ds}{s^3}\, \tilde{W}(s) \int d^4x \sqrt{g} (D_\mu c^\alpha)(D^\mu \bar{c}_\alpha) + \cdots$$

Ghost-improved Einstein-Hilbert truncation: β-functions

ghost-improvement $\Longleftrightarrow$ extra contributions to gravitational β-functions

- encoded in ghost-anomalous dimension $\eta_c \equiv -\partial_t \ln Z_k^c$

autonomous β-functions

$$\partial_t \lambda_k = \beta_\lambda \,, \qquad \partial_t g_k = (2 + \eta_N)\, g_k$$

$$\beta_\lambda = -(2-\eta_N)\lambda_k + \tfrac{g_k}{2\pi}\left[10\Phi_2^{1,0}(-2\lambda_k) - 8\Phi_2^{1,0}(0) - 5\eta_N \tilde{\Phi}_2^{1,0}(-2\lambda_k) + 4\eta_c \tilde{\Phi}_{d/2}^{1,0}(0)\right]$$

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- η_c determined by g, λ, analytic in g!
- new terms: not a “small” correction to EH result
- specify $R_k \Longleftrightarrow$ β-functions can be solved numerically

Ghost-improved Fixed Point structure

Gaussian Fixed Point (GFP):

$$g^* = 0\,, \qquad \lambda^* = 0\,, \qquad \eta_N^* = 0\,, \qquad \eta_c^* = 0$$

- free theory
- saddle point in the g-λ-plane

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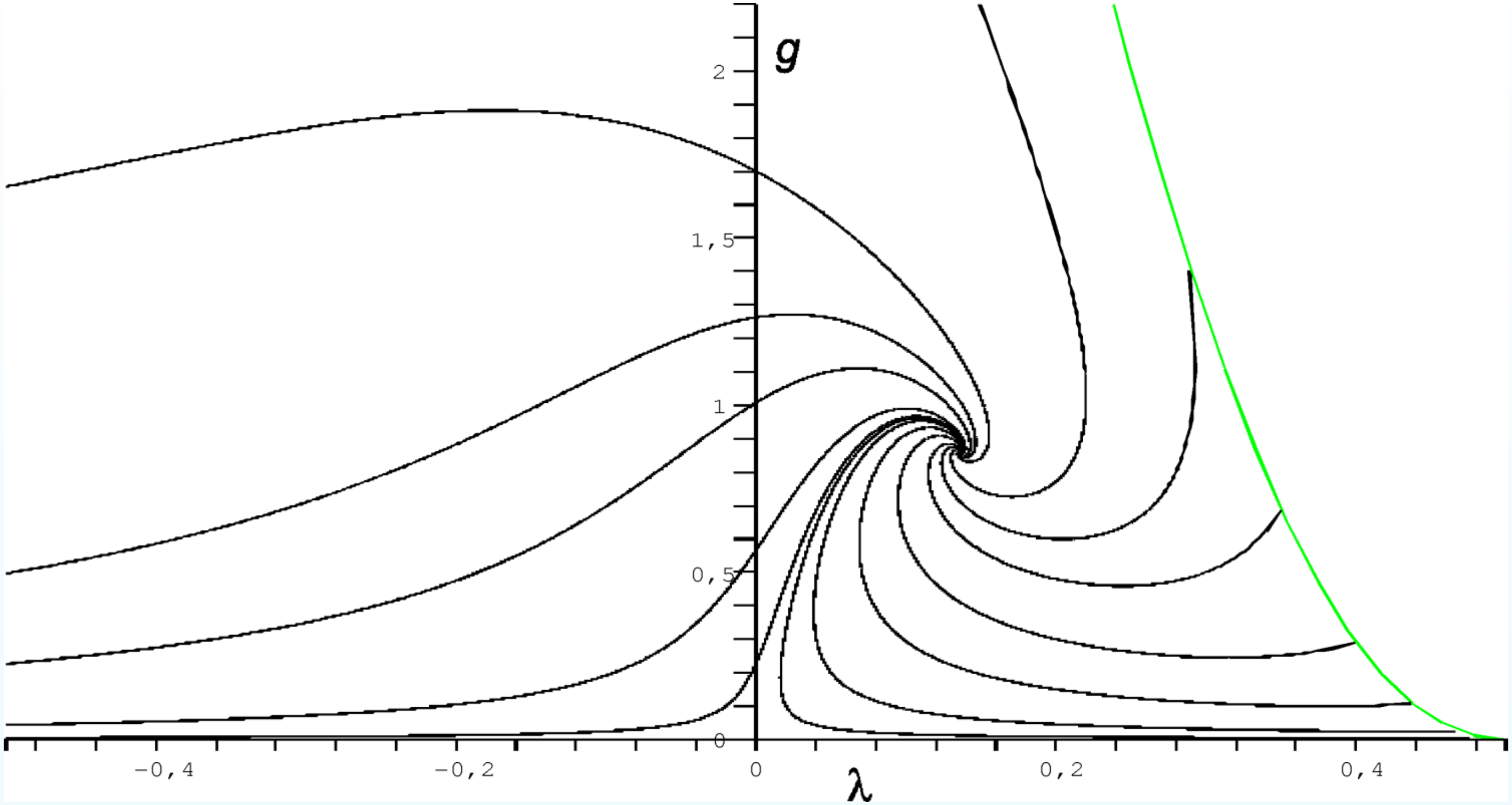
Non-Gaussian Fixed Point (NGFP):

Truncation	λ^*	g^*	$g^* \lambda^*$	η_c^*	Re(θ)	Im(θ)	cutoff
EH + ghost	0.127	0.849	0.107	-1.769	2.148	1.914	opt
EH + ghost	0.250	0.354	0.089	-1.851	2.224	2.331	exp
EH	0.193	0.707	0.136	$-$	1.475	3.043	opt

- supports: non-trivial UV fixed point
- changes numerical values θ, $g^* \lambda^*$ of EH by $\approx 30\%$!

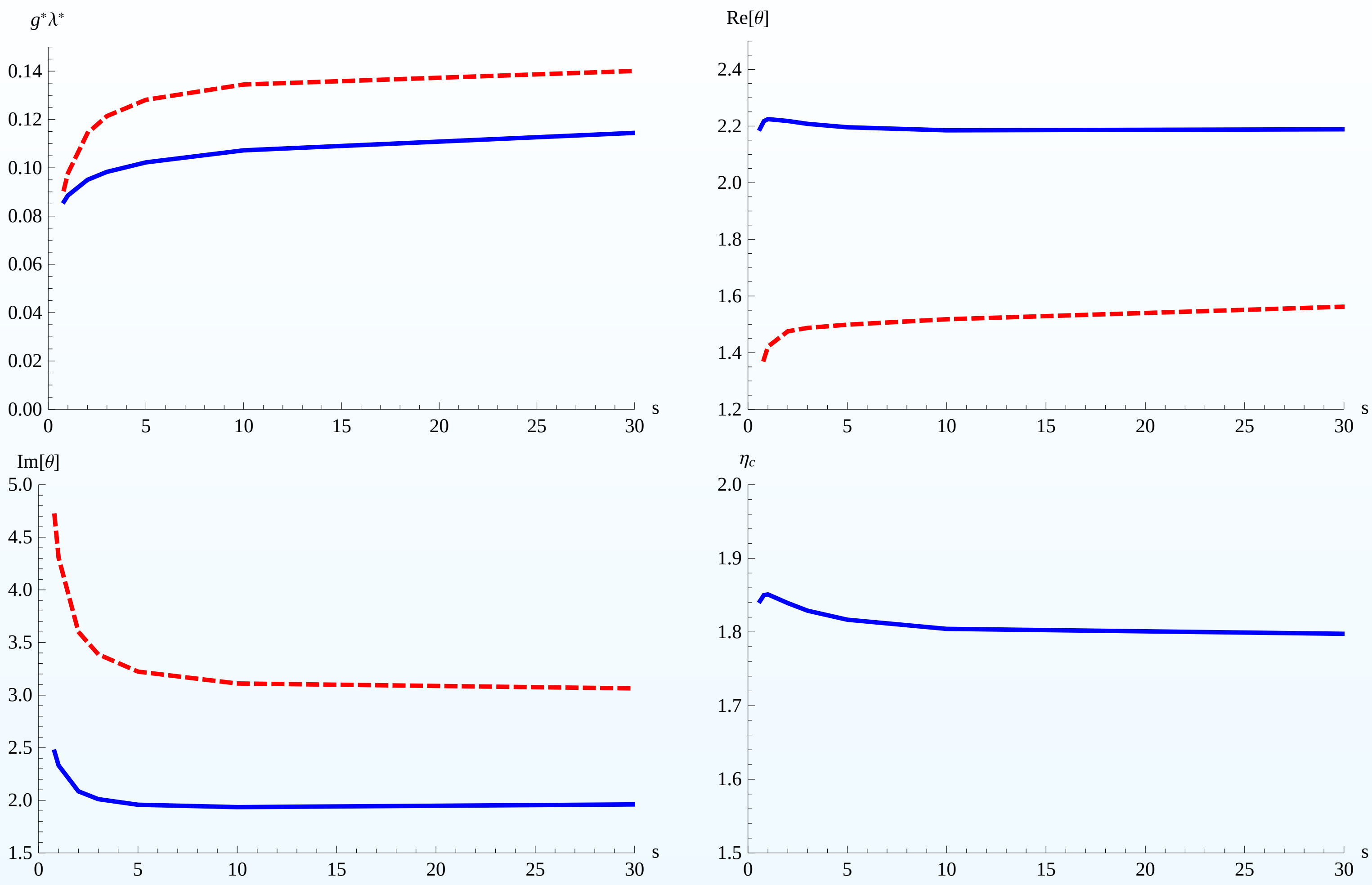
(agreement with A. Eichhorn, H. Gies, arXiv:1001.5033)

Ghost-improved phase portrait



complete confirmation of Einstein-Hilbert result

Ghost-improved Einstein-Hilbert truncation: Scheme-dependence



Ghost-improved Einstein-Hilbert trunaction: Significantly more stable!

Summary

Algorithm for computing the derivative expansion of operator traces:

- allows: systematic exploration of RG flows on theory space
- applicable to gauge theory, gravity, . . .
- easily implemented on computer algebra systems

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springboard for unveiling many physics features encoded by RG flow