Exact Renormalization Group and Φ-derivable approximations

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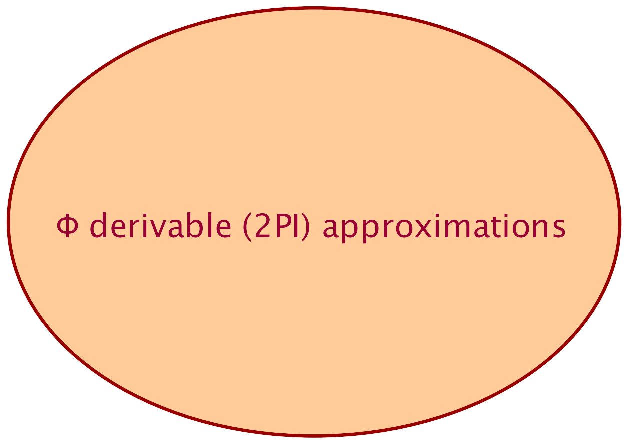
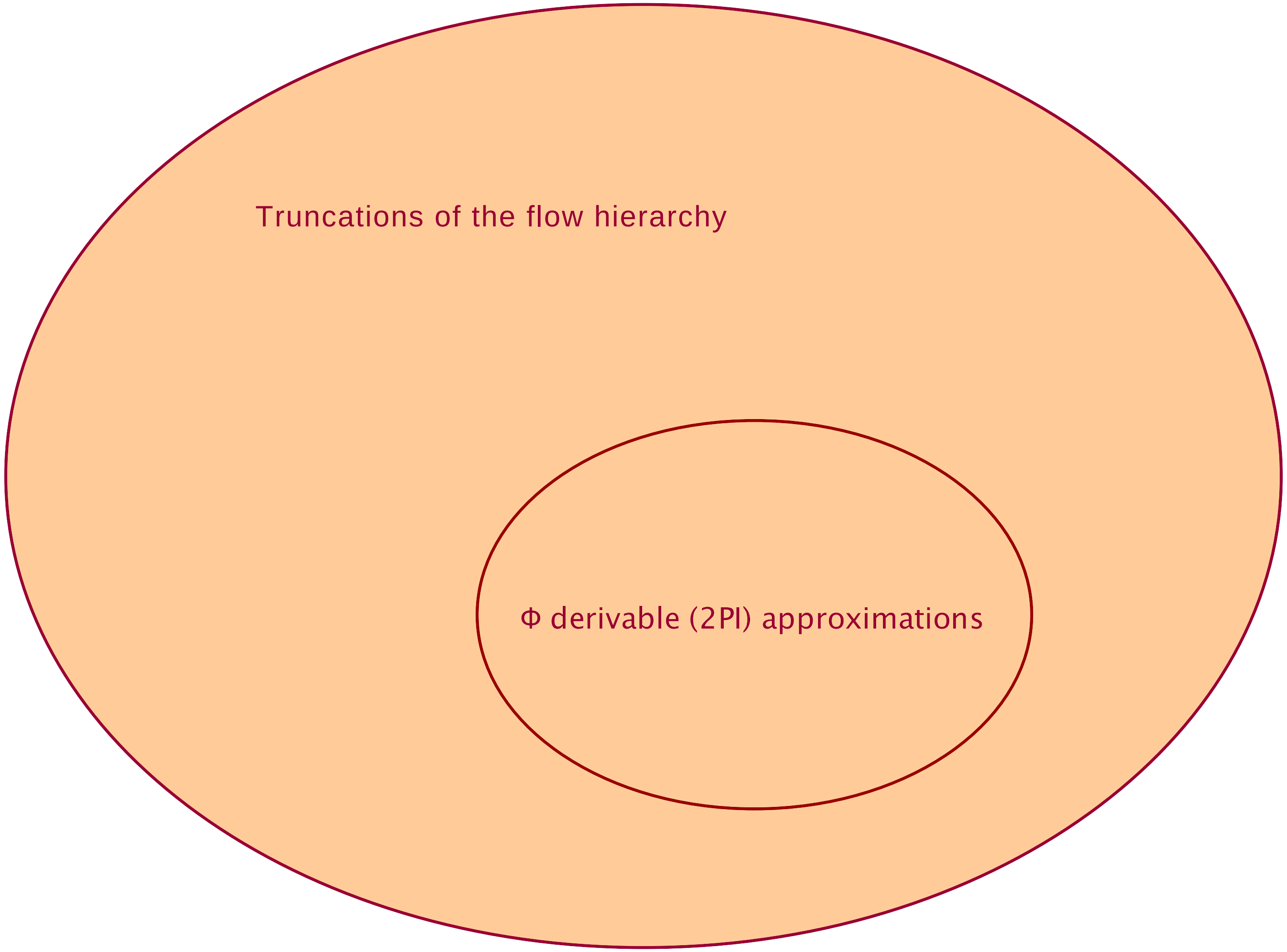
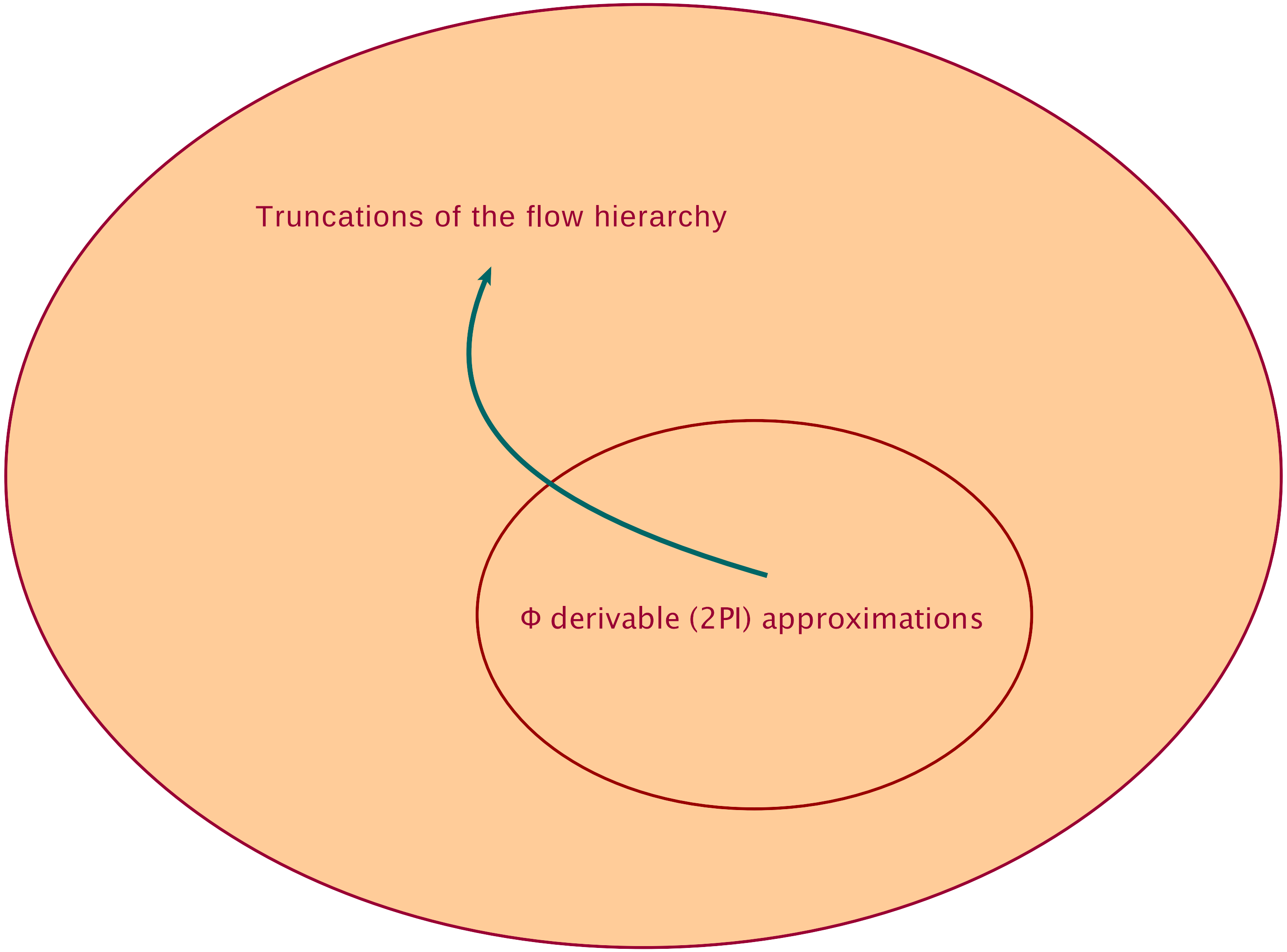
(in collaboration with J.-P. Blaizot & J. M. Pawlowski)


Φ derivable (2PI) approximations


Truncations of the flow hierarchy
Φ derivable (2PI) approximations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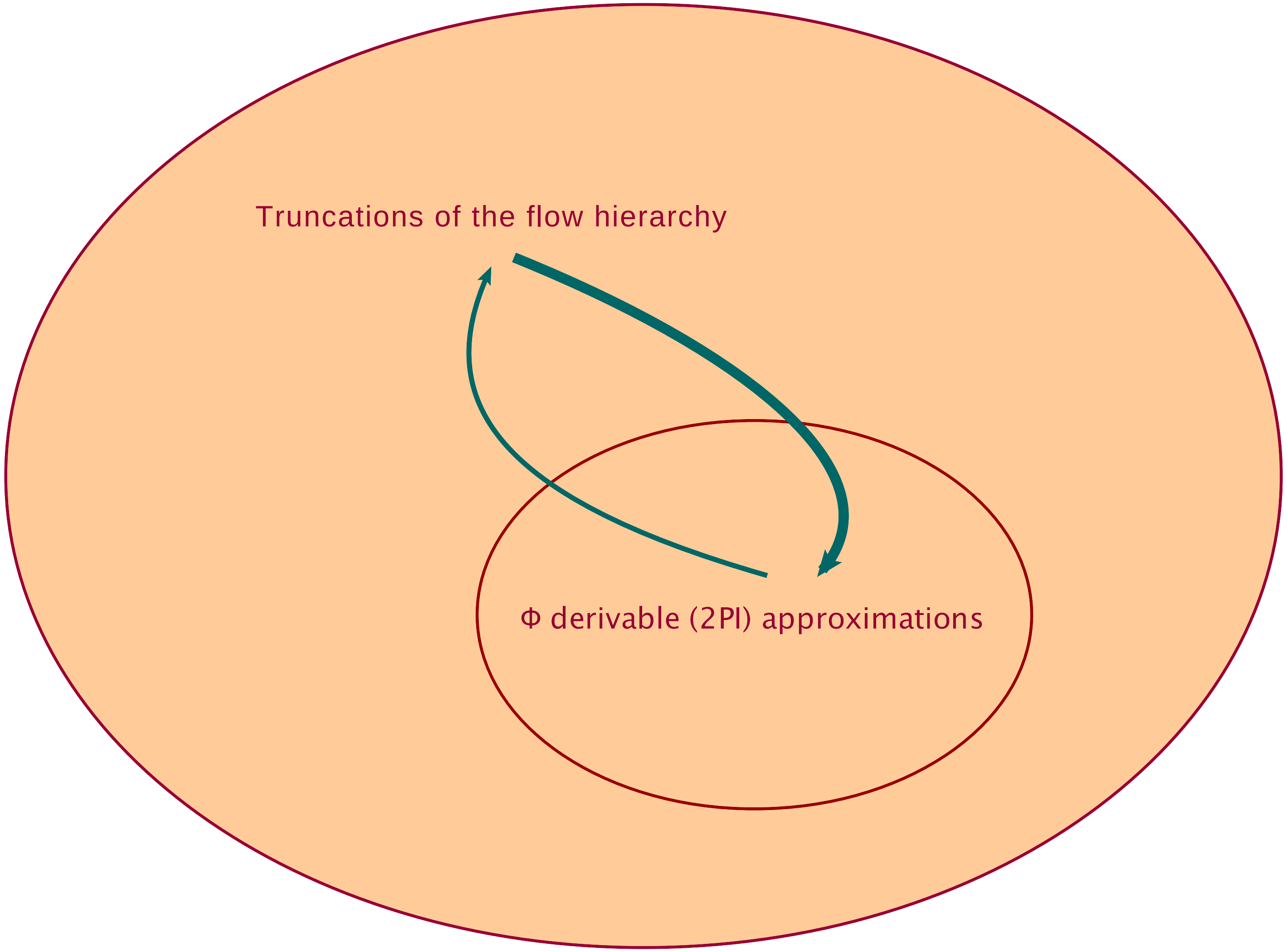
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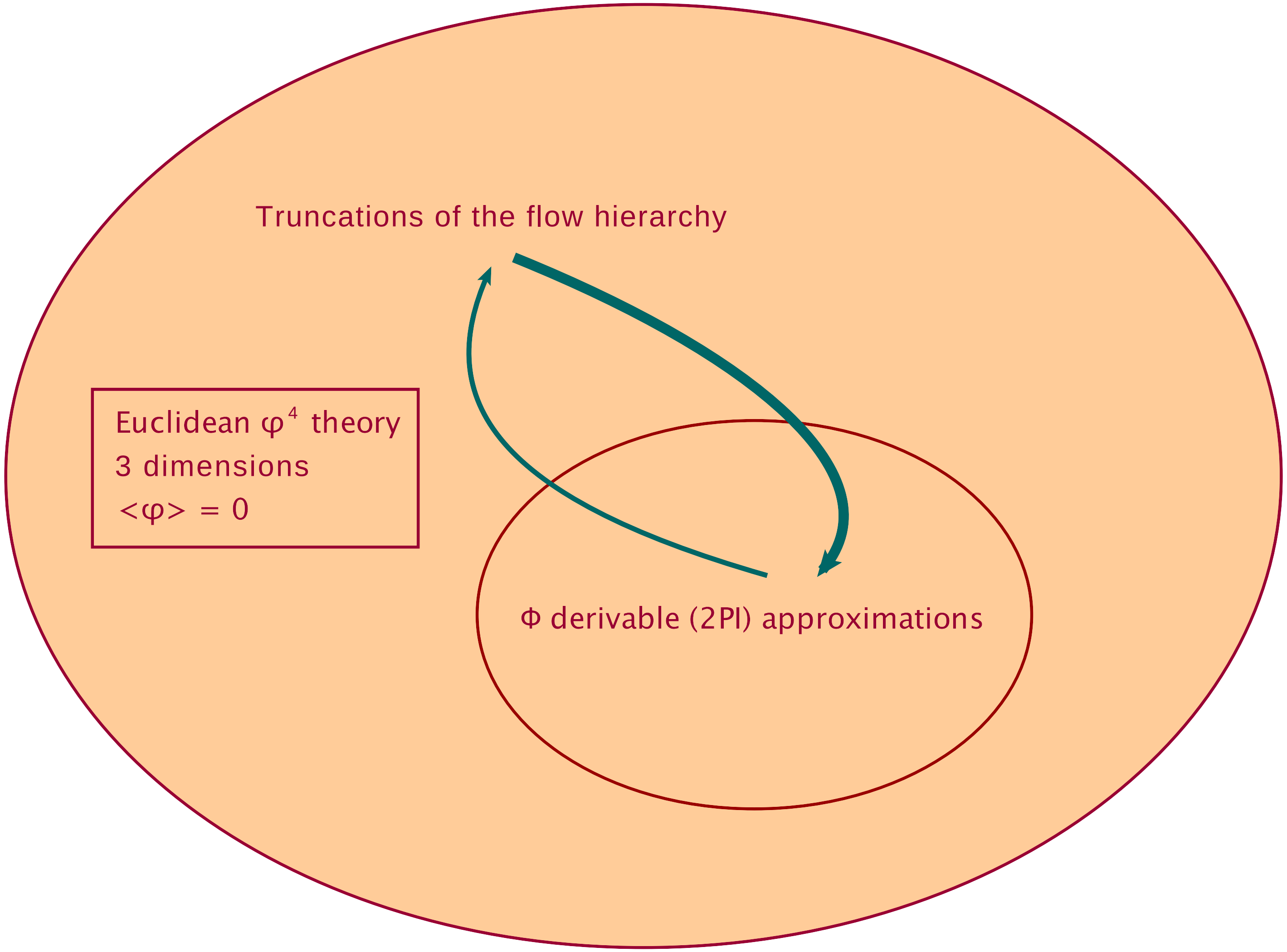

Truncations of the flow hierarchy
Euclidean φ⁴ theory
3 dimensions
<φ> = 0
Φ derivable (2PI) approximations

Basics of 2PI

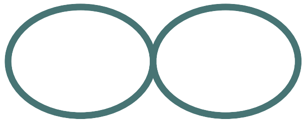
Two-point function

Basics of 2PI
- Two-point function
- Applications
- Gap equation
- Four-point function

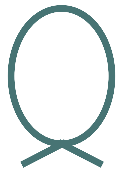
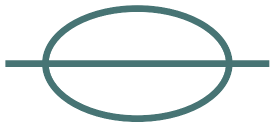
2PI as a truncation of the flow

Results

Basic objet: sum of 2PI (skeleton) diagrams:

$\Phi[G] \equiv$  $+$  $+ \cdots$

Self-energy:

$\Sigma[G] \equiv 2\,\dfrac{\delta \Phi}{\delta G} =$  $+$  $+ \cdots$

Gap equation:

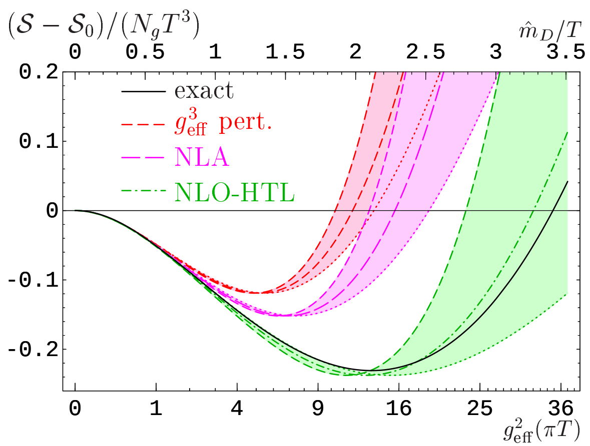
$$G^{-1}(p) = p^2 + m_{\rm b}^2 + \Sigma(p)$$

Applications

Spectral properties:

$$\rho = 2\,\mathrm{Im}\,G$$

Thermodynamics:



Blaizot, Ipp, Rebhan, Reinosa, PRD 72 (2005)

Dynamical aspects:

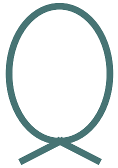
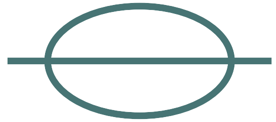
thermalisation,

decoherence

(see talk by J. Serreau)

Gap equation

Example: two-loop *gap equation*

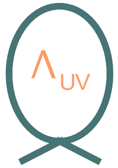
$$G^{-1}(p) = p^2 + m_{\rm b}^2 +$$  $$+$$ 

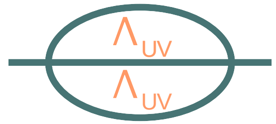
$$= p^2 + m_{\rm b}^2 + \frac{\lambda}{2}\int_q G(q) - \frac{\lambda^2}{6}\int_q\int_l G(l)G(q)G(l+q+p)$$

Non-linear integral equation. Can be tedious to solve.

Gap equation

Example: two-loop *gap equation*

$$G^{-1}(p) = p^2 + m_{\rm b}^2 +$$ 

$$+$$ 

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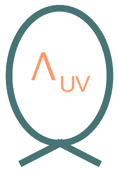
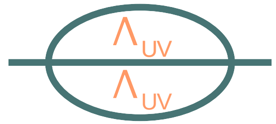
$$G^{-1}(p) = p^2 + m_b^2 +$$ 

$$+$$ 

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In 3d, renormalization is simple to implement:

$$G^{-1}(p) = p^2 + m_b^2 + \Sigma(p)$$

Gap equation

Example: two-loop *gap equation*

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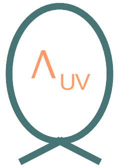
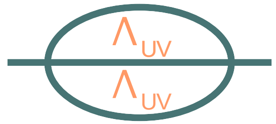


$$+$$



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Non-linear integral equation. Can be tedious to solve.

In 3d, renormalization is simple to implement:

$$G^{-1}(p) = p^2 + m^2 + \big[\Sigma(p) - \Sigma(0)\big]\,, \quad G^{-1}(0) = m^2$$

Gap equation

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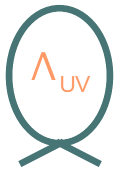
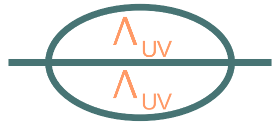
$$G^{-1}(p) \; = \; p^2 + m_{\rm b}^2 +$$ 

$$+$$ 

$$= \; p^2 + m_{\rm b}^2 + \frac{\lambda}{2}\int_q G(q) - \frac{\lambda^2}{6}\int_q\int_l G(l)G(q)G(l+q+p)$$

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In 3d, renormalization is simple to implement:

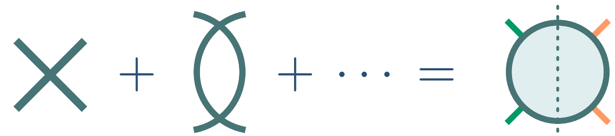
$$G^{-1}(p) = p^2 + m^2 + \left[\Sigma(p) - \Sigma(0)\right], \quad G^{-1}(0) = m^2$$

In 4d, what is the correct four-point function?

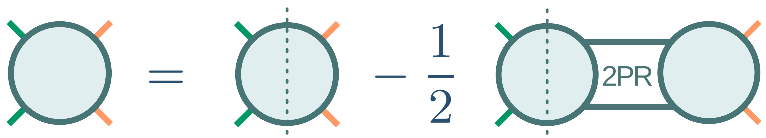
Four-point function

2PI kernel:

$$\mathcal{I}[G] \equiv 4\,\frac{\delta^2\Phi}{\delta G\,\delta G} =$$



Four-point function:



Linear integral equation:

$$\Gamma^{(4)}(q,p) = \mathcal{I}(q,p) - \frac{1}{2}\int_l \mathcal{I}(q,l)\,G^2(l)\,\Gamma^{(4)}(l,p)$$

2PI as a truncation of the flow hierarchy

2PI truncations of the flow

Flow hierarchy:

$$\partial_k \Gamma_k^{(2)}(p) = -\frac{1}{2}\int_q \partial_k R_k(q)\, G_k^2(q)\, \Gamma_k^{(4)}(q,p)$$

$$\partial_k \Gamma_k^{(4)}(q,p) = \mathcal{F}_k^{(4)}\left[\Gamma_k^{(2)}, \Gamma_k^{(4)}, \Gamma_\kappa^{(6)}\right] \quad \cdots$$

2PI truncations of the flow

Flow hierarchy:

$$\partial_k \Gamma_k^{(2)}(p) = -\frac{1}{2}\int_q \partial_k R_k(q)\, G_k^2(q)\, \Gamma_k^{(4)}(q,p)$$

$$\Gamma_k^{(4)}(q,p) = \mathcal{I}_k(q,p) - \frac{1}{2}\int_l \Gamma_k^{(4)}(q,l)\, G_k^2(l)\, \mathcal{I}_k(l,p)$$

$$\mathcal{I}[G] = \frac{4\,\delta^2\Phi}{\delta G \delta G} =$$  $+$  $+\cdots$

2PI truncations are exact

$$\partial_k \Gamma_k^{(2)}(p) \;=\; -\frac{1}{2}\int_q \partial_k R_k(q)\, G_k^2(q)\, \Gamma_k^{(4)}(q,p)$$

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$$G_k^{-1} = \Gamma^{(2)}_k + R_k \;\Rightarrow\; \partial_k G_k = -(\partial_k\Gamma^{(2)}_k + R_k)\, G^2_k$$

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$$\partial_k \Gamma_k^{(2)}(p) = \frac{1}{2}\int_q \partial_k G_k(q)\, \frac{4\,\delta^2\Phi}{\delta G(q)\delta G(p)}\bigg|_{G_k}$$

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$$G_k^{-1} = \Gamma_k^{(2)} + R_k \;\; \Rightarrow \;\; \partial_k G_k = -(\partial_k \Gamma_k^{(2)} + R_k)\, G_k^2$$

$$\partial_k \Gamma_k^{(2)}(p) = \partial_k \left. \frac{2\,\delta\Phi}{\delta G(p)}\right|_{G_k} = \partial_k \Sigma_k(p)$$

Exactness

The 2PI truncated flow:

$$\partial_k \Gamma_k^{(2)}(p) = -\frac{1}{2}\int_q \partial_k R_k(q)\, G_k^2(q)\, \Gamma_k^{(4)}(q,p)$$

$$\Gamma_k^{(4)}(q,p) = \mathcal{I}_k(q,p) - \frac{1}{2}\int_l \Gamma_k^{(4)}(q,l)\, G_k^2(l)\, \mathcal{I}_k(l,p)$$

is exact:

$$\partial_k \Gamma_k^{(2)}(p) = \partial_k \Sigma_k(p)$$

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$$\Gamma_k^{(2)}(p) = p^2 + m^2 + \left[\Sigma_k(p) - \Sigma(0)\right]$$

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In the limit $k \longrightarrow 0$:

$$\Gamma_{k=0}^{(2)}(p) = p^2 + m^2 + \left[\Sigma(p) - \Sigma(0)\right]$$

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$$\partial_k \Gamma_k^{(2)}(p) = -\frac{1}{2}\int_q \partial_k R_k(q)\, G_k^2(q)\, \Gamma_k^{(4)}(q,p)$$

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In the limit $k \longrightarrow 0$:

$$G^{-1}(p) = p^2 + m^2 + \left[\Sigma(p) - \Sigma(0)\right]$$

Exactness

The 2PI truncated flow:

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- ✓ Gap equation reformulated as an initial value problem.
- ✓ The only integral equation to be solved is linear.
- ✓ The four-point function appears explicitely.

Initial condition

3d initial condition:

$$k = \Lambda \to \infty \; : \quad \Gamma^{(2)}_\Lambda(p) \approx p^2 + m^2_\Lambda \,, \quad \Gamma^{(2)}_{k=0}(0) = m^2$$

On the other hand:

$$\Gamma^{(2)}_\Lambda(p) = p^2 + m^2 + \big[\Sigma_\Lambda(p) - \Sigma(0)\big]$$

It follows that, as $\Lambda \to \infty$:

$\big[\Sigma_\Lambda(p) - \Sigma(0)\big]$ becomes p-independent

Initial condition

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It follows that, as $\Lambda \to \infty$:

$\left[\Sigma_\Lambda(p) - \Sigma_\Lambda(0)\right]$ goes to zero

Initial condition

In 3d, the large Λ behavior is ordered perturbatively:

$$\left[\Sigma_\Lambda(p)-\Sigma_\Lambda(0)\right] = \Big[\;\text{(diagram)}\;-\;\text{(diagram)}\;\Big|_{p=0}\Big]+\dots$$

$$= -\frac{\lambda^2}{6}\int_q\int_l G_\Lambda(q)G_\Lambda(l)\left[G_\Lambda(l+q+p)-G_\Lambda(l+q)\right]+\dots$$

Posing $R_\Lambda(q)=\Lambda^2 r(q/\Lambda)$ and rescaling the integration variables by Λ:

$$-\frac{\lambda^2}{6}\int_q\int_l \frac{1}{q^2+\frac{m^2}{\Lambda^2}+r(q)}\,\frac{1}{l^2+\frac{m^2}{\Lambda^2}+r(l)}$$

$$\underbrace{\left[\frac{1}{(l+q+\frac{p}{\Lambda})^2+\frac{m^2}{\Lambda^2}+r(l+q+\frac{p}{\Lambda})}-\frac{1}{(l+q)^2+\frac{m^2}{\Lambda^2}+r(l+q)}\right]}_{\to 0}$$

Physical initial condition

$$\Gamma_k^{(2)}(p) = p^2 + m^2 + \left[\Sigma_k(p) - \Sigma(0)\right]$$

$\Downarrow$ (modified gap equation)

$$\tilde{\Gamma}_k^{(2)}(p) = p^2 + m^2 + \left[\tilde{\Sigma}_k(p) - \tilde{\Sigma}_k(0)\right]$$

Physical value:

$$k \to 0 \;:\; \tilde{\Gamma}_{k=0}^{(2)}(p) = \Gamma_{k=0}^{(2)}(p)$$

Initial condition:

$$k = \Lambda \to \infty \;:\; \tilde{\Gamma}_k^{(2)}(p) \to p^2 + m^2$$

Physical initial condition

$$\Gamma_k^{(2)}(p) = p^2 + m^2 + \left[\Sigma_k(p) - \Sigma(0)\right]$$

$\Downarrow$ (modified gap equation)

$$\tilde{\Gamma}_k^{(2)}(p) = p^2 + m^2 + \left[\tilde{\Sigma}_k(p) - \tilde{\Sigma}_k(0)\right]$$

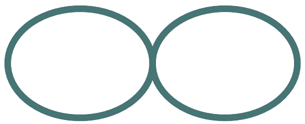
Modified flow:

$$\partial_k \tilde{\Gamma}_k^{(2)}(p) = -\frac{1}{2}\int_q \partial_k R_k(q)\, \tilde{G}_k^2(q)\, \tilde{\Gamma}_k^{(4)}(q,p)$$

$$\tilde{\Gamma}_k^{(4)}(q,p) = \tilde{\mathcal{I}}_k(q,p) - \frac{1}{2}\int_l \tilde{\Gamma}_k^{(4)}(q,l)\, \tilde{G}_k^2(l)\, \tilde{\mathcal{I}}_k(l,p)$$

$$\tilde{\mathcal{I}}_k(q,p) = \mathcal{I}_k(q,p) - \mathcal{I}_k(q,0)$$

Three-loop results

$\Phi[G] =$  $+$ 

Fast Fourier Transform

At three loops, one can conveniently rewrite the flow equation using Fourier Transforms

$$\partial_k \tilde{\Gamma}^{(2)}_k(p) = F_k(p)$$

$$F_k(p) = J_k(p) + \frac{\lambda^2}{2}\int_x \tilde{H}_k(x)\, G^2_k(x)\, e^{-ipx}$$

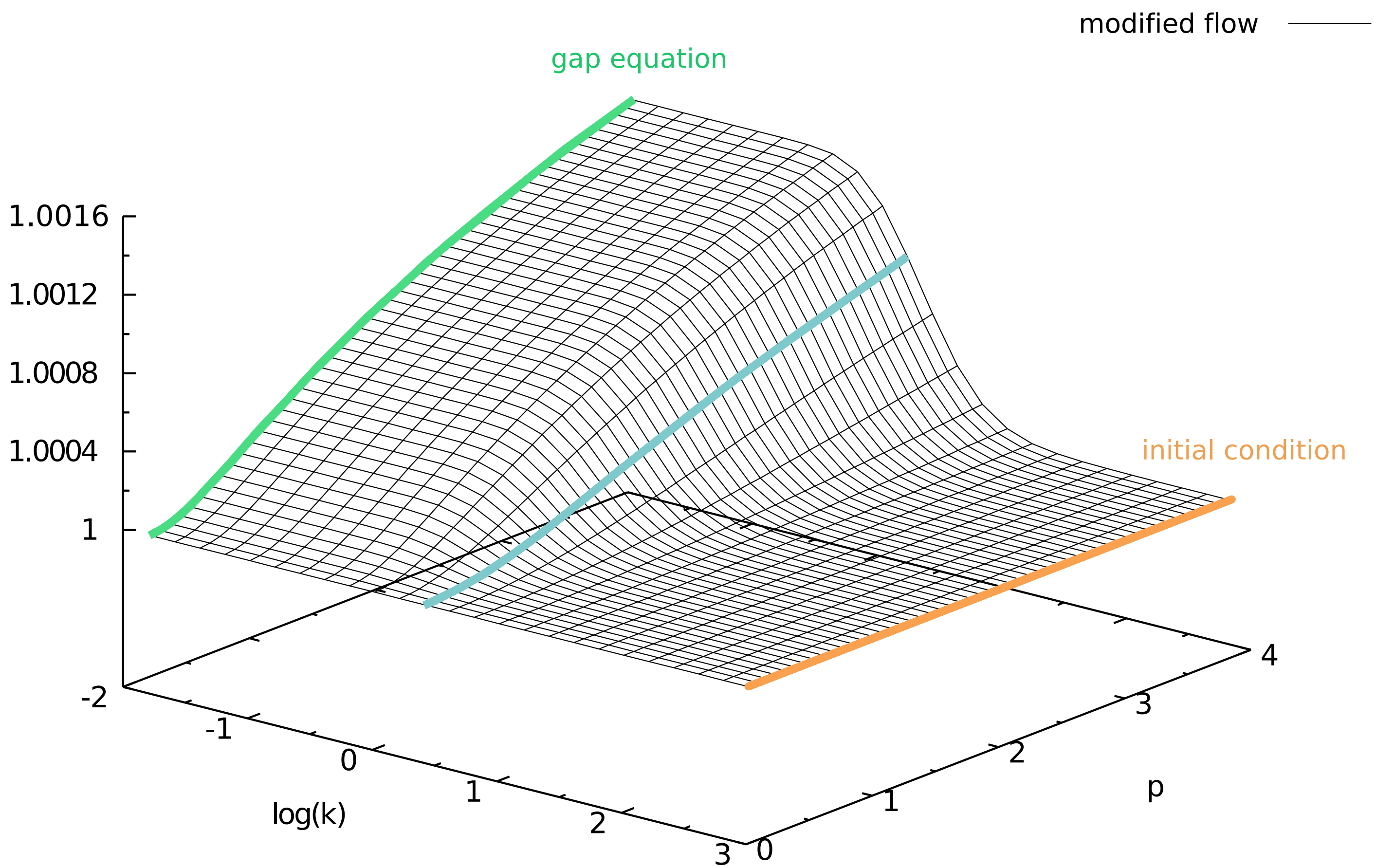
$$J_k(p) = \frac{1}{2}\int_x H_k(x)\, G^2_k(x)\, e^{-ipx} - \frac{1}{2}\int_x H_k(x)\, G^2_k(x)\, e^{-ipx}\Big|_{p=0}$$

$$H_k(x) = \int_q H_k(q)\, e^{iqx}\,, \qquad H_k(q) = \partial_k R_k(q)\, \tilde{G}^2_k(q)$$

$$\tilde{H}_k(x) = \int_q \tilde{H}_k(q)\, e^{iqx}\,, \qquad \tilde{H}_k(q) = F_k(q)\, \tilde{G}^2_k(q)$$

The only integrals are 1d Fourier (sine) transforms for which one can use FFT algorithms.

Results (preliminary)



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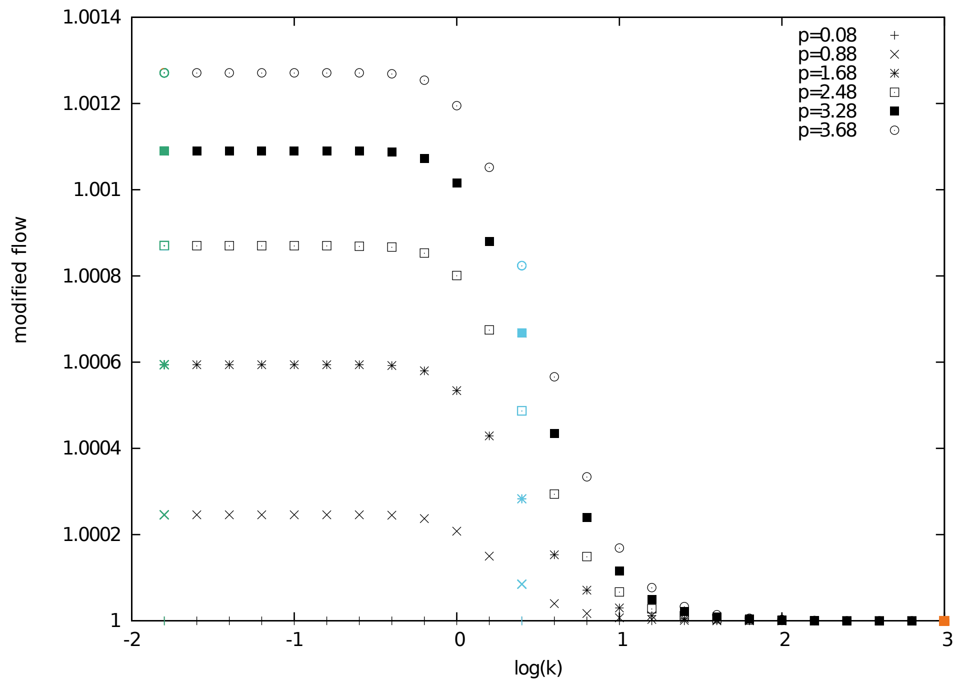

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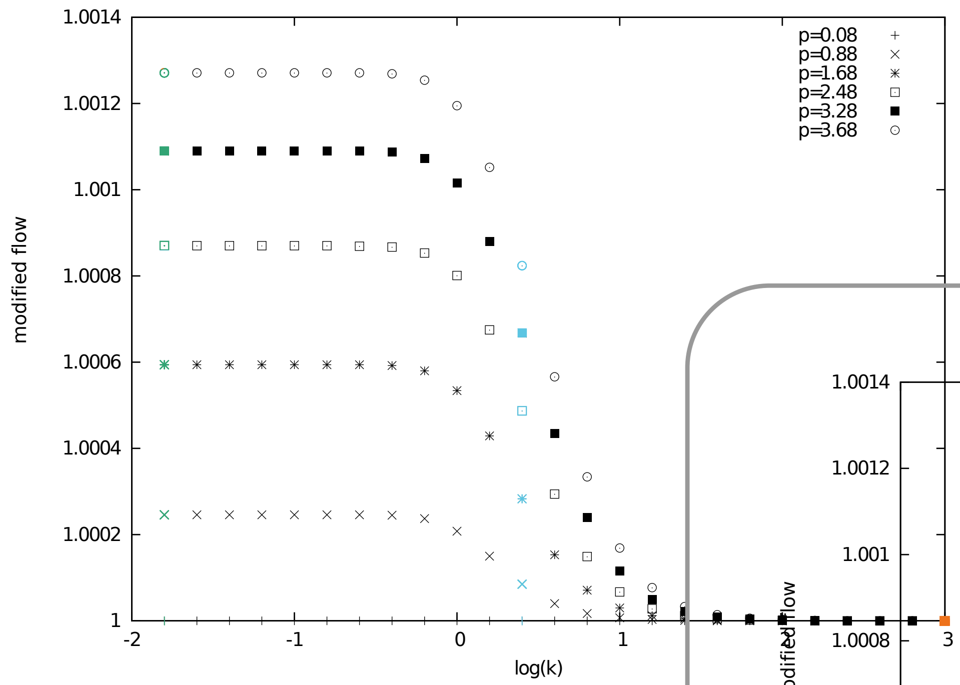

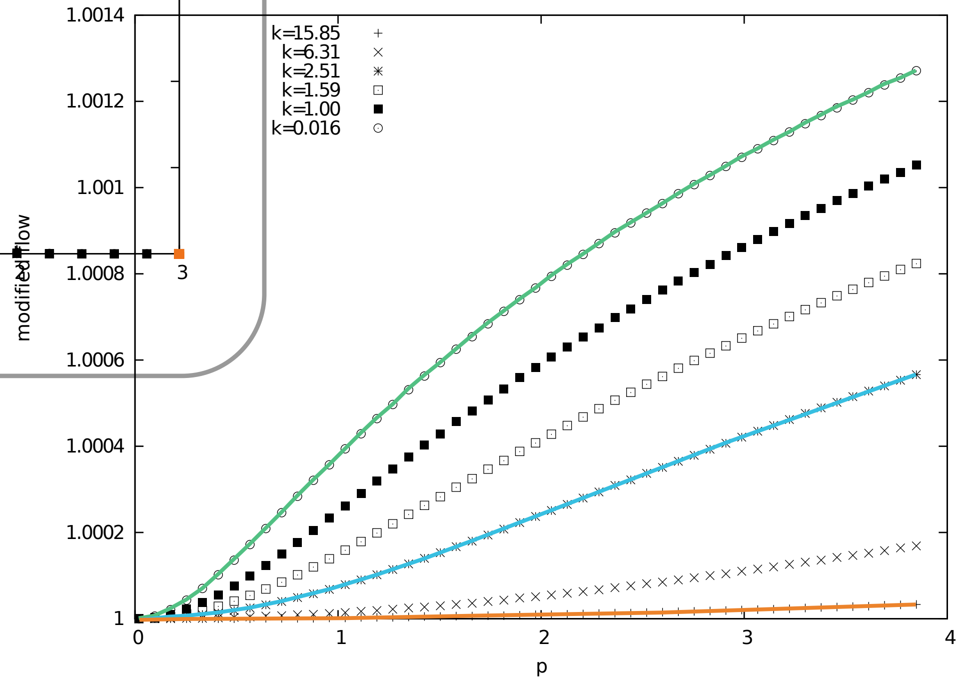

Summary

- ✓ (In equilibrium) Φ-derivable (or 2PI) approximations are nothing but particular truncations of the ERG flow equations.

- ✓ This point of view can simplify the resolution of 2PI approximations for it replaces the underlying non-linear integral gap equation by an initial value problem, coupled to a linear integral equation.

- ✓ This point of view treats the two- and four-point functions on a same footing (this is relevant in particular in four dimensions).

- ✓ In the 3d case, it was possible to find a modified flow equation, reproducing the 2PI result in the limit $k \to 0$, and whose initialisation involves renormalized parameters only.

4d case

2PI truncation of the flow:

$$\partial_k \Gamma_k^{(2)}(p) = -\frac{1}{2}\int_q \partial_k R_k(q)\, G_k^2(q)\, \Gamma_k^{(4)}(q,p)$$

$$\Gamma_k^{(4)}(q,p) = \mathcal{I}_k(q,p)$$

$$-\frac{1}{2}\int_k \Gamma_k^{(4)}(q,k)\, G_k^2(k)\, \mathcal{I}_k(k,p)$$

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$$\Gamma_k^{(4)}(q,p) = \mathcal{I}_k(q,p) - \mathcal{I}_k(0,0)$$

$$-\frac{1}{2}\int_k \Gamma_k^{(4)}(q,k)\, G_k^2(k)\left[\mathcal{I}_k(k,p) - \mathcal{I}_k(k,0)\right]$$

$$-\frac{1}{2}\int_k \left[\mathcal{I}_k(q,k) - \mathcal{I}_k(0,k)\right] G_k^2(k)\, \Gamma_k^{(4)}(k,0)$$