Talk at conference ERG 2010

September 12-19, 2010, Korfu, Greece

Spectral function and quasiparticle damping of interacting bosons

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collaborators: Andreas Sinner (Augsburg) Nils Hasselmann (Natal, Brazil)

reference: Phys. Rev. Lett. 102, 120601 (2009) and arXiv:1008.4521

outline:
1. Introduction: interacting Bose gas
2. Truncation of the FRG vertex expansion
3. Results: spectral function and Beliaev-damping

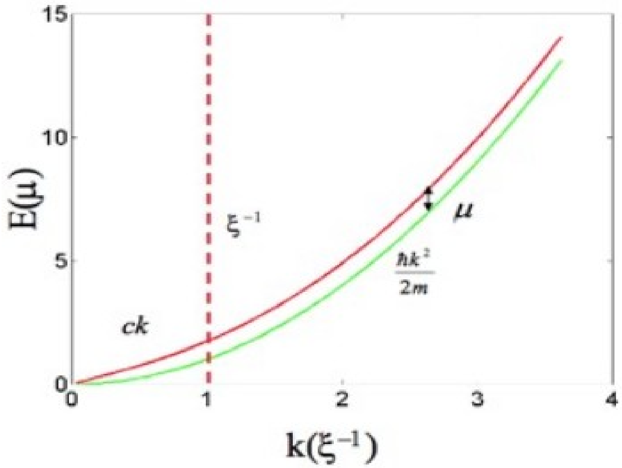
1. Interacting bosons: Bogoliubov theory (1947)

$$H = \sum_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}} + \frac{1}{2V} \sum_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}} U_{\boldsymbol{q}} b^\dagger_{\boldsymbol{k}+\boldsymbol{q}} b^\dagger_{\boldsymbol{k}'-\boldsymbol{q}} b_{\boldsymbol{k}'} b_{\boldsymbol{k}}$$

- Bogoliubov-shift: $b_{\boldsymbol{k}=0} \rightarrow \sqrt{N_0}$ $\qquad \rho_0 = \frac{N_0}{V}$
- Bogoliubov mean-field Hamiltonian:

$$H \approx \sum_{\boldsymbol{k}\neq 0} \left[[\epsilon_{\boldsymbol{k}} + \rho_0(U_0 + U_{\boldsymbol{k}})] b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}} + \frac{\rho_0 U_{\boldsymbol{k}}}{2} (b_{-\boldsymbol{k}} b_{\boldsymbol{k}} + b^\dagger_{\boldsymbol{k}} b^\dagger_{-\boldsymbol{k}}) \right]$$

- condensate density: $\rho_0 = \frac{|b_0|^2}{V} = \frac{\mu}{U_0}$
- excitation energy: $E_{\boldsymbol{k}} = \sqrt{\epsilon_{\boldsymbol{k}}[\epsilon_{\boldsymbol{k}} + 2\rho_0 U_{\boldsymbol{k}}]}$
- long wavelength excitations: sound!



$$E_{\boldsymbol{k}} \sim c|\boldsymbol{k}| \qquad c = \sqrt{\rho_0 U_0/m}$$

beyond mean-field: infrared divergencies

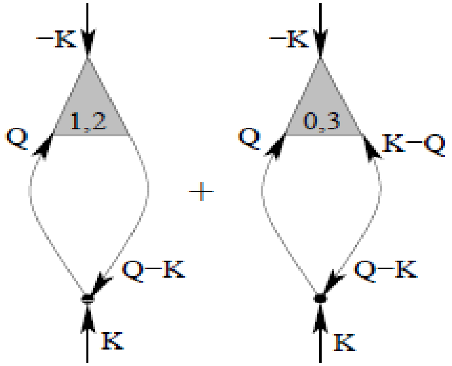
- Bogoliubov mean-field Hamiltonian:

$$H \approx \sum_{\boldsymbol{k} \neq 0} \left[[\epsilon_{\boldsymbol{k}} + \Sigma_N^{(1)}(\boldsymbol{k})] b_{\boldsymbol{k}}^{\dagger} b_{\boldsymbol{k}} + \frac{\Sigma_A^{(1)}(\boldsymbol{k})}{2} (b_{-\boldsymbol{k}} b_{\boldsymbol{k}} + b_{\boldsymbol{k}}^{\dagger} b_{-\boldsymbol{k}}^{\dagger}) \right]$$

normal self-energy: $\Sigma_N^{(1)}(\boldsymbol{k}) = \rho_0 [U_0 + U_{\boldsymbol{k}}]$

anomalous self-energy: $\Sigma_A^{(1)}(\boldsymbol{k}) = \rho_0 U_{\boldsymbol{k}}$

- mean-field fails due to infrared divergent fluctuation corrections:



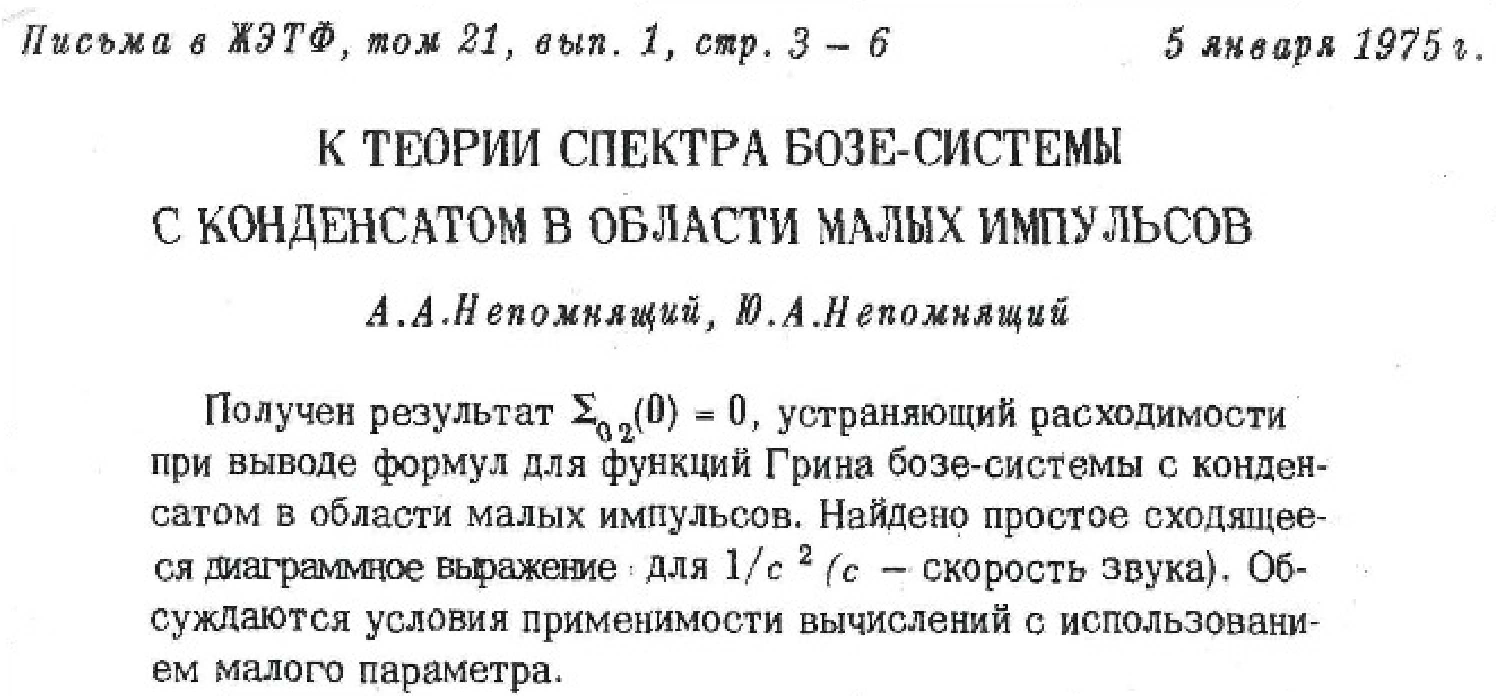
$$\Sigma_A^{(2)}(\boldsymbol{k}) \propto \int_{|\boldsymbol{k}|}^{k_G} \frac{dq}{q^{4-D}} \propto \begin{cases} (k_G/|\boldsymbol{k}|)^{3-D} & \text{for } D < 3 \\ \ln(k_G/|\boldsymbol{k}|) & \text{for } D = 3 \end{cases}$$

$|\Sigma_A^{(2)}(\boldsymbol{k})| \gg |\Sigma_A^{(1)}(\boldsymbol{k})|$ for $|\boldsymbol{k}| \ll k_G$ Ginzburg scale

exact result: Nepomnyashi-identity (1975)

- anomalous self-energy vanishes at zero momentum/frequency:

$$\boxed{\Sigma_A(0) = 0}$$



Письма в ЖЭТФ, том 21, вып. 1, стр. 3 – 6 *5 января 1975 г.*

К ТЕОРИИ СПЕКТРА БОЗЕ-СИСТЕМЫ С КОНДЕНСАТОМ В ОБЛАСТИ МАЛЫХ ИМПУЛЬСОВ

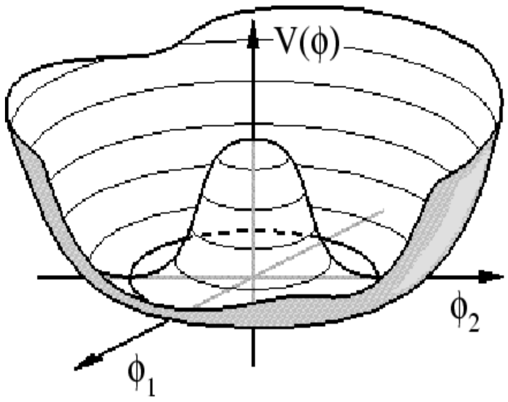
А.А.Непомнящий, Ю.А.Непомнящий

Получен результат $\Sigma_{02}(0) = 0$, устраняющий расходимости при выводе формул для функций Грина бозе-системы с конденсатом в области малых импульсов. Найдено простое сходящееся диаграммное выражение для $1/c^2$ (c – скорость звука). Обсуждаются условия применимости вычислений с использованием малого параметра.

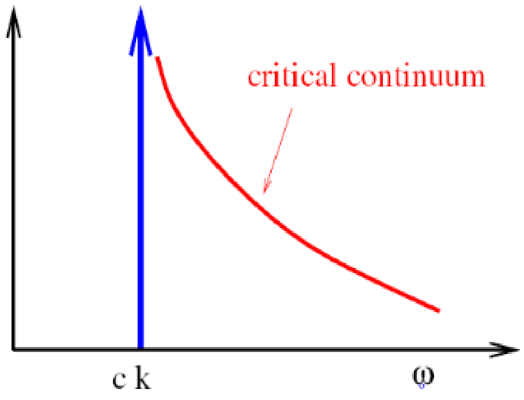
- Bogoliubov approximation wrong: $\Sigma_A^{(1)}(0) = \rho_0 U_0$

- need non-perturbative methods!

origin of infrared divergencies



- reason for IR divergence:
  coupling between transverse and longitudinal fluctuations and resulting divergence of longitudinal susceptibility
  (Patashinskii+Pokrovskii, JETP 1973)



- consequence: critical continuum in longitudinal part of spectral function
  (for bosons not directly measurable!)

- experimentally accessible: longitudinal spin structure factor in quantum antiferromagnets: BEC of magnons! (Kreisel, Hasselmann, PK, PRL 2007)

$$S_{\parallel}(\boldsymbol{k},\omega) = \frac{\chi\, s^2}{M_s^2}\left[\frac{Z_{\parallel}^2}{2}c|\boldsymbol{k}|\delta(\omega - c|\boldsymbol{k}|) + C_D\frac{(mc)^3}{Z_{\rho}^3\rho_0}\frac{\Theta(\omega - c|\boldsymbol{k}|)}{(\omega^2/c^2 - \boldsymbol{k}^2)^{\frac{3-D}{2}}}\right]$$

RG studies of interacting bosons

1.) Field -theoretical RG

Castellani, Di Castro, Pistolesi, Strinati 1997, 2004

- combine field-theoretical RG with Ward-identities to obtain leading IR behavior of correlation functions
- below and in three dimensions interacting bosons at T=0 are controlled by a non-Gaussian fixed point

2.) Functional RG

- ground state properties and IR asymptotics of correlation functions
  Dupuis+Sengupta 2007; Wetterich 2008; Flörchinger+Wetterich 2009
- spectral line shapes, quasi-particle damping
  Sinner+Hasselmann+PK 2009 2010
- IR asymptotics of longitudinal component of spectral function
  Dupuis 2009
- finite temperature thermodynamics in condensed phase
  Eichler+Hasselmann+PK 2009, Flörchinger+Wetterich 2009

2. Truncation of the FRG vertex expansion

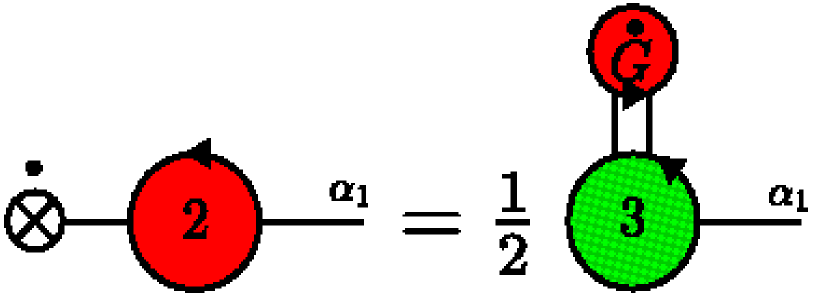
- starting point: “Wetterich equation”

$$\partial_\Lambda \Gamma_\Lambda[\Phi] = \frac{1}{2}\mathrm{Tr}\left[(\partial_\Lambda \boldsymbol{R}_\Lambda)\left(\frac{\delta}{\delta\Phi}\otimes\frac{\delta}{\delta\Phi}\Gamma_\Lambda[\Phi]+\boldsymbol{R}_\Lambda\right)^{-1}\right]$$

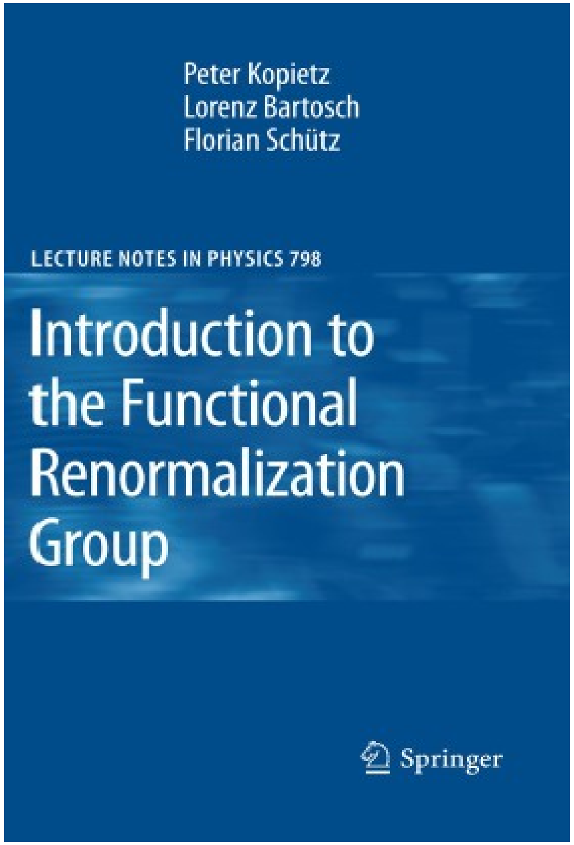
- use vertex expansion in symmetry broken phase: (Schütz, PK, 2006):

get flow of order parameter by demanding that 1-point vertex vanishes identically:

$$\int_{\beta_1}[\mathbf{\Sigma}]_{\alpha_1\beta_1}\partial_\Lambda\bar{\Phi}^0_{\beta_1} = \frac{1}{2}\int_{\beta_1}\int_{\beta_2}[\dot{\mathbf{G}}]_{\beta_1\beta_2}\Gamma^{(3)}_{\beta_1\beta_2\alpha_1}$$



recent review:



Peter Kopietz
Lorenz Bartosch
Florian Schütz

LECTURE NOTES IN PHYSICS 798

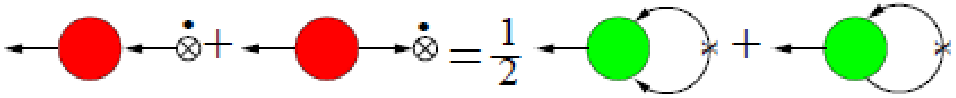
Introduction to the Functional Renormalization Group

Springer

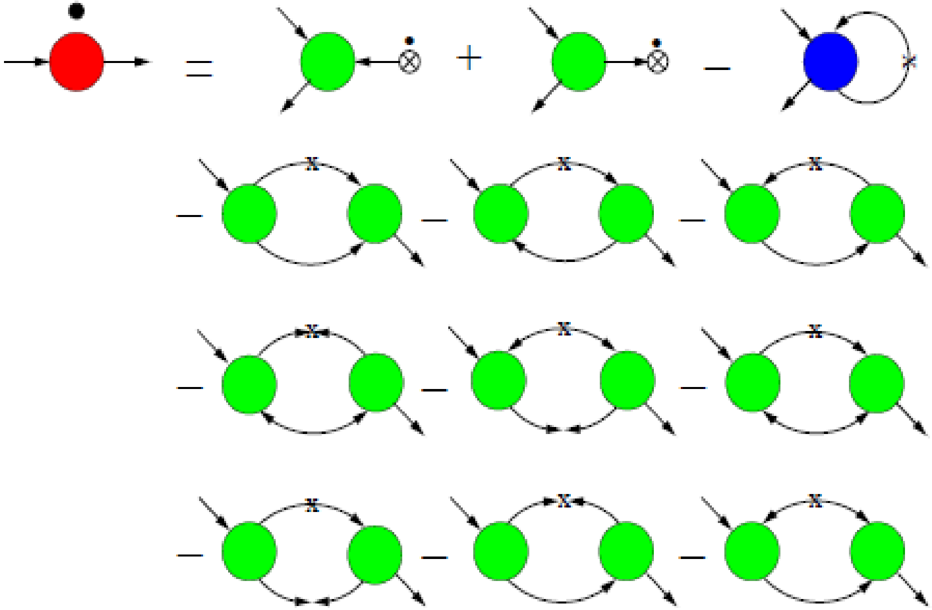
2010. XII, 380 p. (Lecture Notes in Physics, Vol. 798) Hardcover

exact FRG flow equations for order parameter and self-energies

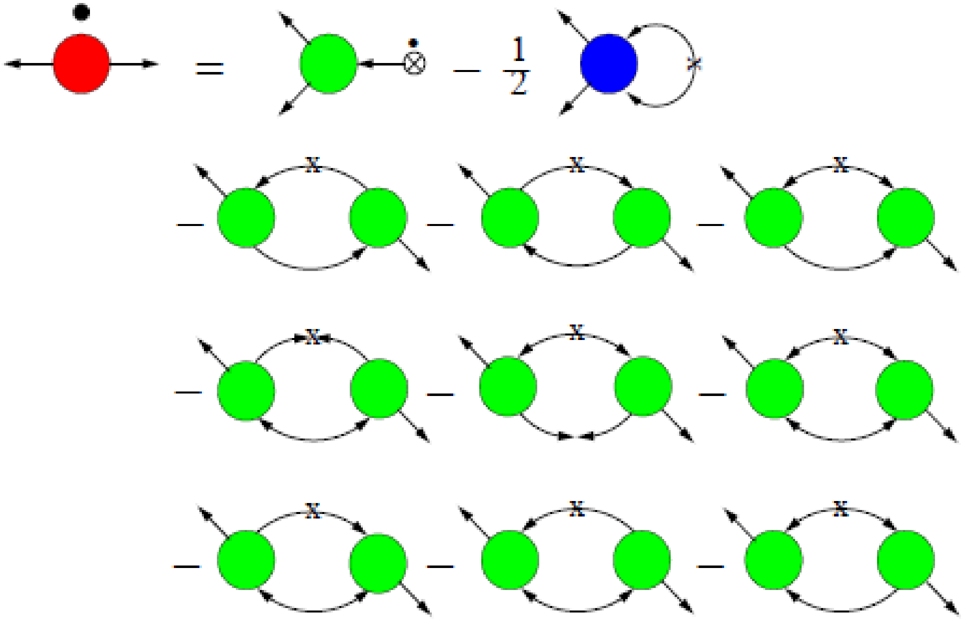
- order parameter:



- normal self-energy:



- anomalous self-energy:



Low-density truncation: retain only two-body interactions

- bare action:

$$S[\bar{\psi}, \psi] = \int d^D x d\tau \left[\bar{\psi}(\partial_\tau - \frac{\boldsymbol{\nabla}^2}{2m} - \mu)\psi + \frac{u_0}{2}(\bar{\psi}\psi)^2 \right]$$

- ansatz for generating functional of irreducible vertices:

$$\Gamma_\Lambda[\bar{\phi}, \phi] \approx \int_K \bar{\phi}_K \sigma_\Lambda(K) \phi_K + \frac{1}{2} \int_K \delta\rho_K u_\Lambda(K) \delta\rho_{-K}$$

$$\delta\rho_K = \int_Q \bar{\phi}_Q \phi_{Q+K} - \delta_{K,0} \rho^0_\Lambda$$

- depends on condensate density ρ^0_Λ and two frequency- and momentum-dependent functions $\sigma_\Lambda(K)$ and $u_\Lambda(K)$

analytic

non-analytic

parametrization of three- and four-point vertices:

- normal and anomalous self-energy:

$$\begin{aligned}\Sigma^N_\Lambda(K) &= \mu + \sigma_\Lambda(K) + \rho^0_\Lambda u_\Lambda(K)\\ \Sigma^A_\Lambda(K) &= \rho^0_\Lambda u_\Lambda(K)\,.\end{aligned}$$

$\sigma_\Lambda(0) = 0$ → Hugenholtz-Pines relation $\Sigma^N_\Lambda(0) - \Sigma^A_\Lambda(0) = \mu$

$\lim\limits_{\Lambda\to 0} u_\Lambda(0) = 0$ → Nepomnyashchy identity $\Sigma^A(0) = 0, \;\; \Lambda \to 0$

- three-legged vertices:

$$\begin{aligned}\Gamma^{(2,1)}_\Lambda(K_1', K_2'; K_1) &= \phi^0_\Lambda[u_\Lambda(K_1') + u_\Lambda(K_2')],\\ \Gamma^{(1,2)}_\Lambda(K_1'; K_2, K_1) &= \phi^0_\Lambda[u_\Lambda(K_1) + u_\Lambda(K_2)].\end{aligned}$$

- effective interaction:

$$\Gamma^{(2,2)}_\Lambda(K_1', K_2'; K_2, K_1) = u_\Lambda(K_1' - K_1) + u_\Lambda(K_2' - K_1)$$

closed FRG equations for condensate density and self-energies

$$\partial_\Lambda \rho_\Lambda^0 = \frac{1}{u_\Lambda(0)} \int_Q \Big\{ \dot{G}_\Lambda^N(Q) [u_\Lambda(Q) + u_\Lambda(0)] + \dot{G}_\Lambda^A(Q) u_\Lambda(Q)] \Big\}$$

$$\begin{aligned}
\partial_\Lambda \Sigma_\Lambda^N(K) = {} & \frac{u_\Lambda(K) + u_\Lambda(0)}{u_\Lambda(0)} \int_Q [\dot{G}_\Lambda^N(Q) + \dot{G}_\Lambda^A(Q)] u_\Lambda(Q) + \int_Q \dot{G}_\Lambda^N(Q) [u_\Lambda(K) - u_\Lambda(K-Q)] \\
& - \rho_\Lambda^0 \int_Q \dot{G}_\Lambda^N(Q) \Big\{ G_\Lambda^N(K-Q) [u_\Lambda(Q) + u_\Lambda(K-Q)]^2 + G_\Lambda^N(Q-K) [u_\Lambda(K) + u_\Lambda(K-Q)]^2 \\
& \quad + G_\Lambda^N(Q+K) [u_\Lambda(Q) + u_\Lambda(K)]^2 + 2 G_\Lambda^A(K-Q) [u_\Lambda(Q) + u_\Lambda(K-Q)] [u_\Lambda(K) + u_\Lambda(K-Q)] \Big\} \\
& - 2\rho_\Lambda^0 \int_Q \dot{G}_\Lambda^A(Q) \Big\{ G_\Lambda^A(K-Q) [u_\Lambda(K) + u_\Lambda(K-Q)] + G_\Lambda^N(K-Q) [u_\Lambda(Q) + u_\Lambda(K-Q)] \Big\} [u_\Lambda(K) + u_\Lambda(Q)]
\end{aligned}$$

$$\begin{aligned}
\partial_\Lambda \Sigma_\Lambda^A(K) = {} & \frac{u_\Lambda(K)}{u_\Lambda(0)} \int_Q \Big\{ \dot{G}_\Lambda^N(Q) [u_\Lambda(Q) + u_\Lambda(0)] + \dot{G}_\Lambda^A(Q) u_\Lambda(Q) \Big\} - \frac{1}{2} \int_Q \dot{G}_\Lambda^A(Q) [u_\Lambda(K+Q) + u_\Lambda(K-Q)] \\
& - \rho_\Lambda^0 \int_Q \dot{G}_\Lambda^N(Q) \Big\{ [G_\Lambda^N(Q-K) + G_\Lambda^N(K-Q)] [u_\Lambda(K) + u_\Lambda(K-Q)] [u_\Lambda(K) + u_\Lambda(Q)] \\
& \quad + \Big(G_\Lambda^A(K+Q) [u_\Lambda(Q) + u_\Lambda(K+Q)] + G_\Lambda^A(K+Q) [u_\Lambda(Q) + u_\Lambda(K+Q)] \Big) [u_\Lambda(Q) + u_\Lambda(K)] \Big\} \\
& - \rho_\Lambda^0 \int_Q \dot{G}_\Lambda^A(Q) \Big\{ G_\Lambda^A(K-Q) \Big([u_\Lambda(Q) + u_\Lambda(K-Q)]^2 + [u_\Lambda(K) + u_\Lambda(K-Q)]^2 \Big) + G_\Lambda^A(K+Q) [u_\Lambda(K) + u_\Lambda(Q)]^2 \\
& \quad + [G_\Lambda^N(K-Q) + G_\Lambda^N(Q-K)] [u_\Lambda(K) + u_\Lambda(K-Q)] [u_\Lambda(Q) + u_\Lambda(K-Q)] \Big\}
\end{aligned}$$

similar truncation strategy for classical O(2) model

- momentum-dependent self-energy in symmetric phase: needed for shift in critical temperature of interacting bosons

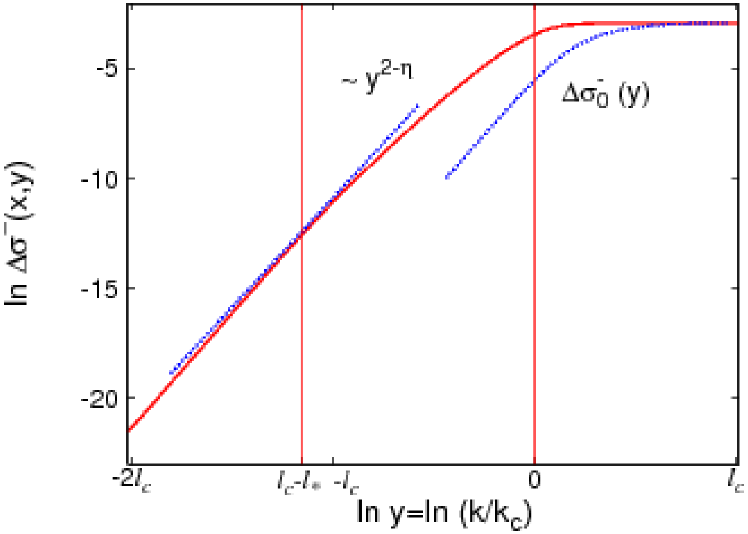
  Ledowski, Hasselmann, PK 2004

- more sophisticated trunction: Blaizot, Mendez-Galain, Wschebor 2006

- self-energy in ordered phase Sinner, Hasselmann, PK 2008

two-parameter scaling form of self-energy:

$$\Sigma(\boldsymbol{k}) = \Sigma(0) + k_G^2 \Delta\sigma^-(|\boldsymbol{k}|\xi, |\boldsymbol{k}|/k_G)$$

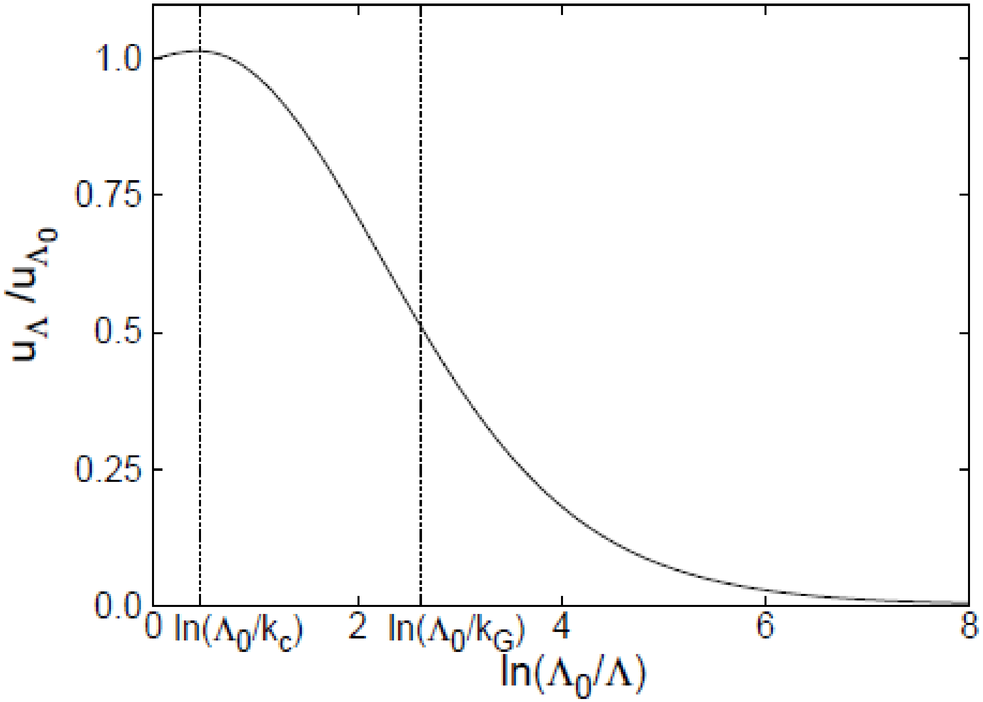


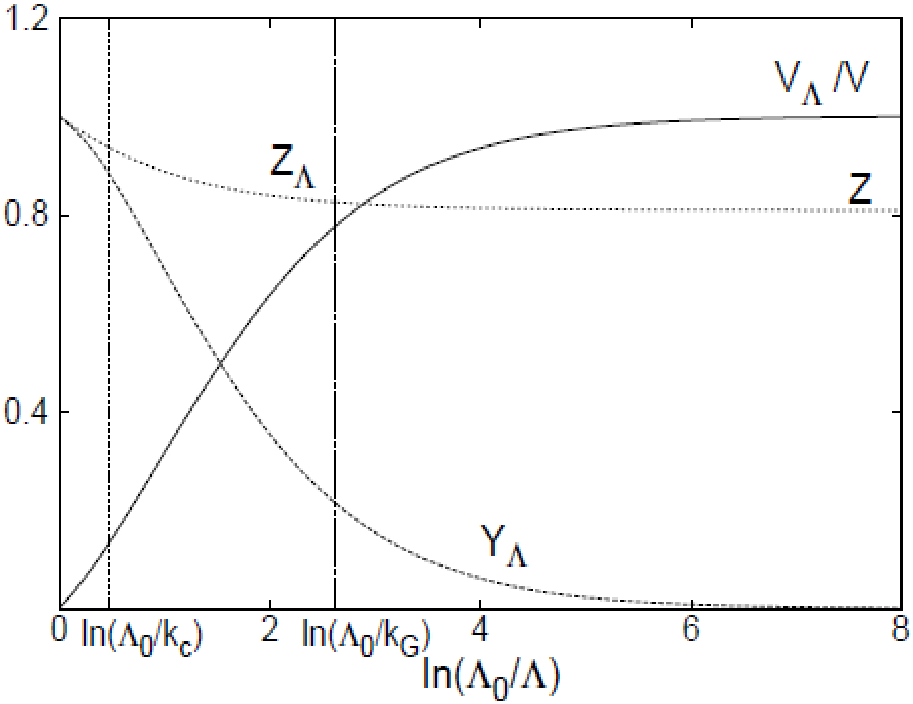
iterative solution of FRG equations

Step 1: derivative expansion:

$$u_\Lambda(K) \approx u_\Lambda(0) = u_\Lambda$$

$$\sigma_\Lambda(K) \approx i\omega\,(1 - Y_\Lambda) + \epsilon_k(Z_\Lambda^{-1} - 1) + \omega^2 V_\Lambda$$





K-dependent self-energies

Step 2: non-selfconsistent iteration

- given the self-consistent values for $u_\Lambda, \rho^0_\Lambda, Y_\Lambda, Z_\Lambda, V_\Lambda$

  approximate Green functions on r.h.s. of flow equations by

$$G^N_\Lambda(K) = \frac{Y_\Lambda i\omega + Z^{-1}_\Lambda \epsilon_k + \Delta_\Lambda + R_\Lambda(k) + V_\Lambda \omega^2}{\mathcal{D}_\Lambda(K)} \qquad G^A_\Lambda(K) = -\frac{\Delta_\Lambda}{\mathcal{D}_\Lambda(K)}$$

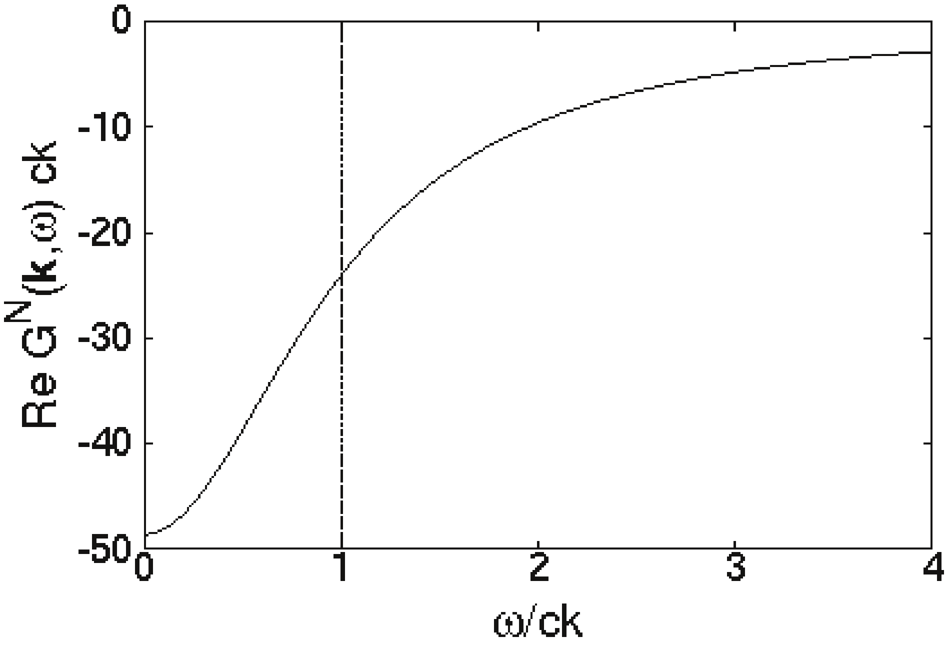
$$\mathcal{D}_\Lambda(K) = Y^2_\Lambda \omega^2 + [V_\Lambda \omega^2 + Z^{-1}_\Lambda \epsilon_k + R_\Lambda(k)]\,[2\Delta_\Lambda + V_\Lambda \omega^2 + Z^{-1}_\Lambda \epsilon_k + R_\Lambda(k)]$$

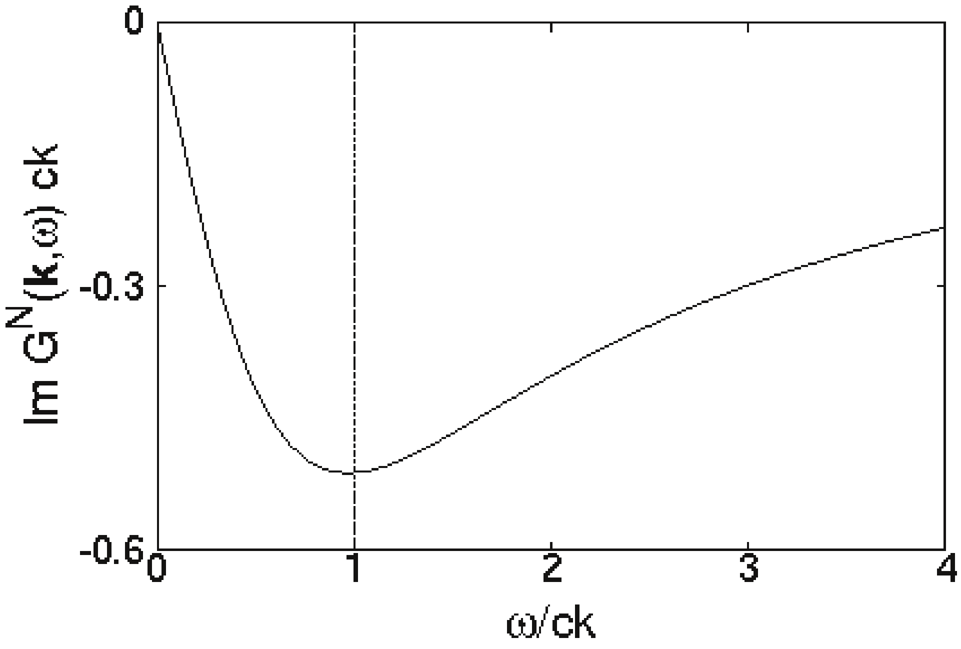
$$\Delta_\Lambda = \rho^0_\Lambda u_\Lambda \qquad R_\Lambda(k) = (1-\delta_{k,0})\,(2mZ_\Lambda)^{-1}\left(\Lambda^2 - k^2\right)\Theta\left(\Lambda^2 - k^2\right)$$

- perform resulting integrations numerically

  important: integrals contain all powers of momentum and frequency!

Matsubara Green function

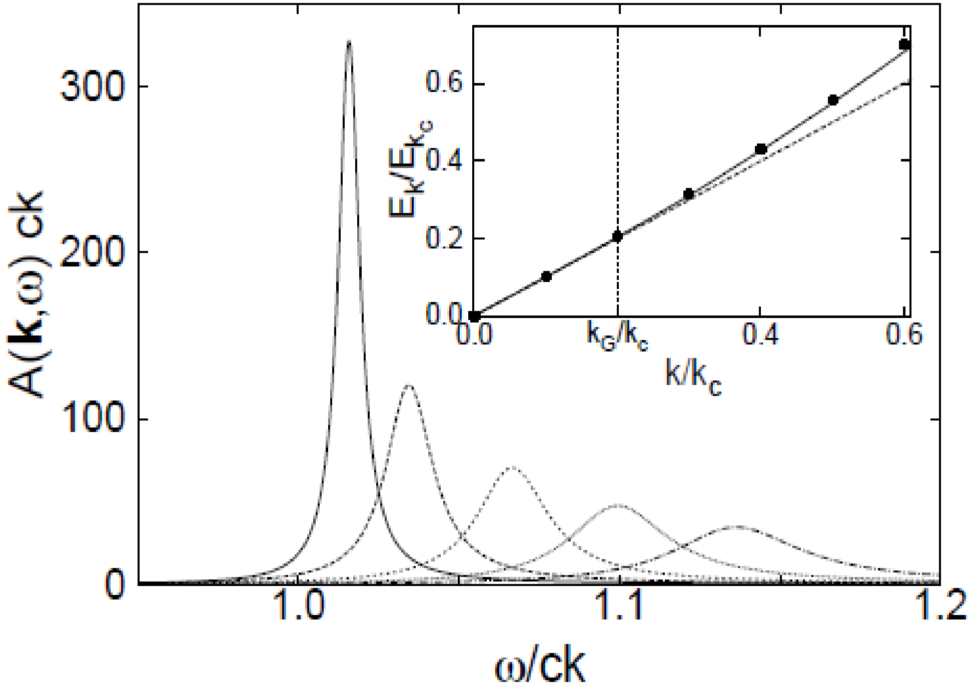




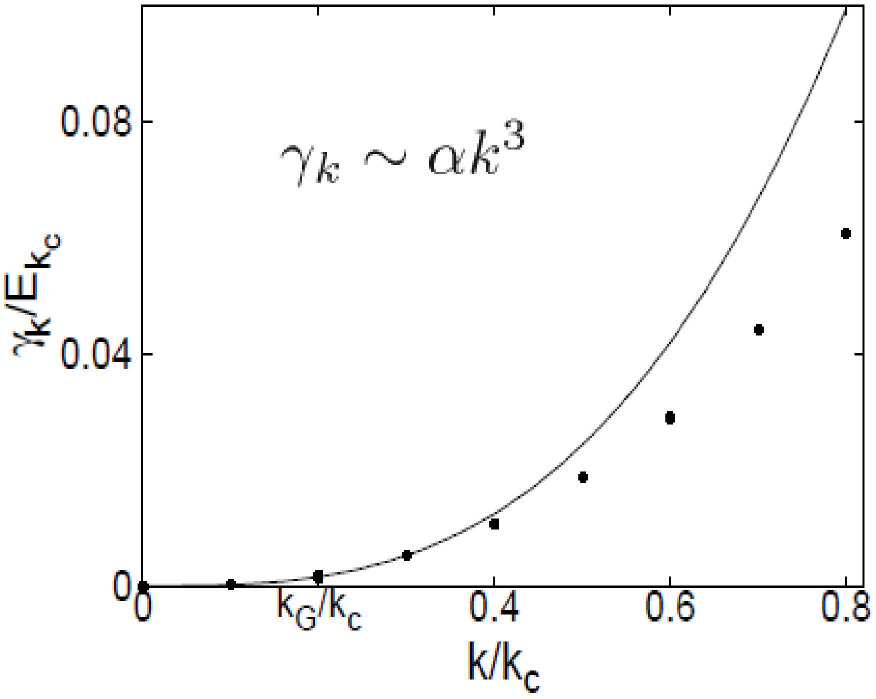
- numerical analytic continuation (Pade) to get spectral function

3.Results: spectral function and quasi-particle damping in 2D

- spectral function:



- quasi-particle damping:

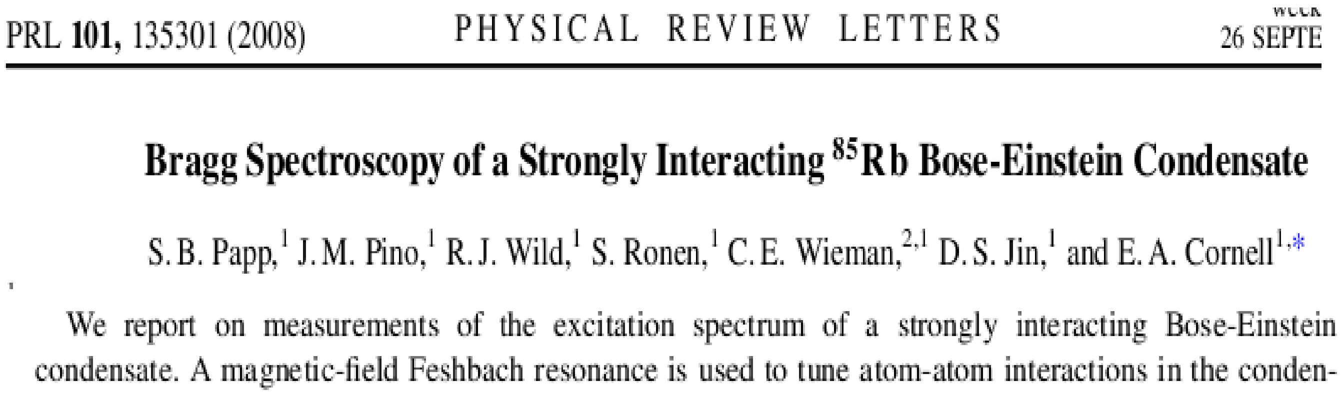


$2mu_{\Lambda_0} = 15$ $\quad k_c = 2mc \quad c/c_0 = 0.67$

$2m\mu/\Lambda_0^2 = 0.15 \quad k_G/k_c \approx 0.2 \quad \alpha/\alpha_0 = 0.53$

what about experiments?

Cold atom experiments can now measure renormalized excitation spectrum of strongly interacting bosons:



PRL **101,** 135301 (2008) PHYSICAL REVIEW LETTERS 26 SEPTE

Bragg Spectroscopy of a Strongly Interacting ^{85}Rb Bose-Einstein Condensate

S. B. Papp,[1] J. M. Pino,[1] R. J. Wild,[1] S. Ronen,[1] C. E. Wieman,[2,1] D. S. Jin,[1] and E. A. Cornell[1,*]

We report on measurements of the excitation spectrum of a strongly interacting Bose-Einstein condensate. A magnetic-field Feshbach resonance is used to tune atom-atom interactions in the conden-

Summary

- First calculation of spectral function of interacting bosons without violating Nepomyachshy identity

- Bogoliubov quasi-particles in 2D are well-defined with damping $\gamma_k \sim \alpha(\mu, u_0)k^3$

- FRG can take into account infinitely many irrelevant couplings, corresponding to all powers in momentum and frequency