Running Couplings in Topologically Massive Gravity

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Outline

Topologically massive gravity

Action

$$S(\gamma) = Z \int d^3x \sqrt{-\gamma} \left(-2\Lambda + R + \frac{1}{2\mu} \varepsilon^{\lambda\mu\nu} \Gamma^{\rho}_{\lambda\sigma} \left(\partial_\mu \Gamma^{\sigma}_{\nu\rho} + \frac{2}{3} \Gamma^{\sigma}_{\mu\tau} \Gamma^{\tau}_{\nu\rho} \right) \right)$$

$$Z = \frac{1}{16\pi G}$$

Dimensionless combinations of couplings

$$\nu = \mu G \; ; \qquad \tau = \Lambda G^2 \; ; \qquad \phi = \mu / \sqrt{|\Lambda|}$$

$$\nu^2 = \tau \phi^2$$

Background field expansion

$$\gamma_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$$

Gauge fixing

$$S_{GF} = -\frac{Z}{2\alpha}\int d^3x\sqrt{-g}\chi_\mu g^{\mu\nu}\chi_\nu \ ,$$

$$\chi_\nu = \partial_\mu h^{\mu\nu} - \frac{\beta+1}{4}\partial_\nu h \ .$$

$$S_{gh} = -\int d^3x\sqrt{-g}\bar{C}^\mu\left(\delta^\nu_\mu\Box + \frac{1-\beta}{2}\nabla_\mu\nabla^\nu + R_\mu{}^\nu\right)C_\nu$$

Later will set $\beta = \frac{2\alpha+1}{3}$

Modified generating functional

$$e^{-W_k[J]} = \int D\phi \exp\left\{-S[\phi] - \Delta S_k[\phi] - \int dx\, J\phi\right\}$$

IR Cutoff

$$\Delta S_k[\phi] = \frac{1}{2}\int dp\, \phi(-p)\, R_k(p^2)\, \phi(p) = \frac{1}{2}\sum_n \phi_n^2 R_k(\lambda_n)\,.$$

Effective average action

$$\Gamma_k[\phi] = W_k\,[J] - \int dx\, J\phi - \Delta S_k[\phi]\,,$$

One loop evaluation

For the EA

$$\Gamma^{(1)} = S + \frac{1}{2}\mathrm{Tr}\log\left(S^{(2)}\right)$$

Similarly

$$\Gamma_k^{(1)} = S + \frac{1}{2}\mathrm{Tr}\log\left(S^{(2)} + R_k\right)$$

therefore

$$k\frac{d\Gamma_k^{(1)}}{dk} = \frac{1}{2}\mathrm{Tr}\left(S^{(2)} + R_k\right)^{-1} k\frac{dR_k}{dk}$$

Derivative expansion

$$\Gamma_k(\phi, g_i) = \sum_{n=0}^{\infty} \sum_i g_i^{(n)}(k) \mathcal{O}_i^{(n)}(\phi)$$

$$k \frac{d\Gamma_k(\phi, g_i)}{dk} = \sum_{n=0}^{\infty} \sum_i \beta_i^{(n)}(k) \mathcal{O}_i^{(n)}(\phi)$$

One loop flow eqn for TMG

$$k\frac{d\Gamma_k^{(1)}}{dk} = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2(S+S_{GF})}{\delta h_{\mu\nu}\delta h_{\rho\sigma}} + \mathcal{R}^{\mu\nu\rho\sigma}\right)^{-1} k\frac{d\mathcal{R}_{\rho\sigma\mu\nu}}{dk} - \mathrm{Tr}\left(\frac{\delta^2 S_{gh}}{\delta\bar{C}^{\mu}\delta C_{\nu}} + \mathcal{R}_{\mu}{}^{\nu}\right)^{-1} k\frac{d\mathcal{R}^{\mu}{}_{\nu}}{dk}$$

Beta functions of G, μ, Λ can be read off calculating $k\frac{d\Gamma_k^{(1)}}{dk}$ on S^3

Second variations

Take g metric on S^3

$$S^{(2)} + S_{GF} =$$

$$\frac{1}{4} Z \int d^3x \sqrt{-g} \Bigg[h_{\mu\nu} \left(\Box - \frac{2R}{3} + 2\Lambda \right) h^{\mu\nu} + \frac{2(1-\alpha)}{\alpha} h_{\mu\nu} \nabla^\mu \nabla_\rho h^{\rho\nu}$$

$$+ \left(2 - \frac{\beta+1}{\alpha} \right) h \nabla^\mu \nabla^\nu h_{\mu\nu} - \left(1 - \frac{(\beta+1)^2}{8\alpha} \right) h \Box h + \frac{1}{6} h (R - 6\Lambda) h$$

$$+ \frac{1}{\mu} \varepsilon^{\lambda\mu\nu} h_{\lambda\sigma} \left(\nabla_\mu \left(\Box - \frac{R}{3} \right) h^\sigma{}_\nu - \nabla_\mu \nabla^\sigma \nabla^\rho h_{\rho\nu} \right) \Bigg]$$

Decomposition

$$h_{\mu\nu} = h^T_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + \nabla_\mu \nabla_\nu \sigma - \frac{1}{3} g_{\mu\nu} \Box \sigma + \frac{1}{3} g_{\mu\nu} h$$

$$\nabla^\mu h^T_{\mu\nu} = 0 \; ; \quad g^{\mu\nu} h^T_{\mu\nu} = 0 \; ; \nabla_\mu \xi^\mu = 0$$

likewise for ghosts

$$C_\mu = V_\mu + \partial_\mu S \; ; \quad \nabla^\mu V_\mu = 0$$

Field redefinitions

$$\sqrt{\Box + \frac{R}{3}} \, \xi_\mu = \hat{\xi}_\mu \; ; \qquad \sqrt{\Box \left(\Box + \frac{R}{2} \right)} \, \sigma = \hat{\sigma} \; ; \qquad \sqrt{\Box} S = \hat{S}$$

$$S^{(2)} + S_{GF} =$$
$$\frac{Z}{4}\int \left[h^{TT}_{\mu\nu}\Delta_2^{\mu\nu\rho\sigma}h^{TT}_{\rho\sigma} + c_1\hat{\xi}_\mu\Delta_1^{\mu\nu}\hat{\xi}_\nu + c_\sigma\hat{\sigma}\Delta_\sigma\hat{\sigma} + c_h h\Delta_h h\right]$$

$$\Delta_{2\mu\nu}{}^{\rho\sigma} = \left(\Box - \frac{2R}{3} + 2\Lambda\right)\delta^{(\rho}_{(\mu}\delta_{\nu)}{}^{\sigma)} + \frac{1}{\mu}\varepsilon_{(\mu}{}^{\lambda(\rho}\delta^{\sigma)}_{\nu)}\nabla_\lambda\left(\Box - \frac{R}{3}\right)$$
$$\Delta_{1\mu}{}^{\nu} = \Box + \frac{1-\alpha}{3}R + 2\alpha\Lambda$$
$$\Delta_\sigma = \Box + \frac{2-\alpha}{4-\alpha}R + \frac{6\alpha\Lambda}{4-\alpha}$$
$$\Delta_h = \Box + \frac{R}{4-\alpha} + \frac{6\Lambda}{4-\alpha}$$
$$c_1 = -\frac{2}{\alpha}, \qquad c_\sigma = \frac{2(4-\alpha)}{9\alpha}, \qquad c_h = -\frac{4-\alpha}{18}$$

Eigenvalues and multiplicities

$$\lambda_n^{T\pm} = \frac{R}{6}(n^2+2n+2) - 2\Lambda \pm \frac{1}{\mu}\left(\frac{R}{6}\right)^{3/2} n(n+1)(n+2)\,, \quad n \geq 2\,,$$

$$\lambda_n^{\xi} = \frac{R}{6}\left(n^2+2n-3+2\alpha\right) - 2\alpha\Lambda\,, \quad n \geq 2\,,$$

$$\lambda_n^{\sigma} = \frac{R}{6}\left(n^2+2n-\frac{6(2-\alpha)}{4-\alpha}\right) - \frac{6\alpha\Lambda}{4-\alpha}\,, \quad n \geq 2\,,$$

$$\lambda_n^{h} = \frac{R}{6}\left(n^2+2n-\frac{6}{4-\alpha}\right) - \frac{6\Lambda}{4-\alpha}\,, \quad n \geq 0$$

$$m_n^{T+} = m_n^{T-} = n^2+2n-3\,,$$

$$m_n^{\xi} = m_n^{V} = 2(n^2+2n)\,,$$

$$m_n^{\sigma} = m_n^{h} = m_n^{S} = n^2+2n+1$$

Cutoff

$$\mathcal{O} = Z \begin{pmatrix} \Delta_2 & & & \\ & c_1 \Delta_1 & & \\ & & c_\sigma \Delta_\sigma & \\ & & & c_h \Delta_h \end{pmatrix}$$

$$\mathcal{R}_k = Z \begin{pmatrix} R_k(\Delta_2) & & & \\ & c_1 R_k(\Delta_1) & & \\ & & c_\sigma R_k(\Delta_\sigma) & \\ & & & c_h R_k(\Delta_h) \end{pmatrix}$$

Collecting

$$\mathrm{Tr}(Zc_i(\Delta_i + R_k(\Delta_i)))^{-1} \frac{d}{dt} (Zc_i R_k(\Delta_i)) = \mathrm{Tr}(W)$$

$$W(x) = \frac{\partial_t R_k(x)}{x + R_k(x)}$$

$$k\frac{d\Gamma_k}{dk} = \frac{1}{2}\left[\mathrm{Tr}_2 W(\Delta_2) + \mathrm{Tr}_1 W(\Delta_1) + \mathrm{Tr}_0 W(\Delta_\sigma) + \mathrm{Tr}_0 W(\Delta_h)\right] - \left[\mathrm{Tr}_1 W(\Delta_V) + \mathrm{Tr}_0 W(\Delta_S)\right]$$

Choice of R_k

$$R_k(x) = (k^2 - x)\theta(k^2 - x)\,; \qquad W(x) = 2\theta(k^2 - x)$$

$$\begin{aligned} k\frac{d\Gamma_k}{dk} &= \sum_{\pm}\sum_n m_n^{T\pm}\theta(1 - |\tilde{\lambda}_n^{T\pm}|) + \sum_n m_n^{\xi}\theta(1 - \tilde{\lambda}_n^{\xi}) \\ &+ \sum_n m_n^{\sigma}\theta(1 - \tilde{\lambda}_n^{\sigma}) + \sum_n m_n^{h}\theta(1 - \tilde{\lambda}_n^{h}) \\ &-2\sum_n m_n^{V}\,\theta(1 - \tilde{\lambda}_n^{V}) - 2\sum_n m_n^{S}\,\theta(1 - \tilde{\lambda}_n^{S}) \end{aligned}$$

(here $\tilde{\lambda} = \lambda/k^2$)

Euler-Maclaurin formula

$$\sum_{n=n_0}^{\infty} F(n) = \int_{n_0}^{\infty} dx\, F(x) + \frac{1}{2}F(n_0) - \frac{B_2}{2!}F'(n_0) - \frac{B_4}{4!}F'''(n_0) + R$$

Perform integrals

$$\int_{n_0}^{\infty} dx\, m(x)\theta(1-\tilde{\lambda}(x)) = \int_{n_0}^{n_{\max}} dx\, m(x)$$

where $\lambda_{n_{\max}} = k^2$ or $\tilde{\lambda}_{n_{\max}} = 1$

What is $n_{\max}$?

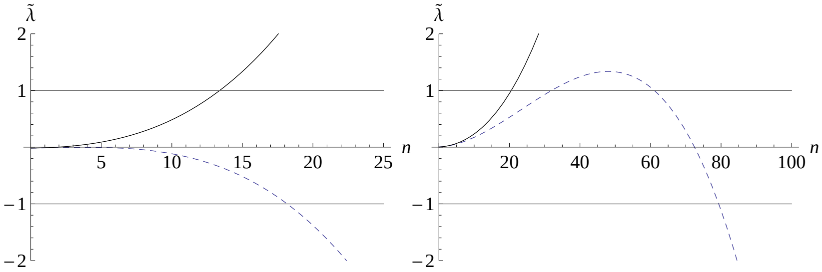


Figure: $\tilde{\lambda}_n^{T+}$ (solid curve) and $\tilde{\lambda}_n^{T-}$ (dashed curve) as functions of n, for $\tilde{R} = \tilde{\Lambda} = 0.01$. Right: large $\tilde{\mu}$ regime (here $\tilde{\mu} = 3$). Left: small $\tilde{\mu}$ regime (here $\tilde{\mu} = 0.3$).

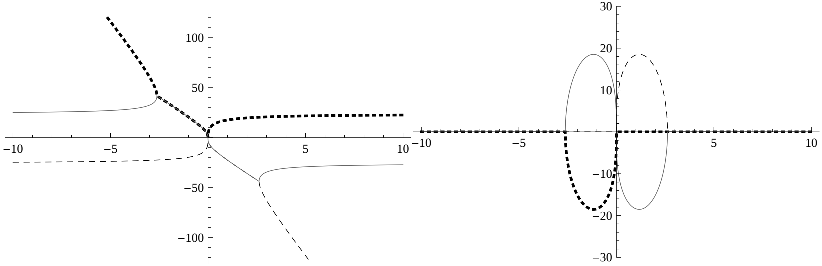


Figure: The real (left) and imaginary (right) parts of the roots of the equation $\tilde{\lambda}_n^{T+} = 1$, for $\tilde{R} = \tilde{\Lambda} = 0.01$, as functions of $\tilde{\mu}$. The solutions of the equation $\tilde{\lambda}_n^{T-} = 1$ are obtained by the reflection $\tilde{\mu} \to -\tilde{\mu}$.

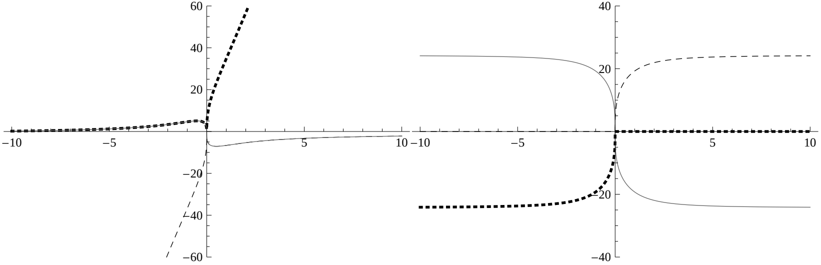


Figure: The real (left) and imaginary (right) parts of the roots of the equation $\tilde{\lambda}_n^{T-} = -1$, for $\tilde{R} = \tilde{\Lambda} = 0.01$, as functions of $\tilde{\mu}$. The solutions of the equation $\tilde{\lambda}_n^{T+} = -1$ are obtained by the reflection $\tilde{\mu} \to -\tilde{\mu}$.

Euler-Maclaurin gives

$$k\frac{d\Gamma_k}{dk} = \sum \left[C_0 R^{-3/2} + C_2 R^{-1/2} + C_{3/2} + \frac{1}{2}F(n_0) - \frac{B_2}{2!}F'(n_0) \right]$$

compare with

$$V(S^3) = 2\pi^2 \left(\tfrac{6}{R}\right)^{3/2} ; \; \int \mathrm{tr}(\omega d\omega + \tfrac{2}{3}\omega^3) = 32\pi^2$$

$$\begin{aligned} \Gamma_k &= V(S^3)\left(\frac{2\Lambda}{16\pi G} - \frac{1}{16\pi G}R + \frac{1}{12\sqrt{6}\pi G\mu}R^{3/2} + \dots\right) \\ &= \frac{3\sqrt{6}\pi}{4}\left(\frac{2\Lambda}{16\pi G}\frac{k^3}{R^{3/2}} - \frac{1}{16\pi G}\frac{k}{R^{1/2}} + \frac{1}{12\sqrt{6}\pi G\mu} + \dots\right) \end{aligned}$$

R-independent terms

	n_0	$C_{3/2}$	$F(n_0)$	$F'(n_0)$
$h^T_{\mu\nu}$	2	12	20	24
$\hat{\xi}^\mu$	2	-24	32	24
$\hat{\sigma}$	2	-18	18	12
h	0	$-\frac{2}{3}$	2	4
C^μ	1	$-\frac{8}{3}$	12	16
$\hat{C}$	1	$-\frac{16}{3}$	8	8

$$\sum \left[C_{3/2} + \frac{1}{2} F(n_0) - \frac{B_2}{2!} F'(n_0) \right] = 0 \rightarrow \beta_\nu = 0$$

Measure couplings in units of cutoff

$$G = \tilde{G} k^{-1}\,, \qquad \Lambda = \tilde{\Lambda} k^2\,, \qquad \mu = \tilde{\mu} k$$

Beta functions

$$\frac{1}{8\pi\tilde{G}}\left(k\frac{d\tilde{\Lambda}}{dk} - \frac{\tilde{\Lambda}}{\tilde{G}}k\frac{d\tilde{G}}{dk}\right) = -\frac{3\tilde{\Lambda}}{8\pi\tilde{G}} + \frac{A}{16\pi}$$

$$\frac{1}{16\pi\tilde{G}^2}k\frac{d\tilde{G}}{dk} = \frac{1}{16\pi\tilde{G}} + \frac{B}{16\pi}$$

$$\frac{1}{12\sqrt{6}\pi\tilde{\mu}\tilde{G}}\left(\frac{1}{\tilde{G}}k\frac{d\tilde{G}}{dk} + \frac{1}{\tilde{\mu}}k\frac{d\tilde{\mu}}{dk}\right) = 0$$

Beta functions of $\tilde{G}$ and $\tilde{\Lambda}$

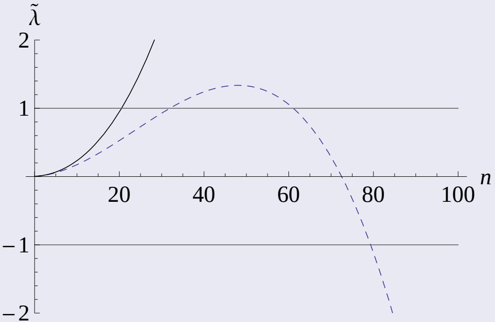
$$\begin{aligned} \beta_{\tilde{G}} &= \tilde{G} + B(\tilde{\mu})\tilde{G}^2 \ , \\ \beta_{\tilde{\Lambda}} &= -2\tilde{\Lambda} + \frac{1}{2}\tilde{G}\left(A(\tilde{\mu},\tilde{\Lambda}) + 2B(\tilde{\mu})\tilde{\Lambda}\right) \end{aligned}$$

Since $\nu = \mu G = \tilde{\mu}\tilde{G}$ is constant

can replace $\tilde{\mu}$ by $\nu/\tilde{G}$

Ascending root cutoff

For $\tilde{\mu} > \sqrt{\frac{27}{4}}$



Beta function coefficients

$$A(\tilde{\Lambda},\tilde{\mu}) = -\frac{16}{3\pi} + \frac{9(2\sqrt{3}\cos 2\theta - \sqrt{3}\cos 4\theta + 8(\cos\theta)^3\sin\theta)}{\pi(\cos 3\theta)^3} + \frac{8(3+11\alpha-2\alpha^2)}{\pi(4-\alpha)}\tilde{\Lambda} + \frac{48(\cos\theta - \sqrt{3}\sin\theta)}{\pi\sin 6\theta}\tilde{\Lambda}\,,$$

$$B(\tilde{\mu}) = -\frac{4(1+\alpha)(11-2\alpha)}{\pi(4-\alpha)} - \frac{2(\sqrt{3}\sin\theta - \cos\theta) + 22(\sqrt{3}\sin 5\theta + \cos 5\theta))}{3\pi\sin 6\theta}\,,$$

$$\theta = \frac{1}{3}\arctan\sqrt{\frac{4\tilde{\mu}^2}{27} - 1}$$

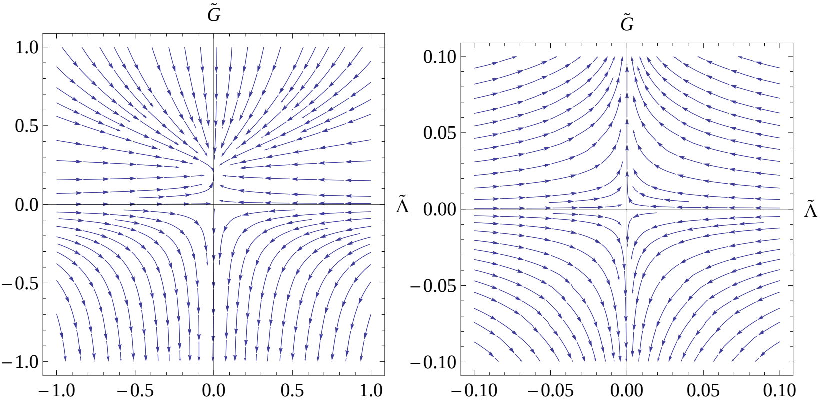


Figure: The flow in the $\tilde{\Lambda}$-$\tilde{G}$ plane for $\alpha = 0$, $\nu = 5$. Right: enlargement of the region around the origin, showing the Gaussian FP. The beta functions become singular at $|\tilde{G}| = 1.9245$.

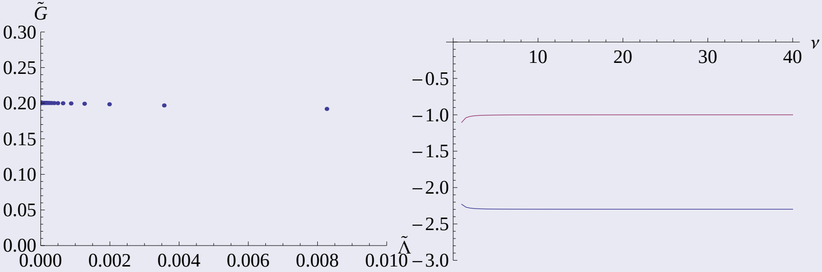
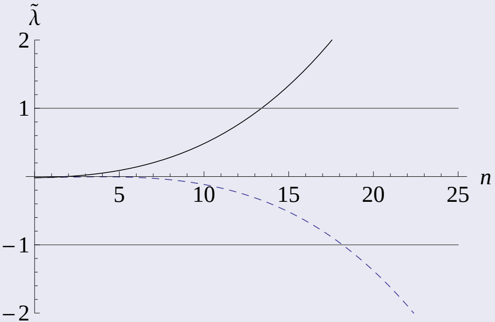


Figure: Position of the FP (left) and eigenvalues of the stability matrix (right) for the nontrivial FP with $\alpha = 0$, $1 < \nu < 40$. Note that for this range of ν the singularity is always above the FP. In the left panel, ν grows from right to left. Note that $\tilde{\Lambda}_* > 0$ in this scheme. For large ν, $\tilde{G}_*$ tends to 0.2005 and the eigenvalues tend to -1 and -2.298.

Descending root cutoff

For $\tilde{\mu} < \sqrt{\frac{27}{4}}$



Descending root cutoff

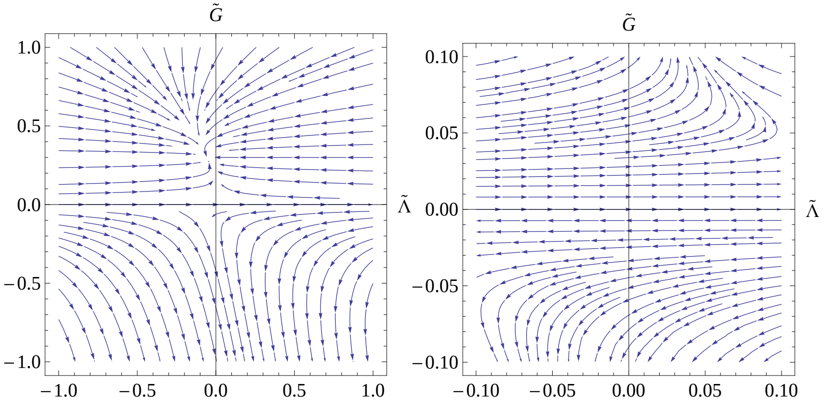


Figure: The flow in the $\tilde{\Lambda}$-$\tilde{G}$ plane for $\alpha = 0$, $\nu = 0.1$. Right: enlargement of the region around the origin, showing that there is no Gaussian FP. The beta functions diverge on the $\tilde{\Lambda}$ axis.

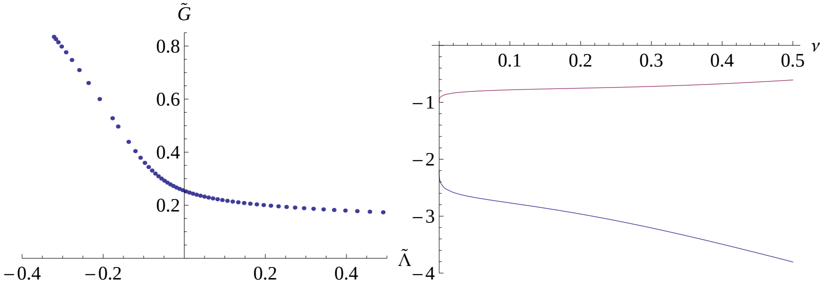
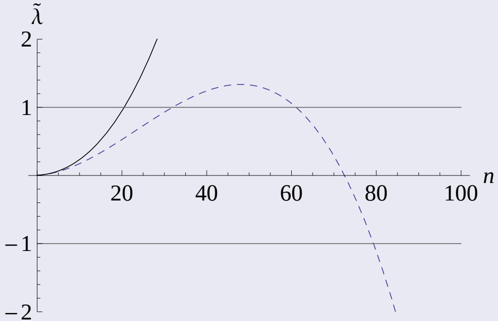


Figure: Position of the FP (panel) and eigenvalues of the stability matrix (panel) for the nontrivial FP with $\alpha = 0$, $10^{-6} < \nu < 0.5$, in the descending root cutoff scheme. In the left panel, ν decreases from right to left. $\tilde{\Lambda}_*$ changes sign for $\nu = 0.18$. The rightmost point ($\nu = 0.5$) has $\tilde{\mu} \approx 3 > \sqrt{27/4}$.

Spectrally balanced cutoff

For any $\tilde{\mu}$ choose same $n_{\max}$ for λ^{TT+} and λ^{TT-}.



Spectrally balanced cutoff

$$A(\tilde{\mu}, \tilde{\Lambda}) = -\frac{16}{3\pi} + \frac{2\sqrt{3}}{\pi(\cosh\eta)^3} + \frac{8(3 + 11\alpha - 2\alpha^2)}{\pi(4-\alpha)}\tilde{\Lambda} + \frac{16\sqrt{3}}{\pi(\cosh 3\eta + 2\cosh\eta)}\tilde{\Lambda}$$

$$B(\tilde{\mu}) = -\frac{4(11 + 9\alpha - 2\alpha^2)}{3\pi(4-\alpha)} - \frac{8\sqrt{3}}{9\pi}\left(\frac{8 + 11\cosh 2\eta}{\cosh 3\eta + 2\cosh\eta}\right)$$

$$\eta = \frac{1}{3}\,\mathrm{arctanh}\sqrt{1 - \frac{4\tilde{\mu}^2}{27}}$$

Spectrally balanced cutoff

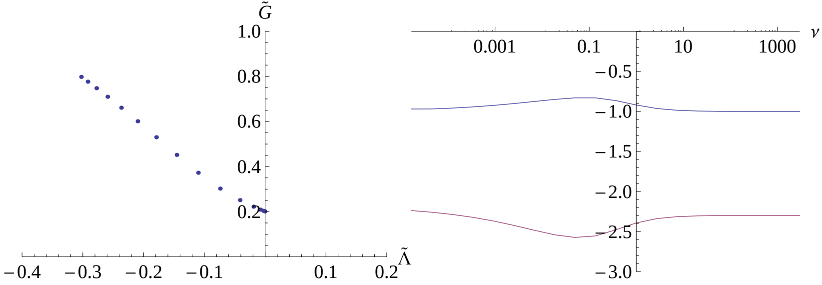


Figure: Left: The position of the FP in the $\tilde{\Lambda}$-$\tilde{G}$ plane, varying ν from 0.002 (upper left) to 1000 (lower right). The point with coordinates (0,0.2005) is the limit $\nu \to \infty$. Right: The eigenvalues of M as functions of ν. For $\nu = 0.002$ they are -1 and -2.298 while for $\nu = 1000$ they are -0.969 and -2.238.

General features

- Simple form of R_k implies spectrally unbalanced cutoff. No choice of roots is good for all $\tilde{\mu}$: ascending root good for large $\tilde{\mu}$ (small $\tilde{G}$), descending root for small $\tilde{\mu}$ (large $\tilde{G}$)
- Spectrally balanced cutoff good for all $\tilde{\mu}$
- Qualitative picture consistent
- Agreement with heat kernel calculation for $\mu \to \infty$

Scheme independent features

- $\beta_\nu = 0$ expected for topological reasons
- GFP with crit exp 1 and -2
- NGFP with crit exp ~ -1 and ~ -2
- $\tilde{G}_*$ is positive

Scheme dependent features

- position of NGFP
- in particular, there is residual uncertainty on sign of cosmological constant at NGFP for large $\tilde{\mu}$
- scaling exponents at NGFP (slightly)

Gauge dependence

EOM gives $R = 6\Lambda$, but we did not use this. Off shell EA gauge dependent, so off shell beta functions are gauge dependent. On shell Hilbert action is proportional to

$$V(S^3)\frac{\Lambda}{16\pi G} \sim \frac{1}{\sqrt{\Lambda}G} \sim \frac{1}{\sqrt{\tau}}$$

so expect β_τ to be gauge independent. Indeed

$$\beta_\tau \sim A + 6B\tilde{\Lambda} = -\frac{16}{3\pi}(1 + 3\tilde{\Lambda}) + \mu\text{-dependent terms}$$

Asymptotic safety

In $d \geq 4$

- large value of $\tilde{G}$
- consistency of truncation

For large μ, TMG is an example of asymptotically safe theory which can be studied perturbatively

Since Riemann can be expressed in terms of Ricci, higher derivative terms can be eliminated order by order in perturbation theory