Renormalization of microscopic Hamiltonians

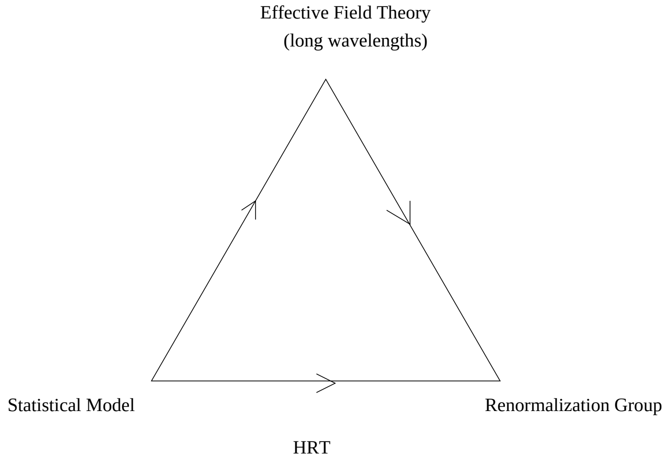
Renormalization Group without Field Theory

Alberto Parola

Università dell'Insubria (Como - Italy)

Renormalization Group ⇒ Universality

Only dimensionality and symmetry matter



Hierarchical Reference Theory

- Keeps track of higher order operators
- Provides information on non universal properties
  (critical temperature, crossovers etc.)

A.P. & L.Reatto: PRL, **53**, 2417 (1984); PRA **31**, 3309 (1985);
Adv. Phys. **44**, 211, (1995).

Outline

- Wilson's momentum-shell integration RG for microscopic models
- Sharp & Smooth cut-off formulations
- Approximate non perturbative closures
- Relation to the Local Potential Approximation
- First order transition and the convexity requirement
- Extension to Quantum Hamiltonians
- Open problems

HRT vs. RG: A short story

(scalar order parameter)

Φ^4 Field Theory

$$S = \int d\mathbf{r} \left[\frac{1}{2} |\nabla \Phi|^2 + \frac{r}{2} \Phi^2 + \frac{u}{4!} \Phi^4 \right]$$

$$Z = \int \mathcal{D}\Phi(\mathbf{r})\, e^{-S}$$

Perturbation theory in u

Statistical Model

$$H = \frac{1}{2} \sum_{i \neq j} v(\mathbf{r}_i - \mathbf{r}_j)$$

$$Z = \int d\mathbf{r}_1 \cdots d\mathbf{r}_N\, e^{-\beta H}$$

Perturbation theory in β

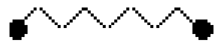
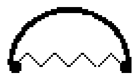
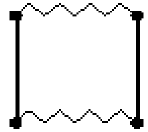
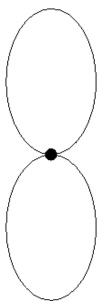
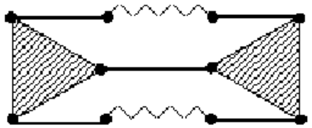
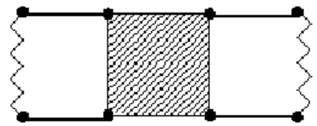
Formal Perturbation Theory

- Split the potential into a Hard Sphere part and an Attractive tail

$$v(r) = v_{HS}(r) + w(r)$$

- Expand the free energy ($\ln Z$) in powers of $w(r)$

- Order the diagrams according to the number of loops

	Φ^4 Field Theory	Statistical Model
Zero loops	$\ln Z_0 = -\frac{V}{2}\int \frac{d\mathbf{q}}{(2\pi)^D} \ln\left[\frac{r+q^2}{2\pi}\right]$	$\ln Z_{MF} = \ln Z_{HS} + \frac{1}{2}$ 
One loop		$\frac{1}{2}$  $+ \frac{1}{4}$  $+ \cdots\cdots$ $= -\frac{V}{2}\int \frac{d\mathbf{q}}{(2\pi)^D} \ln\left[1+\beta\rho\, S_{HS}(q)\tilde{w}(q)\right]$
Two loops	$\frac{1}{8}$ 	$\frac{1}{4}$  $+ \frac{1}{8}$  $+ \cdots\cdots$

Correspondence Field Theory $\Leftrightarrow$ Statistical Model

Propagator: $$\frac{1}{r+q^2} \quad \Longleftrightarrow \quad [-\beta\tilde{w}(q)]\,\frac{[\rho S_{HS}(q)]^2}{1+\beta\rho S_{HS}(q)\tilde{w}(q)}$$

Vertex: $$u_n(\mathbf{r}_1,\cdots\mathbf{r}_n) \quad \Longleftrightarrow \quad c_n(\mathbf{r}_1,\cdots,\mathbf{r}_n) \equiv \frac{\delta^n \ln Z_{HS}[\rho(\mathbf{r})]}{\delta\rho(\mathbf{r}_1)\cdots\delta\rho(\mathbf{r}_n)}$$

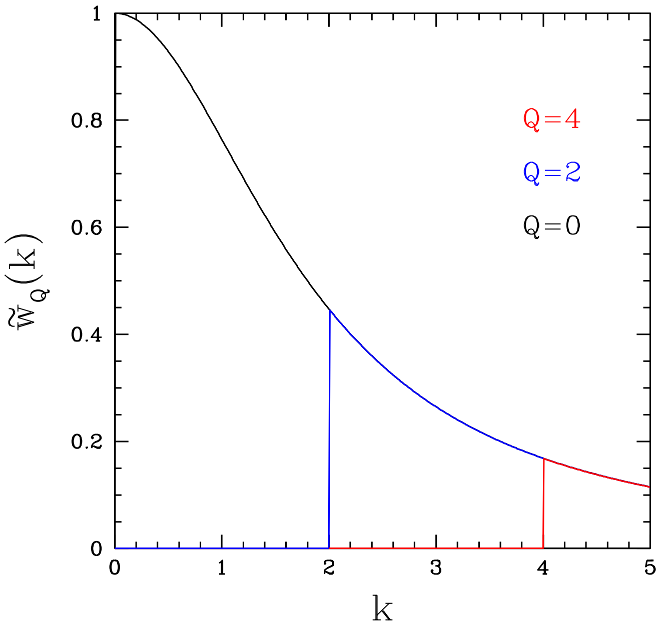
Momentum Shell Integration RG

Cut-off on the propagator $\Longleftrightarrow$ Cut-off on the interaction $\tilde{w}(q)$

Sharp cut-off implementation

Sequence of intermediate Q-systems

$$\tilde{w}_Q(k) = \begin{cases} \tilde{w}(k) & \text{for } k > Q \\ 0 & \text{for } k < Q \end{cases}$$



$Q : \infty \to 0$

$$\lim_{Q\to\infty} \tilde{w}_Q(k) = 0$$

$$\lim_{Q\to 0} \tilde{w}_Q(k) = \tilde{w}(k)$$

The cut-off Q limits the range of density fluctuations included in the Q-system

Liquid-vapor transition inhibited at every $Q \neq 0$

Evolution of the free energy with Q

$$\frac{d\mathcal{A}_Q}{dQ} = -\frac{d}{2}\Omega_d\, Q^{d-1} \ln\left[1 - F_Q(Q)\,\beta\tilde{w}(Q)\right]$$
$$= -\frac{d}{2}\Omega_d\, Q^{d-1} \ln\left[1 + \frac{\beta\tilde{w}(Q)}{\mathcal{C}_Q(Q)}\right]$$

- $-kT\,\mathcal{A}_Q =$ Free energy density of the Q-system
  $+$ Mean field contribution $[\,\Gamma_k\,]$
- $\mathcal{C}_Q(k) =$ Direct correlation function of the Q-system $\left[\Gamma_k^{(2)}(q)\right]$
  $+$ Mean field contribution $\Longrightarrow$ $\mathcal{C}_Q(\mathbf{r}_1 - \mathbf{r}_2) = \dfrac{\delta^2 \mathcal{A}_Q}{\delta\rho(\mathbf{r}_1)\delta\rho(\mathbf{r}_2)}$
- $\Omega_d =$ volume of the unit sphere in d-dimension

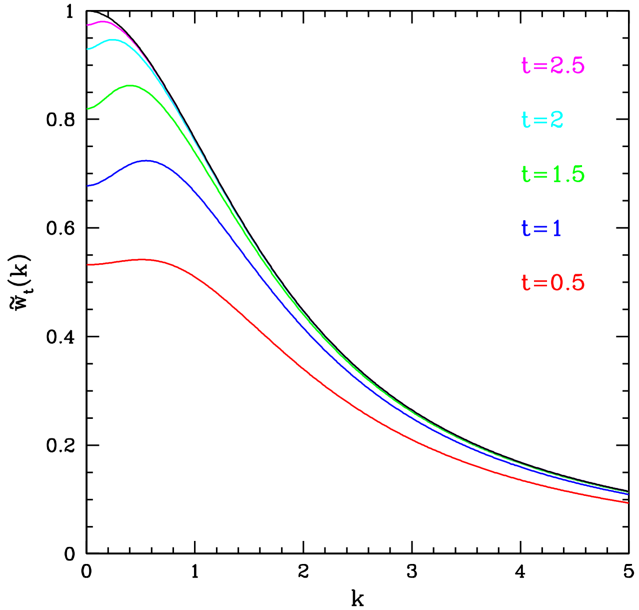
Exact hierarchy of differential equations for $\mathcal{C}_Q$ and $c_n{}^Q$ $(n = 3, \cdots \infty)$

Smooth cut-off implementation

Sequence of intermediate t-systems

$$w_t(r) = w(r) - e^{-(d+2-\eta)t}\, w(r/e^t)$$

$$\tilde{w}_t(k) = \tilde{w}(k) - e^{-(2-\eta)t}\, \tilde{w}(k\, e^t)$$



$t : 0 \to \infty$

$Q \sim e^{-t}$

$$\lim_{t\to 0} \tilde{w}_t(k) = 0$$

$$\lim_{t\to \infty} \tilde{w}_t(k) = \tilde{w}(k)$$

Phase transitions are suppressed at all finite t's

A.P: J. Phys. C **26**, 5071 (1986)

A.P., D.Pini and L.Reatto: PRL **100**, 165704 (2008)

Evolution of the free energy with t

$$\frac{d\mathcal{A}_t}{dt} = \frac{\beta}{2}\int \frac{d^d k}{(2\pi)^d}\,\frac{d\tilde{w}_t(k)}{dt}\,\left[\mathcal{C}_t(k) + e^{-(2-\eta)t}\,\beta\tilde{w}(k\,e^t)\right]^{-1}$$

- $-kT\,\mathcal{A}_t =$ Free energy density of the t-system

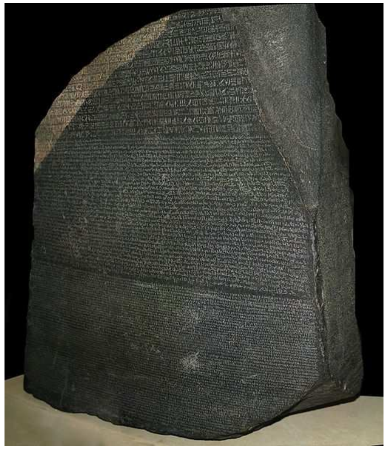
  $+$ Mean field contribution $\quad [\,\Gamma_k\,]$

- $\mathcal{C}_t(k) =$ Direct correlation function of the t-system $\quad \left[\Gamma_k^{(2)}(q)\right]$

  $+$ Mean field contribution $\quad \Longrightarrow \quad \mathcal{C}_t(\mathbf{r}_1 - \mathbf{r}_2) = \dfrac{\delta^2 \mathcal{A}_t}{\delta\rho(\mathbf{r}_1)\delta\rho(\mathbf{r}_2)}$

Exact hierarchy of differential equations for $\mathcal{C}_t$ and ${c_n}^t$ $(n = 3, \cdots \infty)$

The Dictionary



NPRG	Smooth cut-off HRT
k	$Q = e^{-t}$
ϕ	ρ
$R_k(q)$	$-e^{-(2-\eta)t}\beta w(q\, e^t)$
$\Gamma_k[\phi]$	$-\mathcal{A}_t(\rho)$
$\Gamma_k^{(2)}(q)$	$-\mathcal{C}_t(q)$
$\Gamma_k^{(n)}(\mathbf{q}_1 \cdots \mathbf{q}_n)$	$-c_t^n(\mathbf{q}_1 \cdots \mathbf{q}_n)$

Highlights of the exact HRT equations

- For $Q \to 0$ and in the critical region the HRT equations simplify and, through a simple rescaling, reduce to the standard RG hierarchy of equations for a scalar field theory.

- The HRT smooth cut-off prescription depends on the interparticle potential $w(r)$ and corresponds to the smooth cut-off RG with a suitable choice of smearing function

- The full HRT equations retain information on non-universal properties and short range correlations (i.e. the full UV behavior of the statistical model is preserved in the RG procedure)

- The HRT strategy can be trivially generalized to $O(n)$ spin models on a lattice

An approximate closure

$$\mathcal{C}_t(k) = c_{HS}(k) - \lambda_t(\rho)\,\beta\tilde{w}(k)$$

$\lambda(\rho) = 1 \Longrightarrow$ Mean Field

$\lambda_t(\rho)$ defined by the compressibility sum rule $\Rightarrow$ $\mathcal{C}_t(k=0) = \frac{\partial^2 \mathcal{A}_t}{\partial \rho^2}$

Local Potential Approximation at long wavelengths

$$\mathcal{C}_t(k) \underset{k\sim 0}{\longrightarrow} \frac{\partial^2 \mathcal{A}_t}{\partial \rho^2} - b\,k^2$$

- Closed (approximate) partial differential equation for the thermodynamics of the t-system
- To be solved with initial condition $\mathcal{A}_t = -\beta A_{MF}/V$ for $t = 0$
- In the $t \to \infty$ limit $\mathcal{A}_t \to$ Physical free energy density of the fully interacting model

Critical Properties

Formal structure of the HRT evolution equation at large t

$$\frac{d\mathcal{A}_t}{dt} = e^{-dt}\Phi\left(e^{2t}\frac{\partial^2\mathcal{A}_t}{\partial\rho^2}\right)$$

where $\Phi(x)$ is a non-linear function
depending on the choice of sharp/smooth cut-off

- Rescaling: $z = (\rho - \rho_c)\, e^{\frac{d-2}{2}t}$

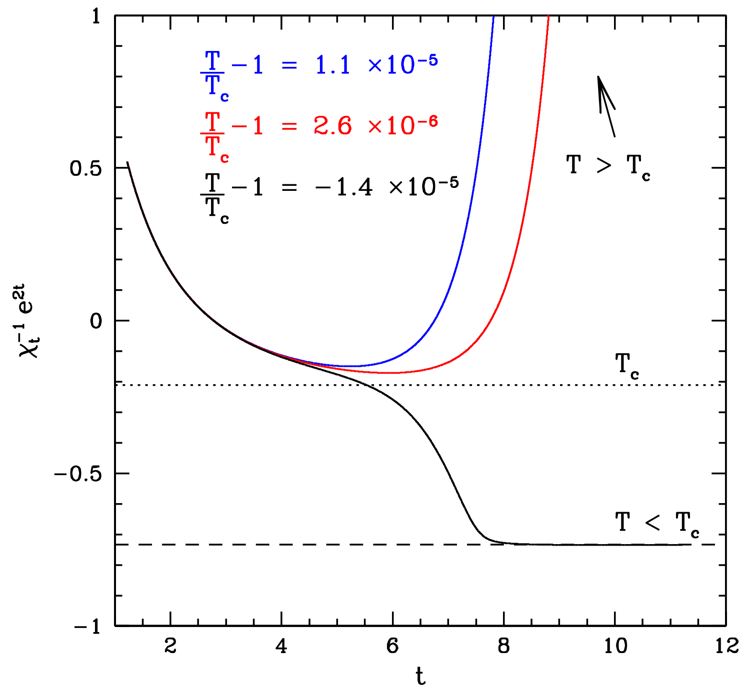
$$H_t(z) = e^{dt}\left[\mathcal{A}_t(\rho) - \mathcal{A}_t(\rho_c)\right]$$

- Fixed point equation: the standard RG structure in LPA is recovered

$$\frac{d-2}{2}\, z\, H'_*(z) - d\, H_*(z) = \Phi(H''_*(z)) - \Phi(H''_*(0))$$

HRT flow of the inverse susceptibility

$$\chi_t^{-1} = -\left.\frac{\partial^2 \mathcal{A}_t(\rho)}{\partial \rho^2}\right|_{\rho_c}$$



- Approach to the fixed point value in the critical region
- Flow to the low temperature fixed point in the two-phase region

Critical exponents and amplitudes

$$\chi^{-1} = -\left.\frac{\partial^2 \mathcal{A}_\infty(\rho)}{\partial \rho^2}\right|_{\rho_c} \sim \frac{1}{C_\pm}\left|\frac{T-T_c}{T_c}\right|^{\gamma}$$

$$|\rho_\times - \rho_c| \propto (T_c - T)^{\beta}$$

ϵ-expansion

$$\gamma = 1 + \frac{\epsilon}{6} + \cdots$$

$$\beta = \frac{1}{2} - \frac{\epsilon}{6} + \cdots$$

$d = 3$

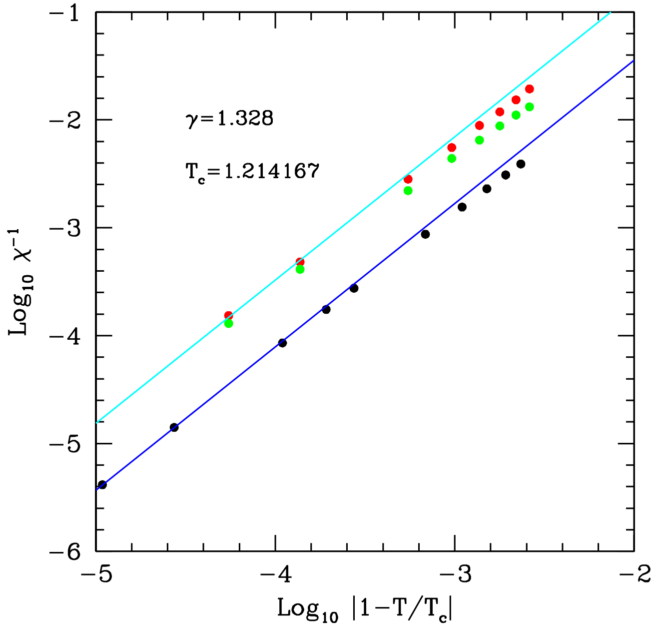
	HRT sharp	HRT smooth	Exact
γ	1.378	1.328	1.237
β	0.345	0.332	0.326
η	0	0	0.036
$U_2 = C_+/C_-$	−	4.16	4.76

PRL, **53**, 2417 (1984); PRL **100**, 165704 (2008)

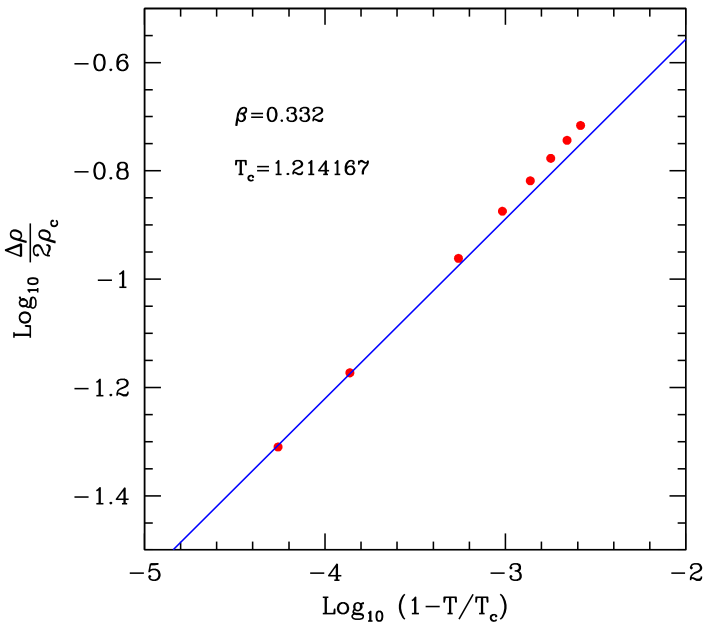
Numerical integration of the full HRT equation

Smooth cut-off

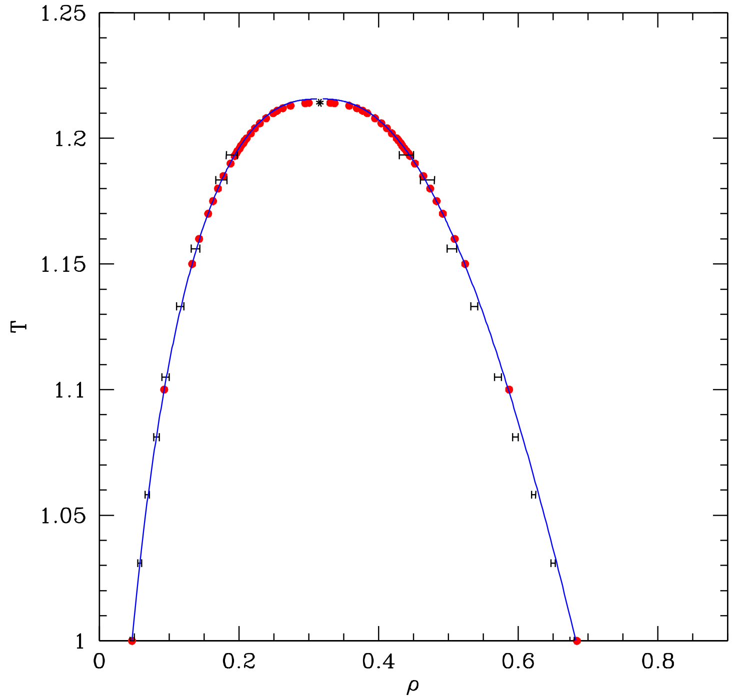
Inverse susceptibility



Coexistence curve

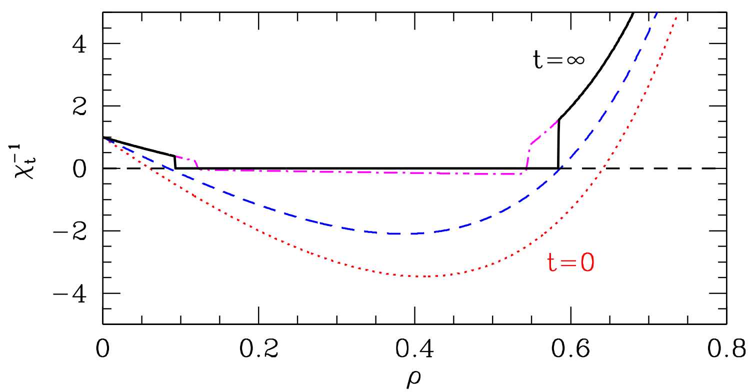


Phase diagram of a fluid



	HRT-smooth	MC
T_c	1.2142	1.212(2)
ρ_c	0.3157	0.312(2)

First order transition



- Fluctuations restore the convexity of the free energy

- For $t \to \infty$:

  In one-phase regions the susceptibility is always positive

  At coexistence the susceptibility identically vanishes

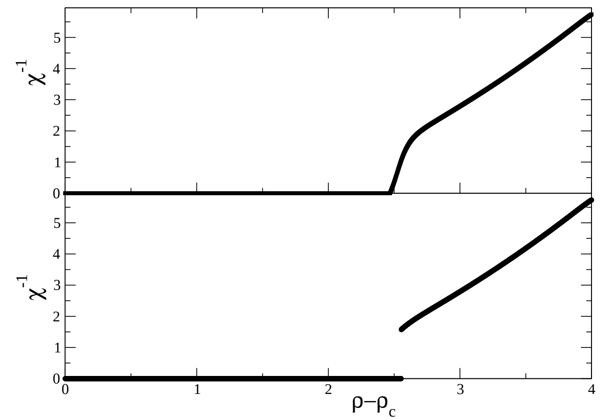
Phys. Rev. E **48**, 3321 (1993); Phys. Rev. E **76**, 031113 (2007)

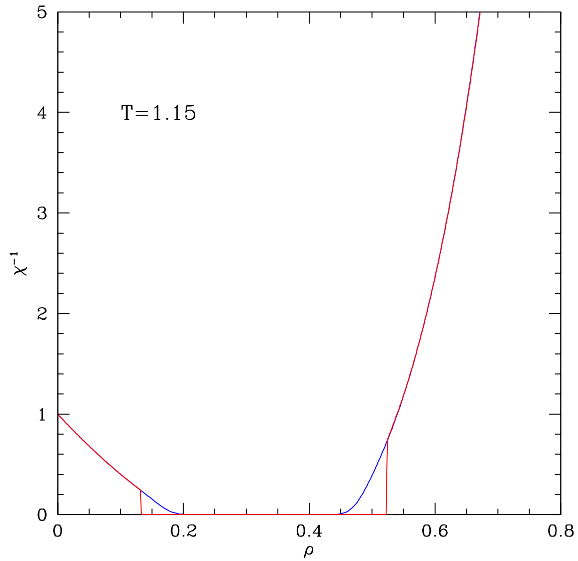
First order transition

Sharp vs Smooth cut-off

Sharp cut-off

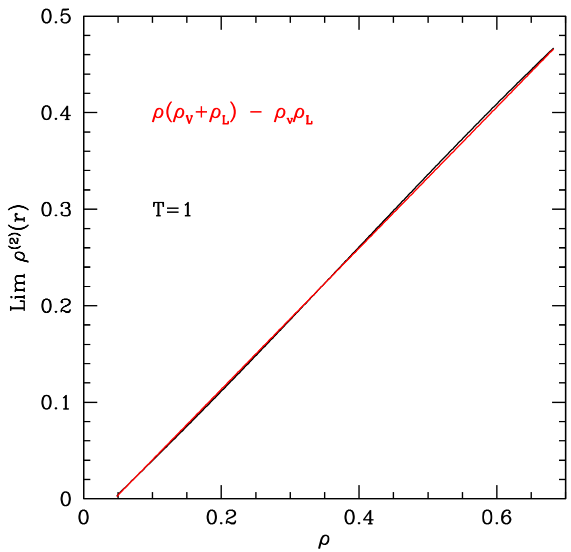
Smooth cut-off





At coexistence

$$\lim_{r\to\infty} \rho^2 g(r) = \rho(\rho_v + \rho_\ell) - \rho_v \rho_\ell$$



Quantum HRT

- General interacting quantum system of bosons/fermions/spins

$$\hat{H} = \hat{H}_R + \hat{V} = \hat{H}_R + \frac{1}{2}\int d\mathbf{x}\, d\mathbf{y}\, \hat{\rho}(\mathbf{x}) w(\mathbf{x}-\mathbf{y}) \hat{\rho}(\mathbf{y})$$

- Order parameter $< \hat{\rho}(\mathbf{x}) > \Longrightarrow$ Perturbative expansion in $w(\mathbf{x})$

$$\frac{\Xi}{\Xi_R} = \frac{\mathrm{Tr}\, e^{-\beta \hat{H}}}{\mathrm{Tr}\, e^{-\beta \hat{H}_R}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\, n!} \int ds_1\, ds'_1 ... ds_n\, ds'_n\, \rho_R(s_1, ...s'_n)\, \phi(s_1, s'_1) ... \phi(s_n, s'_n)$$

$$\phi(s, s') = w(\mathbf{x}-\mathbf{x}')\delta(\tau - \tau') \qquad \tau \in (0, \beta) \qquad s = (\mathbf{x}, \tau)$$

$$\rho_R(\mathbf{x}_1, \tau_1; ...\mathbf{x}_n, \tau_n) = \frac{1}{\Xi_R[J]} \left. \frac{\delta^n \Xi_R[J]}{\delta J(\mathbf{x}_1, \tau_1) ... \delta J(\mathbf{x}_n, \tau_n)} \right|_{J=0}$$

A.P & P.Gianinetti PRB **63**, 104414 (2001)

Formal analogy with a classical model in $d+1$ dimension

- Sharp cut-off on the Fourier components of $w(\mathbf{x})$

- Exact evolution equation for the Helmholtz free energy density

$$\frac{d\mathcal{A}_Q}{dQ} = -Q^{d-1}\,\frac{d}{2}\,\Omega_d \sum_{\omega_n} \ln[1 - F_Q(Q,\omega_n)\tilde{w}(Q)]$$

- Connected two point function in imaginary time:

$$F(\mathbf{x}_1,\tau_1;\mathbf{x}_2,\tau_2) = \left.\frac{\delta^2 \ln \Xi[J]}{\delta J(\mathbf{x}_1,\tau_1)\delta J(\mathbf{x}_2,\tau_2)}\right|_{J=0}$$

- Matsubara frequencies at finite temperature: $\omega_n = \frac{2\pi}{\beta}\,n$

The antiferromagnetic Heisenberg model

$$\hat{H} = J \sum_{<\mathbf{R},\mathbf{R}'>} \hat{\mathbf{S}}_{\mathbf{R}} \cdot \hat{\mathbf{S}}_{\mathbf{R}'} - h \sum_{\mathbf{R}} e^{i\vec{\pi}\cdot\mathbf{R}} \hat{S}^z_{\mathbf{R}}$$

- Order parameter: staggered magnetization $< \hat{S}^z_{\mathbf{R}} >= m\, e^{i\vec{\pi}\cdot\mathbf{R}}$

Parametrization of the two point functions

$$(\alpha_{\perp Q}, \alpha_{||Q})$$

$$F_Q^{xx}(\mathbf{k},\omega) = \frac{\alpha_{\perp Q} - \tilde{w}(\mathbf{k})}{m^{-2}\omega^2 + \alpha_{\perp Q}{}^2 - \tilde{w}(\mathbf{k})^2}$$

$$F_Q^{xy}(\mathbf{k},\omega) = \frac{m^{-1}\omega}{m^{-2}\omega^2 + \alpha_{\perp Q}{}^2 - \tilde{w}(\mathbf{k})^2}$$

$$F_Q^{zz}(\mathbf{k},\omega) = \frac{\delta_{\omega,0}}{\alpha_{||Q} + \tilde{w}(\mathbf{k})}$$

Physical response functions

Single mode approximation

$$\text{Im}\, S^{xx}(\mathbf{k},\omega) = \frac{\pi\omega}{1-\exp(-\beta\omega)} \frac{\delta(\omega-\epsilon_k)+\delta(\omega+\epsilon_k)}{\alpha_\perp + \tilde{w}(\mathbf{k})}$$

$$\text{Im}\, S^{zz}(\mathbf{k},\omega) = \frac{2\pi}{\beta} \frac{\delta(\omega)}{\alpha_{||} + \tilde{w}(\mathbf{k})}$$

- Spin wave dispersion: $\epsilon_k = m\sqrt{{\alpha_\perp}^2 - \tilde{w}(\mathbf{k})^2}$

LPA-like closure

$$\chi^s_\perp = \left(\frac{h}{m}\right)^{-1} = -\beta m \left(\frac{\partial \mathcal{A}}{\partial m}\right)^{-1} = F^{xx}(\vec{\pi}, 0) = (\alpha_\perp - 2Jd)^{-1}$$

$$\chi^s_{||} = \left(\frac{\partial h}{\partial m}\right)^{-1} = -\beta \left(\frac{\partial^2 \mathcal{A}}{\partial m^2}\right)^{-1} = F^{zz}(\vec{\pi}, 0) = (\alpha_{||} - 2Jd)^{-1}$$

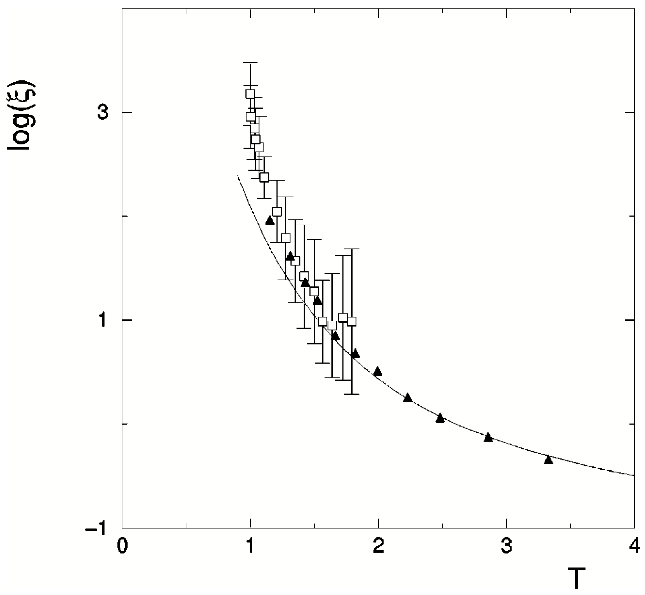
QHRT Equation

$$\frac{d\,\mathcal{A}_Q}{dQ} = -\frac{D_d(Q)}{2}\left\{4\,\ln\left[\frac{\sinh\left(\frac{1}{2}\beta m \alpha_{\perp Q}\right)}{\sinh\left(\frac{1}{2}\beta m\sqrt{\alpha_{\perp Q}{}^2 - 4J^2(d^2 - Q^2)}\right)}\right] + \ln\left[\frac{\alpha_{||Q}{}^2}{\alpha_{||Q}{}^2 - 4J^2(d^2 - Q^2)}\right]\right\}$$

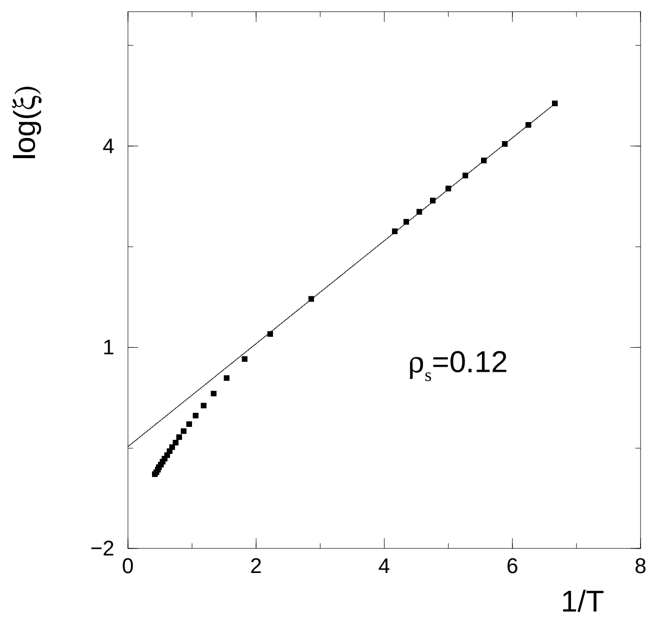
$$Q \in (0, d) \qquad\qquad m \in (0, 1)$$

Numerical results in two dimensions

Spin-1 model



Spin-$\frac{1}{2}$ model



$$\xi(T) \sim \xi_0 \, \mathrm{e}^{\frac{2\pi\bar{\rho}_s}{T}}$$

The Heisenberg model always falls in the renormalized classical regime

Non Linear σ-Model

$$S[\mathbf{\Omega}] = \frac{\rho_s}{2} \int d^d\mathbf{r} \int_0^\beta d\tau \left\{ |\nabla \mathbf{\Omega}|^2 + \frac{1}{c_s^2} \left(\frac{\partial \mathbf{\Omega}}{\partial \tau} \right)^2 \right\} \qquad |\mathbf{\Omega}| = 1$$

One loop RG equations

$$Q\frac{dg}{dQ} = (d-1)g - \frac{K_d}{2} g^2 \coth(g/2T)$$

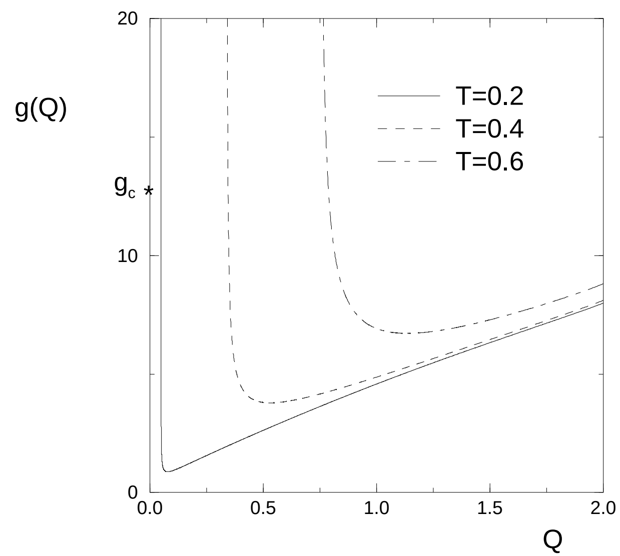
$$\frac{d}{dQ}\left(\frac{g}{T}\right) = \frac{g}{T}$$

Effective coupling constant

$$g = \left(\frac{Q}{\sqrt{d}}\right)^{d-1} \frac{\sqrt{4d}}{m_Q}$$

QHRT evolution of the spontaneous magnetization

$$\frac{dm_Q}{dQ} = K_d \left(\frac{Q}{\sqrt{d}}\right)^{d-2} \coth(\beta Q m_Q)$$



Outlook

Closures

- Beyond LPA: the second equation of the hierarchy
  - η at two loop order
- Better representation of short range physics
  - Repulsive interactions
  - Classical and Quantum Heisenberg models

Order parameters

- Beyond the $O(n)$ model: inhomogeneous systems
- Competing interactions: more complex order parameters
- The Fermi surface in QHRT
- The Coulomb gas (primitive model)

BMW approach

PRL, **53**, 2417 (1984)

VOLUME 53, NUMBER 25 PHYSICAL REVIEW LETTERS 17 DECEMBER 1984

$c(r)=\exp\{-\beta[v_R(r)+w(r)]\}$ for the Ornstein-Zernike direct correlation function. From the free energy Eq. (2) one then finds the correct first virial coefficient. If in the hierarchy we approximate $\tilde{c}_n^Q$ with its ideal-gas value starting from $n=m$ then the truncated system gives the correct low-density expansion for $c(r)$ up to order $m-2$.

In the opposite limit of high density the effect of the many-particle correlations becomes essentially irrelevant. In fact the strong effect of screening of $w(r)$ by the repulsive forces is manifest in (4) because $\tilde{F}(p)$ is very small in the region of small p where $\tilde{\phi}(p)$ is significant. Therefore to lowest order M^Q can be considered as a negligible quantity so that $\tilde{c}^0(k)\approx\tilde{c}^\infty(k)$ but $\tilde{c}^\infty(k)$ corresponds to the ORPA that is known as an excellent approximation for a fluid in the triple-point region.

We discuss now the region of the critical point where $S(0)=\rho^{-1}\tilde{F}^{Q=0}(0)>>1$. Then also $\rho^{-1}\times\tilde{F}^Q(k)$ will be large for small values of k and Q so that the terms unity in the integrands of the hierarchy can be dropped in comparison to $\tilde{\phi}(k)\tilde{F}^Q(k)$. Similarly, terms proportional to $\tilde{F}^Q(k)$ have to be kept only for $k\geqslant Q$, where $\tilde{F}^Q(k)=\tilde{F}^Q(k)$, because for $k<Q$, $\tilde{F}^Q(k)=\tilde{F}^Q(k)/[1+\tilde{\phi}(k)\times\tilde{F}^Q(k)]$ is a bounded quantity. Then in this low-momenta regime the hierarchy simplifies and, for instance, the evolution equation for $\tilde{c}^Q$ can be written in the form

$$\left(\frac{\partial}{\partial \ln Q}+x\frac{\partial}{\partial x}-2+\eta\right)u_2^Q(x)$$
$$=\tfrac{1}{2}(2\pi)^{-d}\int d\Omega_y\{u_4^Q(\vec{x},-\vec{x},\vec{y},-\vec{y})-2[u_3^Q(\vec{x},\vec{y}-\vec{x}-\vec{y})]^2/u_2^Q(\vec{y}+\vec{x})\}/u_2^Q(y), \qquad (7)$$

$\Longleftarrow$ Equation for $\Gamma^{(2)}(k)$

where the momentum integration is over the surface $|\vec{y}|=1$ with the limitation $|y+x|>1$. Here we have introduced the scaled functions

$$u_n^Q(\vec{x}_1,\dots,\vec{x}_n)=-Q^{n(d-2+\eta)/2-d}\check{c}_n^Q(\vec{x}_1Q,\dots,\vec{x}_nQ)\quad\left(\sum_i^n\vec{x}_i=0\right) \qquad (8)$$

for $n>2$ and $u_2^Q(x)$ is defined in the same way in term of $\tilde{c}^Q$. The exponent η is defined as the constant for which $\lim_{Q\to 0}Q^{-2+\eta}[\tilde{c}^Q(xQ)-\tilde{c}^Q(0)]$ is finite at the critical point. The evolution equation (7) for u_2^Q and those for the u_n^Q are equivalent to the RG equations which can be deduced from the theory of Nicoll and Chang.[5] This can be shown by recasting our approximate hierarchy in the form of a differential generator for the free energy $\tilde{A}$ of an inhomogeneous system. In our case this generator involves the second functional derivative of $\beta\tilde{A}$ with respect to the local density, this being equal to $\tilde{c}$, in place of the local magnetization as in the case of Nicoll and Chang. The characteristic momentum Q corresponds to the momentum shell of integration in the RG. Thus the existence of a fixed point for our approximate hierarchy implies a scaling form for the correlation functions in the critical region.[6] The critical behavior given by our approximate hierarchy can be analyzed in the framework of the $\epsilon=4-d$ expansion and in fact, because of the equivalence already discussed, we recover the ϵ expansion for the critical exponents as obtained by RG technique for a one-component order parameter. It is known that to first order in ϵ the presence of vertices of odd order does not modify the Ising universality class.[7]

So far we have not considered the effect of the core condition. When we use (6) in place of (3) we find that the extra term introduced by the core condition vanishes in the $Q\to 0$ limit faster than the other terms provided that

$$\check{c}_4^Q(\vec{q},-\vec{q},0,0)=\partial^2\tilde{c}^Q(q)/\partial\rho^2;$$
$$\check{c}_3^Q(\vec{q},0,-\vec{q})=\partial\tilde{c}^Q(q)/\partial\rho. \qquad (9)$$

These relations are satisfied by the exact correlation functions and we can construct simple decoupling schemes for the full $\check{c}_3^Q$ and $\check{c}_4^Q$ compatible with (9), for instance

$$\check{c}_4^Q(\vec{q},-\vec{q},\vec{k},-\vec{k})$$
$$=\tfrac{1}{2}[\partial^2\tilde{c}^Q(\vec{q}+\vec{k})/\partial\rho^2+\partial^2\tilde{c}^Q(\vec{q}-\vec{k})/\partial\rho^2]$$

$\Longleftarrow$ BMW approximation

and

$$\check{c}_3^Q(\vec{q},\vec{k},-\vec{q}-\vec{k})=\partial\tilde{c}^Q(\vec{q}+\vec{k})/\partial\rho.$$

When we use this closure in (4) we obtain a closed equation for $\tilde{c}^Q$ from which we can deduce the critical exponents in the framework of the ϵ expansion. These turn out to be

$$\gamma=1+\tfrac{1}{6}\epsilon+O(\epsilon^2);$$
$$\nu=\tfrac{1}{2}+\tfrac{1}{12}\epsilon+O(\epsilon^2); \qquad (10)$$
$$\eta=\tfrac{1}{54}\epsilon^2+O(\epsilon^3),$$

and these are equal to the Ising values to leading order. An open question is if the basin of attraction of this fixed point also encompasses initial condi-