Resummation of cosmological perturbations and the cosmological model

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September 18, 2010
Outline

- Effect of inhomogeneities on the average expansion
- Large-scale structures in quintessence cosmology
- Non-linear spectrum of matter perturbations
Standard framework

- Basic assumptions: Homogeneity and isotropy
  \[
  ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \\
  \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \\
  \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)
  \]

- Inhomogeneities can be treated as small perturbations of this background

- Indications for the acceleration of the cosmological expansion
  1. Distant supernovae
  2. Power spectrum of the galaxy distribution
  3. Cosmic microwave background

- For acceleration: \( p < -\rho/3 \)
Question

Could the acceleration of the cosmological expansion be related to the appearance of inhomogeneities in a pressureless cosmological fluid (dark matter)?
Our approach

- All the information about the expansion of the Universe is obtained through light signals.
- Study light propagation in an exact background that mimics a Universe with structure.
- Calculate observables: Luminosity distance of a light source as a function of its redshift.

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**Figure:** *The evolution of the density profile for a central underdensity surrounded by an overdensity.*
Luminosity distance and redshift

- Consider photons emitted within a solid angle $\Omega_s$ by an isotropic source with luminosity $L$. These photons are detected by an observer for whom the light beam has a cross-section $A_o$.
- The redshift factor is
\[
1 + z = \frac{\omega_s}{\omega_o} = \frac{k_s^0}{k_o^0},
\]
- The energy flux $f_o$ measured by the observer is
\[
f_o = \frac{L}{4\pi D_L^2} = \frac{L}{4\pi} \frac{\Omega_s}{(1 + z)^2 A_o}.
\]
- Integrating the optical equations allows the determination of the luminosity distance $D_L$ as a function of the redshift $z$. 
Observer at the center of a large hole

Figure: The distance modulus $\mu = m - M = 5 \log(D_L/\text{Mpc}) + 25$ as a function of redshift $z$.

a) Green line: FRW cosmology with $\Omega_m = 1$, $\Omega_\Lambda = 0$.

b) Blue line: FRW cosmology with $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$.

c) Red line: LTB cosmology with the observer at the center of an underdense region of present size $\sim 800$ Mpc.
Observer and source at random positions

Figure: The distribution of luminosity distances for various redshifts in the LTB Swiss-cheese model for inhomogeneities with length scale $40 \, h^{-1} \text{Mpc}$. 

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Figure: *Same as before for a characteristic scale of 400 $h^{-1}$ Mpc.*
Quintessence cosmology

- It seems unlikely that the acceleration of the cosmological expansion can be attributed to the growth of inhomogeneities.
- Negative pressure is needed.
- The simplest scenario assumes the presence of a cosmological constant.
- The quintessence scenario can provide a dynamical explanation for the smallness of the present value of the vacuum energy.
- We shall discuss coupled quintessence: a quintessence field coupled with dark matter (or neutrinos).
- What kind of new structures can appear in such cosmologies?
- Are they observable?
Basic relations

- **Action**

\[
S = \int d^4 x \sqrt{-g} \left( \frac{1}{16\pi G} R - \frac{1}{2} \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x_\mu} - U(\phi) \right) - \sum_i \int m(\phi(x_i)) d\tau_i,
\]

with \( d\tau_i = \sqrt{-g_{\mu\nu}(x_i) dx_i^\mu dx_i^\nu} \) and the second integral taken over particle trajectories.

- **Equation of motion**

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu\nu} \frac{\partial \phi}{\partial x^\nu} \right) = \frac{dU}{d\phi} - \frac{d \ln m(\phi(x))}{d\phi} \left( T_M \right)_\mu ^\mu.
\]
Cosmological evolution

- Homogeneous background

\[
\ddot{\phi} + 3H\dot{\phi} = \frac{dU}{d\phi} + \frac{d\ln m(\phi)}{d\phi}(\rho - 3p)
\]

\[
\dot{\rho} + 3H\rho = -\frac{d\ln m(\phi)}{d\phi}(\rho - 3p)\dot{\phi}
\]

\[
H^2 = \frac{8\pi G}{3} \left( \frac{1}{2} \dot{\phi}^2 + U(\phi) + \rho \right)
\]
Static spherically symmetric configurations

- Metric:

\[ ds^2 = -B(r) dt^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) + A(r) \, dr^2. \]

- Fermi-Dirac distribution at every point in space:

\[ f(p) = \left[ \exp \left( \frac{\sqrt{p^2 + m^2(\phi(r))} - \mu(r)}{T(r)} \right) + 1 \right]^{-1}. \]

- The Einstein equations give:

\[ T(r) = T_0 / \sqrt{B(r)}, \quad \mu(r) = \mu_0 / \sqrt{B(r)}. \]

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Dark matter in galaxy haloes

- The coupling between DM and the quintessence field generates an attractive force between DM particles.
- The typical DM velocity is larger than in the decoupled case.
- Implications for DM detection.
Compact astrophysical objects made of dark matter

Figure: A typical configuration
Inhomogeneities and expansion

Large-scale structures in quintessence cosmology

Non-linear matter power spectrum

Conclusions

Figure: The mass to radius relation.
Astrophysical objects made of neutrinos

\[ R(Mpc) \]

\[ m_\nu (eV) \]

- I: \( \tilde{R}=2, |\phi|=0.2 \)
- II: \( \tilde{R}=0.65, |\phi|=0.05 \)
- III: \( \tilde{R}=5.4 \cdot 10^{-4}, |\phi|=10^{-4} \)

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Link with observations

- **Study the formation of structure in the distribution of dark (and baryonic) matter.**
- The evolution of inhomogeneities depends on the cosmological background.
- The matter spectrum at various redshifts reflects the detailed structure of the cosmological model.
- Comparison with observations of the galaxy distribution can differentiate between models.
- Baryon acoustic oscillations: 100 Mpc range.
- Analytical calculation of the matter spectrum beyond the linear level.
Formalism

- **Action**

\[
S = \int d^4 x \sqrt{-g} \left( \frac{1}{2M^2} R - \frac{1}{2} \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\mu} - U(\phi) \right) - \sum_i \int m(\phi(x_i)) d\tau_i,
\]

with \( d\tau_i = \sqrt{-g_{\mu\nu}(x_i) dx_i^\mu dx_i^\nu} \) and the second integral taken over particle trajectories.

- **Equation of motion**

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu\nu} \frac{\partial \phi}{\partial x^\nu} \right) = \frac{dU}{d\phi} - \frac{d \ln m(\phi(x))}{d\phi} (T_{CDM})^\mu_\mu.
\]

- \( M = 1 \)
- \( \beta(\phi) = -d \ln m(\phi)/d\phi. \)
Ansatz for the metric

\[ ds^2 = a^2(\tau) \left[ (1 + 2\Phi(\tau, \vec{x})) d\tau^2 - (1 - 2\Phi(\tau, \vec{x})) d\vec{x} d\vec{x} \right], \]

with \( \Phi \ll 1 \).

Scalar field

\[ \phi(\tau, \vec{x}) = \bar{\phi}(\tau) + \delta\phi(\tau, \vec{x}), \]

with \( \delta\phi/\bar{\phi} \ll 1 \). In general, \( \bar{\phi} = \mathcal{O}(1) \) in units of \( M \).

Density field

\[ \rho(\tau, \vec{x}) = \bar{\rho}(\tau) + \delta\rho(\tau, \vec{x}), \]

with \( \delta\rho/\bar{\rho} \ll 1 \).

Velocity field

\[ |\delta\vec{v}| \ll 1. \]
Hierarchy of scales

- Our aim is to take into account the effect of the local velocity field $\delta \vec{v}$, when this becomes significant because of large field gradients.

- For subhorizon perturbations with momenta $k \gg \mathcal{H} = \dot{a}/a$, the linear analysis predicts $|\delta \vec{v}| \sim (k/\mathcal{H})\Phi \sim (\mathcal{H}/k)(\delta \rho/\bar{\rho})$.

- We assume the hierarchy of scales: $\Phi, \delta \phi/\bar{\phi} \ll |\delta \vec{v}| \ll \delta \rho/\bar{\rho} \lesssim 1$.

- As we are dealing with subhorizon perturbations, we assume that the spatial derivatives of $\Phi, \delta \phi, \delta \vec{v}$ dominate over their time derivatives. We also assume that a spatial derivative acting on $\Phi, \delta \phi$ or $\delta \vec{v}$ increases the position of that quantity in the hierarchy by one level: $\vec{\nabla}\Phi$ is comparable to $\delta \vec{v}$, while $\nabla^2 \Phi$ is comparable to $\bar{\rho}$. 
Equations of motion for several non-relativistic species

- The evolution of the **homogeneous background** is described by

\[
\mathcal{H}^2 = \frac{1}{3} \left[ a^2 \sum_{i=1,2} \bar{\rho}_i + \frac{1}{2} \dot{\phi}^2 + a^2 U(\bar{\phi}) \right] \equiv \frac{1}{3} a^2 \rho_{\text{tot}}
\]

\[
\dot{\bar{\rho}}_i + 3\mathcal{H}\bar{\rho}_i = -\beta_i \dot{\phi} \bar{\rho}_i
\]

\[
\ddot{\phi} + 2\mathcal{H} \dot{\phi} = -a^2 \left( \frac{dU}{d\phi}(\bar{\phi}) - \sum_{i=1,2} \beta_i \bar{\rho}_i \right),
\]

with \( \rho_{\text{tot}} \equiv \sum_i \bar{\rho}_i + \dot{\phi}^2/(2a^2) + U(\bar{\phi}) \).

- For the CDM we set \( \beta_1 = \beta \), while for BM, because of strong observational constraints, we set \( \beta_2 = 0 \).
Equations for the perturbations

- Poisson equations

\[ \nabla^2 \delta \phi = -a^2 \sum_i \beta_i \delta \rho_i \equiv -3 \sum_i \beta_i H^2 \Omega_i \delta_i \]
\[ \nabla^2 \Phi = \frac{1}{2} a^2 \sum_i \delta \rho_i \equiv \frac{3}{2} H^2 \sum_i \Omega_i \delta_i, \]

with \( \Omega_i(\tau) \equiv \bar{\rho}_i a^2 / (3H^2) \).

- Continuity and Euler equations

\[ \delta \dot{\rho}_i + 3H \delta \rho_i + \nabla [ (\bar{\rho}_i + \delta \rho_i) \delta \vec{v}_i ] = - \beta_i \dot{\phi} \delta \rho_i \]
\[ \delta \dot{\vec{V}}_i + (\dot{H} - \beta_i \dot{\phi}) \delta \vec{V}_i + (\delta \vec{V}_i \cdot \vec{V}) \delta \vec{V}_i = - \vec{V} \Phi + \beta_i \vec{V} \delta \phi. \]
Time renormalization group (Pietroni (2008))

- Fourier-transformed density contrast and velocity field:
  \( \delta_i \equiv \delta \rho_i(k, \tau) / \bar{\rho}_i \) and \( \theta_i(k, \tau) \equiv \vec{\nabla} \cdot \vec{\delta} \vec{v}_i(k, \tau) \).

- They obey

\[
\dot{\delta}_i(k, \tau) + \theta_i(k, \tau) = - \int d^3k_1 \, d^3k_2 \, \delta_D(k-k_1-k_2) \, \tilde{\alpha}(k_1, k_2) \, \theta_i(k_1, \tau) \, \delta_i(k_2, \tau),
\]

where \( \tilde{\alpha}(k_1, k_2) = k_1 \cdot (k_1 + k_2) / k_1^2 \), and

\[
\dot{\theta}_i(k, \tau) + (\mathcal{H} - \beta_i \dot{\phi}) \theta_i(k, \tau) + \frac{3 \mathcal{H}^2 \sum_j \Omega_j (2 \beta_i \beta_j + 1) \delta_j(k, \tau)}{2} = - \int d^3k_1 \, d^3k_2 \, \delta_D(k-k_1-k_2) \, \tilde{\beta}(k_1, k_2) \, \theta_i(k_1, \tau) \, \theta_i(k_2, \tau),
\]

where \( \tilde{\beta}(k_1, k_2) = (k_1 + k_2)^2 k_1 \cdot k_2 / (2k_1^2 k_2^2) \).
Define the quadruplet

\[
\begin{pmatrix}
\varphi_1(k, \eta) \\
\varphi_2(k, \eta) \\
\varphi_3(k, \eta) \\
\varphi_4(k, \eta)
\end{pmatrix}
= e^{-\eta} \begin{pmatrix}
\delta_{CDM}(k, \eta) \\
-\theta_{CDM}(k, \eta) \\
\delta_{BM}(k, \eta) \\
-\theta_{BM}(k, \eta)
\end{pmatrix},
\]

where \( \eta = \ln a(\tau) \).

The equations of motion become

\[
\partial_\eta \varphi_a(k, \eta) + \Omega_{ab} \varphi_b(k, \eta) = e^n \gamma_{abc}(k, -k_1, -k_2) \varphi_b(k_1, \eta) \varphi_c(k_2, \eta).
\]

The indices \( a, b, c \) take values 1, \ldots, 4. Repeated momenta are integrated over, while repeated indices are summed over.
The non-zero components of the effective vertices $\gamma$ are

$$
\begin{align*}
\gamma_{121}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) &= \frac{\tilde{\alpha}(\mathbf{k}_1, \mathbf{k}_2)}{2} \delta_D(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2) = \gamma_{112}(\mathbf{k}, \mathbf{k}_2, \mathbf{k}_1) \\
\gamma_{222}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) &= \tilde{\beta}(\mathbf{k}_1, \mathbf{k}_2) \delta_D(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2) \\
\gamma_{343}(\mathbf{k}, \mathbf{k}_3, \mathbf{k}_4) &= \frac{\tilde{\alpha}(\mathbf{k}_3, \mathbf{k}_4)}{2} \delta_D(\mathbf{k} + \mathbf{k}_3 + \mathbf{k}_4) = \gamma_{334}(\mathbf{k}, \mathbf{k}_4, \mathbf{k}_3) \\
\gamma_{444}(\mathbf{k}, \mathbf{k}_3, \mathbf{k}_4) &= \tilde{\beta}(\mathbf{k}_3, \mathbf{k}_4) \delta_D(\mathbf{k} + \mathbf{k}_3 + \mathbf{k}_4).
\end{align*}
$$

The $\Omega$-matrix, that defines the linear evolution, is

$$
\Omega(\eta) = \begin{pmatrix}
1 & -1 & 0 & 0 \\
-\frac{3}{2} \Omega_{CDM} (2\beta^2 + 1) & 2 - \beta \phi' + \frac{\mathcal{H}'}{\mathcal{H}} & -\frac{3}{2} \Omega_{BM} & 0 \\
0 & 0 & 1 & -1 \\
-\frac{3}{2} \Omega_{CDM} & 0 & -\frac{3}{2} \Omega_{BM} & 2 + \frac{\mathcal{H}'}{\mathcal{H}}
\end{pmatrix}.
$$

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Define the spectra, bispectra and trispectra as

\[
\langle \varphi_a(k, \eta) \varphi_b(q, \eta) \rangle \equiv \delta_D(k + q) P_{ab}(k, \eta)
\]
\[
\langle \varphi_a(k, \eta) \varphi_b(q, \eta) \varphi_c(p, \eta) \rangle \equiv \delta_D(k + q + p) B_{abc}(k, q, p, \eta)
\]
\[
\langle \varphi_a(k, \eta) \varphi_b(q, \eta) \varphi_c(p, \eta) \varphi_d(r, \eta) \rangle \equiv \delta_D(k + q) \delta_D(p + r) P_{ab}(k, \eta) P_{cd}(p, \eta)
\]
\[
+ \delta_D(k + p) \delta_D(q + r) P_{ac}(k, \eta) P_{bd}(q, \eta)
\]
\[
+ \delta_D(k + r) \delta_D(q + p) P_{ad}(k, \eta) P_{bc}(q, \eta)
\]
\[
+ \delta_D(k + p + q + r) Q_{abcd}(k, p, q, r, \eta).
\]
Essential approximation: Neglect the effect of the trispectrum on the evolution of the bispectrum.

In this way we obtain

$$\partial_\eta P_{ab}(k, \eta) = -\Omega_{ac}P_{cb}(k, \eta) - \Omega_{bc}P_{ac}(k, \eta)$$

$$+ e^{\eta} \int d^3q \left[ \gamma_{acd}(k, -q, q - k)B_{bcd}(k, -q, q - k) + \gamma_{bcd}(k, -q, q - k)B_{acd}(k, -q, q - k) \right],$$

$$\partial_\eta B_{abc}(k, -q, q - k) = -\Omega_{ad}B_{dbc}(k, -q, q - k) - \Omega_{bd}B_{adc}(k, -q, q - k) - \Omega_{cd}B_{abd}(k, -q, q - k)$$

$$+ 2e^{\eta} \int d^3q \left[ \gamma_{ade}(k, -q, q - k)P_{db}(q, \eta)P_{ec}(k - q, \eta) + \gamma_{bde}(-q, q - k, k)P_{dc}(k - q, \eta)P_{ea}(k, \eta) + \gamma_{cde}(q - k, k, -q)P_{da}(k, \eta)P_{eb}(q, \eta) \right].$$
Coupled quintessence

- The field has a potential $V(\phi) \sim \phi^{-\alpha}$, with $\alpha = 0.143$.
- The present-day energy content of the Universe has $\Omega_{DE} = 0.743$, $\Omega_{CDM} = 0.213$, $\Omega_{BM} = 0.044$.
- The Universe is assumed to have vanishing spatial curvature ($\Omega_k = 0$), current expansion rate $H_0 = 71.9 \text{ km s}^{-1} \text{ Mpc}^{-1}$.
- The mass variance is taken $\sigma_8 = 0.769$, as calculated from the linear spectrum.

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Figure: Comparison of results from N-body simulations and our calculation ($\beta = 0.05$). We display the ratio of the non-linear and linear spectra for $z = 0$. 

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Figure: Baryonic matter density-density spectra for various $\beta$ at $z = 0$, normalized with respect to the a smooth spectrum.
Figure: Spectra for $\beta = 0.1$ at $z = 0$, normalized with respect to a smooth spectrum.
Variable equation of state

- No coupling between dark matter and dark energy.
- One massive species: baryonic+dark matter
- Dark energy fluctuations are negligible.
- Equation of state $p/\rho = w(z)$

$$w(a) = \frac{a \tilde{w}(a)}{a + a_{\text{trans}}} - \frac{a_{\text{trans}}}{a + a_{\text{trans}}}$$

$a = 1/(1 + z)$ is the scale factor.

$\tilde{w}(a) = \tilde{w}_0 + (1 - a) \tilde{w}_a$

$a_{\text{trans}}$ corresponds to the “transition epoch”

- At small redshifts

$$w(a) = w_0 + (1 - a) w_a.$$ 

We describe the various models through $w_0 \equiv w(z = 0)$, $w' \equiv dw/dz|_{z=0}$, and $a_{\text{trans}}$.

Figure: Linear and non-linear spectra at $z = 0$, for $w_0 = -0.8$, $w' = -0.7$, normalized with respect to a smooth spectrum.
Figure: The fractional shift of the maximum, minima and nodes of the non-linear spectrum, as a function of redshift, for $w_0 = -0.8$, $w' = -0.7$. 

\[ \frac{\delta k_i}{k_i,\text{lin}} \]
Figure: The fractional shift of the first maximum from its location for $\Lambda$CDM, as a function of $w_0$ and $w'$, at a redshift $z = 0.366$. 
Conclusions

- The inhomogeneities in the matter distribution have a very small effect on the average acceleration.
- Novel large-scale structures can appear in quintessence cosmology.
- The spectrum of matter perturbations is a very useful tool in order to differentiate between cosmological models. The non-linear corrections to the spectrum must be evaluated carefully in order to compare with astrophysical data.