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Functional RG for few-body physics

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Review of results from:

Schmidt and Moroz, arXiv:0910.4586
Krippa, Walet and Birse, arXiv:0911.4608
Krippa, Walet and Birse, in preparation

Outline

- Background: effective field theories
  in nuclear and cold-atom physics
- Functional RG
- Three-body systems (unitary limit – scaling)
- Four-body systems
- Summary

Background

Effective field theory: now mainstream tool for nuclear forces

- symmetries of underlying dynamics (QCD)
- systematic expansions in powers of low-energy scales (momenta, pion mass ...)
- RG methods used to derive power countings

$\longrightarrow$ classify terms as perturbations around fixed points (Wilsonian approach, sharp cut-offs)

Background

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2-nucleon scattering very strong at low-energies
(inverse scattering lengths $1/a \sim 10 - 40$ MeV)

- expand around nontrivial fixed point "unitary limit" (zero-energy bound state: scale free)
- corresponds to effective-range expansion [Bethe (1949)]
- also describes atomic systems with Feshbach resonance tuned to threshold

3-body physics in unitary limit

Momentum space: one-variable integral equation

[Skornyakov and Ter-Martirosian (1956)]

Faddeev equation in hyperspherical coordinates

($R^2 = |\mathbf{r}_1 - \mathbf{r}_2|^2 + |\mathbf{r}_2 - \mathbf{r}_3|^2 + |\mathbf{r}_3 - \mathbf{r}_1|^2$)

- Schrödinger equation with $1/R^2$ potential [Efimov, 1971]

$$-\frac{1}{M}\left[\frac{\mathrm{d}^2}{\mathrm{d}R^2} + \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R} - \frac{\nu^2}{R^2}\right] u(r) = p^2\, u(R)$$

- hyperangular eigenvalue ν^2 fixed by boundary condition (*S*-waves)

$$1 = \sigma \frac{4}{\sqrt{3\pi}\,\nu} \frac{\Gamma\left(\frac{1-\nu}{2}\right)\Gamma\left(\frac{1+\nu}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \sin\left(\frac{\pi\nu}{6}\right)$$

- spatially symmetric: $\sigma = +1$; mixed-symmetry $\sigma = -\frac{1}{2}$ ("particle-exchange interaction" between pair and third particle)

Three bosons or more than two “species” of fermion

(spatially symmetric state, eg triton)

- attractive $1/R^2$ potential ($\nu^2 < 0$)
- RG flow tends to limit cycle (discrete remnant of scale invariance)
  [Moroz, talk at this meeting]

$\longrightarrow$ Efimov effect: tower of bound states with constant ratio between energies

- momentum scale factor $e^{\pi/s_0} \simeq 23$ where $s_0 = 1.00624$
- leading 3-body force is marginal
- fixes starting point on cycle or energy of one bound state
  (Phillips line relating ^{3}H binding to *nd* scattering length)
- Efimov states now observed for various alkali atoms

Other fermionic 3-body systems less interesting

- 3-body forces irrelevant
  (no role in very-low-energy physics)
- but noninteger anomalous dimensions
- mixed symmetry state of two or more species of fermion:
  leading 3-body force scales as $Q^{4.33244}$ (Q: momentum)
- determine energy eigenvalues in harmonic traps
  [Werner and Castin (2005)]

4-body systems: no exact results

- 4-body forces irrelevant, even in Efimov systems
  [Platter, Hammer and Meissner (2004)]
- low-energy 4-body physics determined by 2- and 3-body forces
  (Tjon line relating ^{4}He and ^{3}H; atomic Efimov systems)

Wilsonian RG methods used to obtain these

- exact results only for problems with 2-body-like forms
  2-body systems; 3-body with contact interactions
  not 3-body with long-range forces; 4-body

EFTs based on contact interactions

- not well suited for standard many-body techniques
- difficult to apply to larger systems, dense matter

$\longrightarrow$ new approaches needed

Functional ("exact") RG

RG for the Legendre-transformed effective action Γ [Wetterich (1993)]
(1-particle-irreducible generating function)

Exact evolution equation has 1-loop structure

$$\begin{aligned}\partial_k \Gamma &= +\frac{i}{2}\,\mathrm{Tr}\left[(\partial_k \mathbf{R}_F)\left((\mathbf{\Gamma}^{(2)} - \mathbf{R})^{-1}\right)_{FF}\right] \\ &\quad -\frac{i}{2}\,\mathrm{Tr}\left[(\partial_k \mathbf{R}_B)\left((\mathbf{\Gamma}^{(2)} - \mathbf{R})^{-1}\right)_{BB}\right]\end{aligned}$$

$\mathbf{\Gamma}^{(2)}$: matrix of second derivatives of the action
$\mathbf{R}_F(q;k)$: regulator for fermion fields; $\mathbf{R}_B(q;k)$: for bosons
(suppress modes with momenta $q < k$)

$\Gamma \longrightarrow$ full effective action as $k \rightarrow 0$

Example: 2 species of fermion

Fermion field: $\psi(x)$ (neutrons or spin-$\frac{1}{2}$ atoms)

Boson "dimer" field: $\phi(x)$ (strongly interacting pairs)

Truncated ansatz for action in vacuum: 2-body, local

$$\begin{aligned}
&\Gamma[\psi,\psi^\dagger,\phi,\phi^\dagger;k] \\
&= \int \mathrm{d}^4x \left[\psi(x)^\dagger\left(\mathrm{i}\,\partial_0 + \frac{\nabla^2}{2M}\right)\psi(x)\right. \\
&\qquad + Z_\phi(k)\,\phi(x)^\dagger\left(\mathrm{i}\,\partial_0 + \frac{\nabla^2}{4M}\right)\phi(x) - u_1(k)\,\phi(x)^\dagger\phi(x) \\
&\qquad \left. - g\left(\frac{\mathrm{i}}{2}\phi(x)^\dagger\psi(x)^{\mathrm{T}}\sigma_2\psi(x) + \mathrm{H\,c}\right)\right]
\end{aligned}$$

g: AA$\rightarrow$D coupling

$u_1(k)$: dimer self-energy (u_1/g^2: only physical parameter)

$Z_\phi(k)$: dimer wave-function renormalisation

Regulators

- fermions: sharp cutoff

$$R_F(\boldsymbol{q}, k) = \frac{k^2 - q^2}{2M}\,\theta(k - q)$$

- nonrelativistic version of "optimised" cutoff [Litim (2001)]
- fastest convergence at this level of truncation
- bosons

$$R_B(\boldsymbol{q}, k) = c_B\,\frac{k^2 - q^2}{4M}\,\theta(k - q)$$

- optimised choice $c_B = 1$ [cf Pawlowski (2007)]
  (no mismatch between fermion and boson cutoffs)

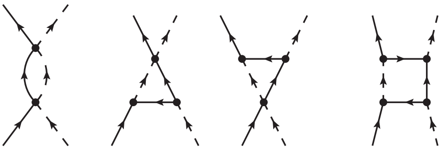
Unitary limit

- $u_1(k) \to 0$ as $k \to 0$ ($a = -Mg^2/(4\pi u_1(0)) \to \infty$)
- $Z_\phi(k) \to 0$ as $k \to \infty$ (dimer auxiliary field at starting scale)

3-body interaction (AD scattering)

$$\Gamma[\psi,\psi^\dagger,\phi,\phi^\dagger;k] = \cdots - \lambda(k)\int d^4x\,\psi^\dagger(x)\phi^\dagger(x)\phi(x)\psi(x)$$

Evolution driven by terms corresponding to “skeleton” diagrams



(loops with AD contact interaction and single-A exchange)

Evolution equation

$$\partial_k\lambda = \frac{28\,k}{125\,g^2\,M}\lambda^2 + \frac{156}{125\,k}\lambda + \frac{128\,g^2\,M}{125\,k^3}$$

Rescale: $\widehat{\lambda} = \frac{k^2}{g^2 M} \lambda$

- dimensionless equation

$$k\partial_k \widehat{\lambda} = \frac{28}{125}\widehat{\lambda}^2 + \frac{406}{125}\widehat{\lambda} + \frac{128}{125}$$

$\longrightarrow$ two fixed point solutions (roots of RHS)

- expand around IR stable point: $\widehat{\lambda} - \widehat{\lambda}_0 \propto k^{3.10355}$
- compare exact solution: $Q^{4.33244}$

$\longrightarrow$ strongly irrelevant but only agree at 30% level

Bosons

Very similar action and evolution equations

- different numerical coefficients
  $\partial_k \lambda$ term linear in λ gets factor of -2 (cf Faddeev equation)
- rescaled equation

$$k\partial_k \widehat{\lambda} = \frac{56}{125}\widehat{\lambda}^2 - \frac{62}{125}\widehat{\lambda} + \frac{256}{125}$$

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$$k\partial_k \widehat{\lambda} = \frac{56}{125}\widehat{\lambda}^2 - \frac{62}{125}\widehat{\lambda} + \frac{256}{125}$$

$\longrightarrow$ two complex roots – fixed points

- expand around either: $\widehat{\lambda} - \widehat{\lambda}_0 \propto k^{\pm 2\mathrm{i} s_0}$
- imaginary exponent $\longrightarrow$ limit cycle [Moroz, talk at this meeting]
- real solutions periodic under scaling k by factor e^{π/s_0}
  where $s_0 = 0.92503$
- agrees with Efimov $s_0 = 1.00624$ to $\sim 5\%$

4-body interactions (2 species of fermion)

Need DD→DD, DD→DAA, DAA→DAA terms

(dimer "breakup" terms allow 3-body physics to feed in)

$$\Gamma[\psi,\psi^\dagger,\phi,\phi^\dagger;k] = \cdots - \int d^4x \left[\frac{1}{2}u_2(k)\left(\phi^\dagger\phi\right)^2 + \frac{1}{4}v(k)\left(\phi^{\dagger 2}\phi\,\psi^{\mathrm{T}}\psi + \mathrm{H\,c}\right) + \frac{1}{4}w(k)\,\phi^\dagger\phi\,\psi^\dagger\psi^{\dagger\mathrm{T}}\psi^{\mathrm{T}}\psi\right]$$

→ coupled evolution equations for u_2, v, w (27 distinct skeletons)

Rescaled 4-body evolution equations

- 4 fixed-point solutions
- only one IR stable
- smallest eigenvalue $\longrightarrow k^{4.19149}$ (irrelevant)

Bosons

- 4 complex fixed points (since λ complex)
- only one IR stable
- eigenvalue with smallest real part $\longrightarrow k^{0.055165+3.50440\mathrm{i}}$

$\longrightarrow$ only weakly irrelevant ??

- couplings flow to cycle driven by $\lambda(k)$

Summary

First full applications of functional RG to 3- and 4-body systems

- local truncation, “optimised” cutoffs
- unitary limit: scaling behaviours agree with exact 3-body
  qualitatively for 2 species of fermion
  much more accurately for bosons (Efimov effect)
- first estimates of anomalous dimensions for 4-body forces
  bosons: real part puzzlingly small

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Future work

- away from unitary limit [Krippa, talk at this meeting]
- 4 species of fermion – nucleons
  SU(4) symmetry: evolution same as either bosons or 2 species
- use these 3-, 4-body interactions as input into calculations of
  dense matter (nuclear and cold atomic)
  [Floerchinger, talk at this meeting]