Gauge fields on truncated Heisenberg space

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Outline

Motivation

Noncommutativity of coordinates was originally introduced to regularize divergences in QFT. One can think intuitively that noncommutativity introduces natural cutoffs in divergent momentum integrals, or that it introduces discretization of space adjusted to symmetries.

However in practice not many noncommutative renormalizable models are found.

Motivation

In part this is because in principle we wish to recover the complete structure of commutative description:

space

geometry
(differential, integral)
eqs. of motion

symmetries
(gauge fields,
gravity)

fields

In addition, to quantize,
and have correct commutative limit!

To obtain a model and work out all details one needs not only NC space (algebra $[x^\mu, x^\nu] = iJ^{\mu\nu}$) and fields (functions $f(x^\mu)$) but also a concrete representation

Commutative limit to Minkowski space somehow singles out the Moyal plane, though of course other models are known

Renormalizable ϕ^4 models on the Moyal plane are obtained effectively by a change of propagator:

- $\frac{1}{p^2+m^2} \longrightarrow \frac{1}{p^2+m^2+x^2}$ Grosse, Wulkenhaar
- $\frac{1}{p^2+m^2} \longrightarrow \frac{1}{p^2+m^2+1/p^2}$ Gurau, Magnen, Rivasseau, Tanasa

On the other hand, commutative limit is anyway always singular.

Idea here: use matrices to regularize.

Further: use geometric structure of matrix spaces along with field-theoretic notions as a guide.

We start with the algebra

$$[\mu x, \mu y] = i\epsilon(1 - \mu' z),$$
$$[\mu x, \mu' z] = i\epsilon(\mu y\, \mu' z + \mu' z\, \mu y),$$
$$[\mu y, \mu' z] = -i\epsilon(\mu x\, \mu' z + \mu' z\, \mu x).$$

- for $\epsilon = 0$ it is commutative
- for $\mu' = 0$ it reduces to the Heisenberg algebra; or alternatively, $z = 0$ is a two-dimensional 'subspace'
- for $\epsilon = 1,\ \mu' = \mu$ it has finite-dimensional representations for any $n \times n$ matrices: truncated Heisenberg algebra

Finite-dimensional representations

$$x = \frac{1}{\mu\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & . & . & . \\ 1 & 0 & \sqrt{2} & . & . & . \\ 0 & \sqrt{2} & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & 0 & \sqrt{n-1} \\ . & . & . & . & \sqrt{n-1} & 0 \end{pmatrix}$$

$$y = \frac{i}{\mu\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 & . & . & . \\ 1 & 0 & -\sqrt{2} & . & . & . \\ 0 & \sqrt{2} & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & 0 & -\sqrt{n-1} \\ . & . & . & . & \sqrt{n-1} & 0 \end{pmatrix}$$

$$z = \frac{n}{\mu}\begin{pmatrix} 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & 0 & 0 \\ . & . & . & . & 0 & 1 \end{pmatrix}$$

The limit $z \to 0$ is weak limit from finite matrices to infinite-matrix representation of the Heisenberg algebra. Therefore in field theory, instead of the Moyal plane, we could use matrix truncations and impose the limit at the end

Idea:

- analyze three-dimensional finite-matrix space: it is curved; find connection, curvature; build field models
- induce models on the subspace $z = 0$:
  - scalar model has a coupling to the curvature (GW action)
  - gauge fields undergo a Kaluza-Klein decomposition: they are coupled to a scalar
- check renormalizability

Differential geometry is not uniquely defined, not even the differential d.

We choose to work with noncommutative frames, in particular because the formalism is adjusted to matrix geometries.

- Cotangent space: basis of frame 1-forms θ^α, $[f, \theta^\alpha] = 0$
- Locally flat inverse metric: $g^{\alpha\beta} = g(\theta^\alpha \otimes \theta^\beta) = \mathrm{const}$
- Dual derivations e_α: $\theta^\alpha(e_\beta) = \delta^\alpha_\beta$
- Derivations given by momenta p_α: $e_\alpha f = [p_\alpha, f]$
- Differential d: $df = (e_\alpha f)\,\theta^\alpha$

We obtain different differentials by choosing different sets of momenta p_α. The choice of momenta is not completely arbitrary. Imposing the Leibniz rule and $d^2 = 0$ we get

$$[p_\alpha, p_\beta] = \frac{1}{i\epsilon} K_{\alpha\beta} + F^\gamma{}_{\alpha\beta} p_\gamma - 2i\epsilon Q^{\gamma\delta}{}_{\alpha\beta} p_\gamma p_\delta .$$

Momenta satisfy a quadratic algebra.

Exterior algebra has to be consistent with d. If we define exterior multiplication by

$$\theta^\gamma \theta^\delta = P^{\gamma\delta}{}_{\alpha\beta} \theta^\alpha \otimes \theta^\beta ,$$

where $P^{\gamma\delta}{}_{\alpha\beta}$ is a projector, then

$$P^{\gamma\delta}{}_{\alpha\beta} = \frac{1}{2}(\delta^\gamma_\alpha \delta^\delta_\beta - \delta^\gamma_\beta \delta^\delta_\alpha) + i\epsilon Q^{\gamma\delta}{}_{\alpha\beta} .$$

Momenta

For the truncated Heisenberg algebra we introduce momenta as

$$\epsilon p_1 = i\mu^2 y, \quad \epsilon p_2 = -i\mu^2 x, \quad \epsilon p_3 = i\mu(\mu z - \frac{1}{2}).$$

p_1 and p_2 are the same as for the Heisenberg algebra. We have

$$[p_1, p_2] = \frac{\mu^2}{2i\epsilon} + \mu p_3$$
$$[p_2, p_3] = \mu p_1 - i\epsilon(p_1 p_3 + p_3 p_1)$$
$$[p_3, p_1] = \mu p_2 - i\epsilon(p_2 p_3 + p_3 p_2)$$

The nonvanishing structure coefficients are

$$K_{12} = \frac{\mu^2}{2}, \quad F^1{}_{23} = \mu, \quad Q^{13}{}_{23} = \frac{1}{2}, \quad Q^{23}{}_{31} = \frac{1}{2}.$$

As a basis in the space of 2-forms we can take anticommutators if 1-forms. The basic relations in the exterior algebra are

$$
\begin{aligned}
&(\theta^1)^2 = 0, \quad (\theta^2)^2 = 0, \quad (\theta^3)^2 = 0, \\
&\{\theta^1, \theta^2\} = 0, \\
&\{\theta^1, \theta^3\} = i\epsilon(\theta^2\theta^3 - \theta^3\theta^2), \\
&\{\theta^2, \theta^3\} = i\epsilon(\theta^3\theta^1 - \theta^1\theta^3),
\end{aligned}
$$

space of 2-forms is three-dimensional.

To define integration we need volume form Θ. Then by definition the integral of a 3-form $\alpha = f\Theta$ is $\int \alpha = \mathrm{Tr}\, f$.

Algebra of 3-forms is obtained from algebra of 2-forms and associativity:

$$\theta^1\theta^3\theta^1 = \theta^2\theta^3\theta^2,$$

$$\theta^1\theta^2\theta^3 = -\theta^2\theta^1\theta^3 = \theta^3\theta^1\theta^2 = -\theta^3\theta^2\theta^1 = i\,\frac{\epsilon^2-1}{2\epsilon}\,\theta^2\theta^3\theta^2,$$

$$\theta^1\theta^3\theta^2 = -\theta^2\theta^3\theta^1 = i\,\frac{\epsilon^2+1}{2\epsilon}\,\theta^2\theta^3\theta^2.$$

$$\theta^3\theta^1\theta^3 = 0, \qquad \theta^3\theta^2\theta^3 = 0.$$

The volume form is unique; we define it as $\Theta = -\frac{i}{2\epsilon}\,\theta^2\theta^3\theta^2$.

A metric compatible connection $\omega^{\alpha}{}_{\beta} = \omega^{\alpha}{}_{\gamma\beta}\theta^{\gamma}$ can be defined as

$$\omega_{\alpha\beta\gamma} = \frac{1}{2}(C_{\alpha\beta\gamma} - C_{\beta\gamma\alpha} + C_{\gamma\alpha\beta}),$$

$C^{\alpha}{}_{\beta\gamma}$ are the Ricci rotation coefficients, $C^{\gamma}{}_{\alpha\beta} = F^{\gamma}{}_{\alpha\beta} - 4i\epsilon Q^{\gamma\delta}{}_{\alpha\beta}p_{\delta}$.

From $\omega^{\alpha}{}_{\beta}$ one obtains curvature $\Omega^{\alpha}{}_{\beta}$

$$\Omega^{\alpha}{}_{\beta} = d\omega^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\gamma}\omega^{\gamma}{}_{\beta} = R^{\alpha}{}_{\beta\rho\sigma}\theta^{\rho}\theta^{\sigma}.$$

In our case,

$$R = \frac{11}{4}\mu^2 - 2\mu^2(\mu z - \frac{1}{2}) - 4\mu^4(x^2 + y^2).$$

U_1 gauge fields are defined as usual. Vector potential is a 1-form, field strength is a 2-form:

$$A = A_\alpha \theta^\alpha, \qquad F = dA + A^2 = \frac{1}{2} F_{\alpha\beta}\theta^\alpha\theta^\beta.$$

Components of the field strength satisfy $F_{\zeta\eta} = F_{\alpha\beta} P^{\alpha\beta}{}_{\zeta\eta}$, which means that in our case they are antisymmetric. We find

$$F_{\zeta\eta} = e_{[\zeta} A_{\eta]} - A_\alpha C^\alpha{}_{\zeta\eta} + [A_\zeta, A_\eta] + 2i\epsilon(e_\beta A_\gamma) Q^{\beta\gamma}{}_{\zeta\eta} + 2i\epsilon A_\beta A_\gamma Q^{\beta\gamma}{}_{\zeta\eta}$$

or, when connection is torsion-free, $\omega^\alpha{}_{[\beta\gamma]} = C^\alpha{}_{\beta\gamma}$,

$$F_{\zeta\eta} = \nabla_{[\zeta} A_{\eta]} + [A_\zeta, A_\eta] + 2i\epsilon(e_\beta A_\gamma) Q^{\beta\gamma}{}_{\zeta\eta} + 2i\epsilon A_\beta A_\gamma Q^{\beta\gamma}{}_{\zeta\eta},$$

$\nabla_\zeta A_\eta = e_\zeta A_\eta - A_\alpha \omega^\alpha{}_{\zeta\eta}$ is gravity-covariant derivative.

When calculus is based on inner derivations there is a special 1-form θ,

$$\theta = -p_\alpha \theta^\alpha,$$

a connection which is invariant under the gauge group. Differential can be expressed as $df = -[\theta, f]$. Difference between connections A and θ, $\mathrm{X}_\alpha = p_\alpha + A_\alpha$, transforms in the adjoint representation: X_α are covariant coordinates.

The field strength in covariant coordinates is

$$F_{\alpha\beta} = 2P^{\gamma\delta}{}_{\alpha\beta}\mathrm{X}_\gamma \mathrm{X}_\delta - F^\gamma{}_{\alpha\beta}\mathrm{X}_\gamma - \frac{1}{i\epsilon}K_{\alpha\beta}.$$

On subspace $z = 0$ we have $p_3 = -\frac{i\mu}{2\epsilon}$, $e_3 = 0$ and component A_3 transforms as a scalar field in the adjoint representation. We denote

$$\mathsf{A}_3 = \phi, \quad \mathsf{A}_1 = A_1, \quad \mathsf{A}_2 = A_2$$

and

$$\mathsf{F}_{12} = F_{12} - \mu\phi = [\mathsf{X}_1, \mathsf{X}_2] + \frac{i\mu^2}{\epsilon} - \mu\phi,$$
$$\mathsf{F}_{13} = D_1\phi - i\epsilon\{p_2 + A_2, \phi\} = [\mathsf{X}_1, \phi] - i\epsilon\{\mathsf{X}_2, \phi\},$$
$$\mathsf{F}_{23} = D_2\phi + i\epsilon\{p_1 + A_1, \phi\} = [\mathsf{X}_2, \phi] + i\epsilon\{\mathsf{X}_1, \phi\}.$$

A_α, $F_{\alpha\beta}$ are gauge fields which would be intrinsically defined in 2d.

To write the Yang-Mills action we need the symmetrized product of forms,

$$\mathcal{S}_{YM} = \frac{1}{16}\int (\mathsf{F}^*\mathsf{F} +^* \mathsf{F}\mathsf{F}),$$

where the Hodge-dual is defined with respect to the volume form Θ. For $\epsilon = 0$ this reduces to the standard expression, while in our case due to normalization we have

$$\mathcal{S}_{YM} = \frac{1}{2}\,\mathrm{Tr}\,\left((1-\epsilon^2)\,\mathsf{F}_{12}\mathsf{F}^{12} + \mathsf{F}_{13}\mathsf{F}^{13} + \mathsf{F}_{23}\mathsf{F}^{23}\right).$$

In components, Yang-Mills action is

$$\mathcal{S}_{YM} = \frac{1}{2}\,\mathrm{Tr}\,((1-\epsilon^2)(F_{12})^2 - 2(1-\epsilon^2)\mu F_{12}\phi + (5-\epsilon^2)\mu^2\phi^2 + 4i\epsilon F_{12}\phi^2$$
$$+ (D_1\phi)^2 + (D_2\phi)^2 - \epsilon^2\{p_1 + A_1, \phi\}^2 - \epsilon^2\{p_2 + A_2, \phi\}^2)$$

or in covariant coordinates

$$\mathcal{S}_{YM} = \frac{1}{2}\,\mathrm{Tr}\,((1-\epsilon^2)([\mathrm{X}_1, \mathrm{X}_2]^2 + \mu^2\phi^2 - \frac{2i\mu^3}{\epsilon}\phi - 2\mu\,[\mathrm{X}_1, \mathrm{X}_2]\,\phi)$$
$$+ 4i\epsilon\,[\mathrm{X}_1, \mathrm{X}_2]\,\phi^2 + [\mathrm{X}_1, \phi]^2 + [\mathrm{X}_2, \phi]^2 - \epsilon^2\{\mathrm{X}_1, \phi\}^2 - \epsilon^2\{\mathrm{X}_2, \phi\}^2).$$

Classical YM equations of motion:

$$D^\alpha D_\alpha\phi+\epsilon^2\{p^\alpha-A^\alpha,\{p_\alpha+A_\alpha,\phi\}\}+(1-\epsilon^2)\,\mu F_{12}-(5-\epsilon^2)\mu^2\phi-2i\epsilon\{F_{12},\phi\}=0,$$

$$(1-\epsilon^2)\,\epsilon^{\alpha\beta}D_\beta(F_{12}-\mu\phi)+2i\epsilon\epsilon^{\alpha\beta}\{D_\beta\phi,\phi\}-[D_\alpha\phi,\phi]-\epsilon^2\{\{p^\alpha+A^\alpha,\phi\},\phi\}=0.$$

They are difficult to solve. Confining to constant solutions we get

$$A_1=0,\ A_2=0,\ \phi=0,$$
$$\mathsf{X}_1=0,\ \mathsf{X}_2=0,\ \phi=\frac{i\mu}{\epsilon}.$$

First solution is the usual vacuum.

Because of the properties of exterior multiplication the only viable Chern-Simons action is

$$\mathcal{S}_{CS} = \alpha \int \mathsf{X}^3 = \alpha \int \mathsf{X}_\alpha \mathsf{X}_\beta \mathsf{X}_\gamma \, \Delta^{\alpha\beta\gamma}_{\zeta\eta\xi} \, \theta^\zeta \theta^\eta \theta^\xi$$

In components,

$$\mathcal{S}_{CS} = \frac{\alpha\mu}{3} \operatorname{Tr} \left((3-\epsilon^2) \, [\mathsf{X}_1, \mathsf{X}_2] \, \mathsf{X}_3 + 2i\epsilon \, (\mathsf{X}_1^2 + \mathsf{X}_2^2) \, \mathsf{X}_3 \right) .$$

CS action can be included, but it does not simplify equations of motion significantly.

The simplest gauge-fixing term is

$$\mathcal{G} = e_\alpha A^\alpha = \partial_1 A^1 + \partial_2 A^2$$

According to the usual procedure the quantum action is given by

$$\mathcal{S} = \mathcal{S}_{YM} + \mathcal{S}_{gf}$$

with

$$\mathcal{S}_{gf} = \mathrm{Tr}\left(B\mathcal{G} + \frac{\alpha}{2}BB - \bar{c}e_\alpha D^\alpha c\right) = \mathrm{Tr}\, s\,(\bar{c}\mathcal{G} + \frac{\alpha}{2}\bar{c}B)$$

BRS

where we introduced the auxilliary field B, ghosts $c, \bar{c}$ and the BRST transformation s which is nilpotent:

$$sA_\alpha = D_\alpha c = e_\alpha c + i[A_\alpha, c],$$

$$sF_{\alpha\beta} = [F_{\alpha\beta}, c],$$

$$s\phi = [\phi, c],$$

$$s(\mathrm{X}_\alpha) = [\mathrm{X}_\alpha, c] = e_\alpha c + i[A_\alpha, c] = sA_\alpha,$$

and

$$sc = -c^2, \qquad s\bar{c} = B, \qquad sB = 0.$$

Conclusions

- On truncated Heisenberg space we obtained the gauge action

$$\mathcal{S}_{YM} = \frac{1}{2}\int (D_1\phi)^2 + (D_2\phi)^2 + 4\mu^2\phi^2$$
$$+ 4iF_{12}\phi^2 - \{p_1 + A_1, \phi\}^2 - \{p_2 + A_2, \phi\}^2$$

- It has the vacuum $A_1 = 0,\ A_2 = 0,\ \phi = 0$
- One can define a nilpotent BRST transformation s and show that the quantum action is BRST invariant
- Explicit quantization? Propagators have the form of the Mehler kernel