Anatomy of a deformed symmetry: field quantization curved momentum space

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for inertial observers

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Less known context in which **free** QFT manifests non-trivial features
field quantization on curved momentum space

Outline

- **Relativistic particles, fields and quantization: a (pedantic) review**
- **Bending phase space and a new quantization ambiguity**
- κ**-quantum fields, the "fine structure" of** κ**-Fock space and hidden entanglement**

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$\phi(x) \in \mathcal{S}$ equipped with $\omega(\phi_1, \phi_2) = \int_\Sigma (\phi_2 \nabla_\mu \phi_1 - \phi_1 \nabla_\mu \phi_2) d\Sigma^\mu$

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Phase space of a classical Klein-Gordon field $\{\mathcal{S}, \omega\}$

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- The coproduct Δ extends the action of elements of $\mathcal{P}$ to multiparticle states

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consequence for translation generator observables: $\Delta P_\mu \neq P_\mu \otimes 1 + 1 \otimes P_\mu$, and

$$\pi_{12}\Delta P_\mu \neq \Delta P_\mu\,, \qquad \pi_{12}(a\otimes b) \equiv (b \otimes a)$$

i.e. **"non-Leibniz" and "non-symmetric" action on multi-particle states**

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- upon **quantization** $j_a \to J_a$, $p_a \to P_a$, observables which belong to $U(\mathfrak{p}_3)$

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(see also Noui et al. 0806.4121, Freidel & Livine hep-th/0512113)

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- for **plane waves** in B consider a one-parameter splitting

$$e_p \equiv e^{-i\frac{1-\beta}{2}p^0 P^*_0}\, e^{ip^j P^*_j}\, e^{-i\frac{1+\beta}{2}p^0 P^*_0}\,.$$

$0 \le |\beta| \le 1$, with momentum composition rules and "antipodes"

$$p \oplus_\beta q = (p^0 + q^0;\, p^j\; e^{\frac{1-\beta}{2\kappa}q^0} + q^j\; e^{-\frac{1+\beta}{2\kappa}p^0})\,, \qquad \ominus_\beta p = (-p^0;\, -e^{\frac{-\beta}{\kappa}p^0} p^i)\,.$$

each choice of β corresponds to a *choice of coordinates* on the manifold B.

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for $\beta = 1$ we have "flat slicing" coordinates

$$\begin{aligned}
\eta_0(p_0, \mathbf{p}) &= \kappa \sinh p_0/\kappa + \frac{\mathbf{p}^2}{2\kappa}\, e^{p_0/\kappa}, \\
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\end{aligned}$$

corresponds to "bicrossproduct" basis of $U(\mathfrak{k})$ introduced in Majid-Ruegg ('94) to prove that

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in the limit $\kappa \longrightarrow \infty$ recover ordinary Poincaré algebra

A new quantization ambiguity

Functions on the deformed mass-shell $\phi \in C^{\infty}(M^{\kappa}_{m})$ defined by the "wave equation"

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No preferred choice of translation generators from which we can define an **energy** coordinate on M_m^κ and thus **no preferred choice of J and P^+** to define one-particle Hilbert space.

κ-one-particle Hilbert space

Hilbert space construction for $\beta = 1$ (bicrossproduct basis), massless case

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κ-**one-particle Hilbert space**: $\mathcal{H}_\kappa$ functions on $M^{\kappa+}$, **positive energy**, equipped with $(\cdot,\cdot)_\kappa$ and modes truncated at κ

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with **same energy** and different linear momentum

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  with **same energy** and different linear momentum

$$\mathbf{K_{12}} = \mathbf{k_1} \oplus \mathbf{k_2} = \mathbf{k_1} + (1-\epsilon_1)\mathbf{k_2}$$
$$\mathbf{K_{21}} = \mathbf{k_2} \oplus \mathbf{k_1} = \mathbf{k_2} + (1-\epsilon_2)\mathbf{k_1}$$

given n-different modes one has $n!$ different n-particle states, one for each permutation of the n modes $\mathbf{k_1}\,, \mathbf{k_2} \ldots \mathbf{k_n}$

Hidden entanglement at the Planck scale

The non-trivial algebraic structure of κ-translations endows the Fock space with a "fine structure"

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$$\begin{aligned} |\epsilon\rangle \otimes |\uparrow\rangle &= |\mathbf{k_1 k_2}\rangle_\kappa \\ |\epsilon\rangle \otimes |\downarrow\rangle &= |\mathbf{k_2 k_1}\rangle_\kappa \end{aligned}$$

with $\epsilon = \epsilon(\mathbf{k_1}) + \epsilon(\mathbf{k_2})$

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- e.g. the state superposition of two total "classical" energies $\epsilon_A = \epsilon(\mathbf{k_1}_A) + \epsilon(\mathbf{k_2}_A)$ and $\epsilon_B = \epsilon(\mathbf{k_1}_B) + \epsilon(\mathbf{k_2}_B)$ can be entangled with the additional hidden modes e.g.

$$|\Psi\rangle = 1/\sqrt{2}(|\epsilon_A\rangle \otimes |\uparrow\rangle + |\epsilon_B\rangle \otimes |\downarrow\rangle)$$

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a new window to phenomenological effects??

Conclusions

- Relativistic symmetries can be deformed to allow **“curvature” for momentum space**
- Motivations to look at such deformations come form **QG and NCFT** scenarios
- Quantization of (free) field theories with group valued momenta leads to **ambiguities** related to the different choices on coordinate functions on the momentum manifold
- Physical interpretation of such ambiguities?
- At the multiparticle level the non-trivial behaviour of field modes leads to a **deformed Fock space**: interesting **entanglement** phenomena can take place
- What role for “trans-planckian” issues in semiclassical gravity (BH evaporation, Inflation)??

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