$\boldsymbol{U(1)}$ gauge field theory on $\boldsymbol{\kappa}$-Minkowski space-time revisited

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- M. Dimitrijević, L. J., work in progress
- M. Dimitrijević, L. J., L. Moeller, $\boldsymbol{U(1)}$ gauge field theory on kappa-Minkowski space, JHEP 0509 (2005) 068
- Talks given here: A. Sitarz, A. Schenkel

PLAN OF THE TALK

- OLD RESULTS
  - The algebra
  - $\star$-product and Seiberg-Witten Mapp
  - The action
- GEOMETRIC PICTURE
- CONCLUDING REMARKS

ALGEBRA

$$[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}_\lambda \hat{x}^\lambda$$

$$C^{\mu\nu}_\lambda = a^n(\delta^\mu_n \delta^\nu_\lambda - \delta^\nu_n \delta^\mu_\lambda)$$

$\mu = 0, 1, \ldots, n; j = 0, \cdots, n-1$
(n+1)-dimensional Minkowski space, $\eta = diag(1, -1, \ldots, -1)$,
$a^n = a = 1/\kappa$

$$[\hat{x}^n, \hat{x}^j] = ia\hat{x}^j$$

$$[\hat{\partial}_i, \hat{x}^\mu] = \eta^\mu_i - ia\eta^\mu_n \hat{\partial}_i$$

$$[M^{in}, \hat{x}^\mu] = \eta^{\mu n}\hat{x}^i - \eta^{\mu i}\hat{x}^n + iaM^{i\mu}$$

$$[M^{in}, \hat{\partial}_j] = \eta^i_j \frac{e^{2ia\hat{\partial}_n} - 1}{2ia} - \frac{ia}{2}\eta^i_j \hat{\partial}^l \hat{\partial}_l + ia\hat{\partial}^i \hat{\partial}_j$$

Algebra of Lorentz generators is unchanged. Both, derivatives and generators of rotations have non-trivial coproduct (deformed Leibniz rule)!

$$\hat{\partial}_i(\hat{f} \cdot \hat{g}) = (\hat{\partial}_i \hat{f}) \cdot \hat{g} + (e^{ia\hat{\partial}_n} \hat{f}) \cdot (\hat{\partial}_i \hat{g})$$

$$M^{in}(\hat{f} \cdot \hat{g}) = (M^{in} \hat{f}) \cdot \hat{g} + (e^{ia\hat{\partial}_n} \hat{f}) \cdot (M^{in} \hat{g}) + ia(\hat{\partial}_k \hat{f}) \cdot (M^{ik} \hat{g})$$

Construct derivatives with usual transformation properties under rotations

$$[M^{\mu\nu}, \hat{D}_\rho] = \eta^\nu_\rho \hat{D}^\mu - \eta^\mu_\rho \hat{D}^\nu$$

$$\hat{D}_n = \frac{1}{a} \sin(a\hat{\partial}_n) - \frac{ia}{2} \hat{\partial}^l \hat{\partial}_l e^{-ia\hat{\partial}_n}$$

$$\hat{D}_i = \hat{\partial}_i e^{-ia\hat{\partial}_n}$$

- Fields as formal power series in coordinates

$$\hat{\Phi}(\hat{x}) = \sum_\mu \hat{\Phi}_{\mu_1 \ldots \mu_n} : (\hat{x}^1)^{\mu_1} \cdots (\hat{x}^n)^{\mu_n} :$$

- Gauge theories are defined by

$$\delta_{\{\alpha\}}\hat{\psi}(\hat{x}) = i\hat{\Lambda}_{\{\alpha\}}\hat{\psi}(\hat{x})$$

$$\delta_{\{\alpha\}}(\hat{\mathcal{D}}_\mu\hat{\psi}(\hat{x})) = i\hat{\Lambda}_{\{\alpha\}}\hat{\mathcal{D}}_\mu\hat{\psi}(\hat{x});\ \hat{\mathcal{D}}_\mu = \hat{D}_\mu - i\hat{V}_\mu$$

$$(\delta_{\{\alpha\}}\delta_{\{\beta\}} - \delta_{\{\beta\}}\delta_{\{\alpha\}})\hat{\psi}(\hat{x}) = \delta_{\{\alpha\times\beta\}}\hat{\psi}(\hat{x})$$

- Gauge fields are enveloping-algebra-valued (due to noncommutativity) and derivative-valued (due to deformed Leibniz rule)!

$$\hat{\mathcal{F}}_{\mu\nu} = i[\hat{\mathcal{D}}_\mu, \hat{\mathcal{D}}_\nu]$$

$$\hat{\mathcal{F}}_{\mu\nu} = \hat{F}_{\mu\nu} + \hat{T}^{\rho}_{\mu\nu}\hat{\mathcal{D}}_\rho + \cdots + \hat{T}^{\rho_1\cdots\rho_l}_{\mu\nu} : \hat{\mathcal{D}}_{\rho_1}\ldots\hat{\mathcal{D}}_{\rho_l} : + \ldots$$

Use only curvature-like terms $\hat{F}_{\mu\nu}$ to construct an effective action!

$\star$-PRODUCT AND SEIBERG-WITTEN MAP

Symmetric $\star$-product, up to first order

$$f(x) \star g(x) = f(x)g(x) + \tfrac{i}{2}C^{\mu\nu}_{\lambda}x^{\lambda}\partial_{\mu}f(x)\partial_{\nu}g(x)$$

Realize the algebra on space of ordinary commutative variables multiplied with an associative $\star$-product (PBW property)
e.g.

$$D^*_n f(x) = \left(\frac{1}{a}\sin(a\partial_n) - \frac{\cos(a\partial_n) - 1}{ia\partial_n^2}\partial_j\partial^j\right)$$

Use Seiberg-Witten map to express nc variables (gauge parameter, fields) in terms of commutative ones, keeping the same number of degrees of freedom as in commutative (Lie algebra-valued!) case. One assumes that both, transformation parameter $\Lambda_{\{\alpha\}}$ and matter field ψ depend on the usual Lie algebra-valued gauge field $A^0_{\mu a}$.

Expliticit solutions are constructed using consistency condition

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi = \delta_{-i[\alpha,\beta]}\psi$$

and

$$\delta_\alpha \psi(x) = i\Lambda_\alpha(x) \star \psi(x)$$

$$\delta_\alpha(\tilde{\mathcal{D}}_\mu \psi(x)) = i\Lambda_\alpha(x) \star \tilde{\mathcal{D}}_\mu \psi(x)$$

order by order (no closed expressions).

In the first order we have for example

$$\Lambda_\alpha = \alpha - \frac{1}{4} C^{\rho\sigma}_\lambda x^\lambda \{A^0_\rho, \partial_\sigma \alpha\}$$

$$\psi = \psi^0 - \frac{1}{2} C^{\rho\sigma}_\lambda x^\lambda A^0_\rho \partial_\sigma \psi^0 + \frac{i}{8} C^{\rho\sigma}_\lambda x^\lambda [A^0_\rho, A^0_\sigma] \psi^0$$

$$\tilde{V}_i = A^0_i - iaA^0_i \partial_n - \frac{ia}{2} \partial_n A^0_i - \frac{a}{4} \{A^0_n, A^0_i\} + \frac{1}{4} C^{\rho\sigma}_\lambda x^\lambda \Big(\{F^0_{\rho i}, A^0_\sigma\} - \{A^0_\rho, \partial_\sigma A^0_i\} \Big)$$

$$\tilde{V}_n = A^0_n - iaA^{0j} \tilde{\partial}_j - \frac{ia}{2} \partial_j A^{0j} - \frac{a}{2} A^0_j A^{0j} + \frac{1}{4} C^{\rho\sigma}_\lambda x^\lambda \Big(\{F^0_{\rho n}, A^0_\sigma\} - \{A^0_\rho, \partial_\sigma A^0_n\} \Big)$$

$$
\begin{aligned}
F_{ij} &= F^0_{ij} - ia\mathcal{D}^0_n F^0_{ij} \\
&\quad + \frac{1}{4} C^{\rho\sigma}_{\lambda} x^{\lambda} \Big(2\{F^0_{\rho i}, F^0_{\sigma j}\} + \{\mathcal{D}^0_{\rho} F^0_{ij}, A^0_{\sigma}\} - \{A^0_{\rho}, \partial_{\sigma} F^0_{ij}\} \Big) \\
T^{\mu}_{ij} &= -2ia\delta^{\mu}_n F^0_{ij} \\
F_{nj} &= F^0_{nj} - \frac{ia}{2} \mathcal{D}^{\mu 0} F^0_{\mu j} \\
&\quad + \frac{1}{4} C^{\rho\sigma}_{\lambda} x^{\lambda} \Big(2\{F^0_{\rho n}, F^0_{\sigma j}\} + \{\mathcal{D}^0_{\rho} F^0_{nj}, A^0_{\sigma}\} - \{A^0_{\rho}, \partial_{\sigma} F^0_{nj}\} \Big) \\
T^{\mu}_{nj} &= -ia\eta^{\mu l} F^0_{lj} - ia\delta^{\mu}_n F^0_{nj}
\end{aligned}
$$

ACTION

First we need an integral with trace property

$$\int d^{n+1}x\ \mu(x)(f \star g) = \int d^{n+1}x\ \mu(x)(g \star f)$$

where

$$\partial_n \mu(x) = 0, \qquad x^j \partial_j \mu(x) = -n\mu(x)$$

Introduce derivatives antihermitean under the integral with a measure μ

$$\int d^{n+1}x\, \mu(x)\, \bar{f} \star \tilde{D}^*_\alpha g = -\int d^{n+1}x\, \mu(x)\, \overline{\tilde{D}^*_\alpha f} \star g$$

The derivatives $\tilde{D}^*_\alpha$ are obtained by substituting

$$\partial_i \longrightarrow \tilde{\partial}_i = \partial_i + \frac{\partial_i \mu}{2\mu}, \qquad \partial_n \longrightarrow \tilde{\partial}_n = \partial_n$$

Now define the action as

$$S = \int d^{n+1}x\ \mu(x)\mathcal{L}$$

and Lagrangian density as

$$\lim_{a\to 0}\mu(x)\mathcal{L} = \mathcal{L}^0$$

Matter part,

$$S_m = \int d^{n+1}x\ \mu(x)\left(\tilde{\bar{\psi}}\star(i\gamma^\mu\tilde{D}^*_\mu + \gamma^\mu\tilde{V}_\mu\star - m)\tilde{\psi}\right)$$

where $\tilde{\psi} = \mu^{-1/2}\psi$

Gauge part

$$S_g = -\frac{1}{4}\int d^{n+1}x\ \mu(x)\mathrm{Tr}\left(X_2\star F_{\mu\nu}\star F^{\mu\nu}\right)$$

where

$$\delta_\alpha X_2 = i[\Lambda_\alpha \overset{\star}{,} X_2]$$

After rescaling, expanding ⋆-product, inserting SW map:

$$\begin{aligned}
S \;=\; \int d^{n+1}x \Big\{ & \bar{\psi}^0 (i\gamma^\mu \mathcal{D}^0_\mu - m)\psi^0 - \frac{1}{4} F^0_{\mu\nu} F^{0\mu\nu} - \\
& -\frac{1}{4} C^{\rho\sigma}_\lambda \left(\bar{\psi}^0 \gamma_\rho \mathcal{D}^0_\sigma \mathcal{D}^{0\lambda} \psi^0 + \overline{\mathcal{D}^0_\sigma \mathcal{D}^{0\lambda} \psi^0} \gamma_\rho \psi^0 \right) - \\
& -\frac{1}{4} (1 - 8b_1) C^{\rho\sigma}_\lambda x^\lambda \bar{\psi}^0 F^0_{\rho\sigma} (i\gamma^\mu (\mathcal{D}^0_\mu \psi^0) - m\psi^0) - \\
& -\frac{i}{2} (1 - 2b_2) C^{\rho\sigma}_\lambda x^\lambda \bar{\psi}^0 \gamma^\mu F^0_{\mu\rho} (\mathcal{D}^0_\sigma \psi^0) + \\
& + i\, b_3 C^{\rho\sigma}_\lambda x^\lambda \bar{\psi}^0 \gamma^\mu F^0_{\rho\sigma} (\mathcal{D}^0_\mu \psi^0) - \\
& -2i \left(b_1 - \tfrac{1}{4}(b_2 - 2b_3)\right) C^{\rho\sigma}_\lambda x^\lambda \bar{\psi}^0 \gamma^\mu (\mathcal{D}^0_\sigma F^0_{\mu\rho}) \psi^0 - \\
& -\frac{ia}{4} (n + 8b_1 - 2nb_2 + 4b_3 - 1) \bar{\psi}^0 \gamma^\mu F^0_{\mu n} \psi^0 - \\
& -\frac{1}{2} (1 - 2b_2) C^{\rho\sigma}_\lambda x^\lambda F^{0\mu\nu} F^0_{\mu\rho} F^0_{\nu\sigma} + \\
& + \frac{1}{8} (1 - 8b_3 - 2b_4) C^{\rho\sigma}_\lambda x^\lambda F^{0\mu\nu} F^0_{\mu\nu} F^0_{\rho\sigma} \Big\}
\end{aligned}$$

All constants b_i are undertemined, they are consequence of ambiguity of the SW map!

TWIST APPROACH IN GENERAL

Take bi-differential operator

$$\boldsymbol{\mathcal{F}^{-1} := \exp\left(\frac{1}{2}\theta^{ab} X_a \otimes X_b\right)}$$

with $[\boldsymbol{X_a}, \boldsymbol{X_b}] = \boldsymbol{0}$

Define star-product

$$\boldsymbol{f * g = \mu\left\{\mathcal{F}^{-1} f \otimes g\right\}}$$

Differential calculus include deformed wedge product

$$\boldsymbol{\omega \wedge_* \omega' := \wedge\{\mathcal{F}^{-1}\omega \otimes \omega'\}}$$

undeformed exterior derivative $\boldsymbol{d}$

$$\boldsymbol{d(\omega \wedge_* \omega') = (d\omega) \wedge_* \omega' + (-1)^{\deg(\omega)} \omega \wedge_* (d\omega')}$$

pairing/contraction

$$\boldsymbol{\langle \omega, X \rangle_* := \langle ., . \rangle \left(\mathcal{F}^{-1}\omega \otimes X\right)}$$

Gauge theory defined (almost) as before

$$\boldsymbol{\delta_{\{\alpha\}}\psi(x) = i\Lambda_{\{\alpha\}}\psi(x)}$$

$$(\delta_{\{\alpha\}}\delta_{\{\beta\}} - \delta_{\{\beta\}}\delta_{\{\alpha\}})\psi(x) = \delta_{\{\alpha\times\beta\}}\psi(x)$$

$$F = dA - iA \wedge_* A$$

$$D\psi = d\psi - iA \star \psi$$

TWIST APPROACH - PARTICULAR EXAMPLE

We take

$$\mathcal{F}^{-1} = \exp\left(\frac{ia}{2}(\partial_n \otimes x^j\partial_j - x^j\partial_j \otimes \partial_n)\right)$$

Star-product expanded

$$f \star g = f \cdot g + \frac{ia}{2}x^j((\partial_n f)\partial_j g - (\partial_j f)\partial_n g) + \mathcal{O}(a^2)$$

We have undeformed wedge product

$$\omega \wedge_* \omega' = \omega \wedge \omega'$$

However,

$$dx^j \star f = e^{-ia\partial_n} f \star dx^j$$

and

$$df = \partial_\mu f \cdot dx^\mu = \partial^*_\mu f \star dx^\mu$$

SEIBERG-WITTEN MAP and ACTION

We solve SW map, for gauge parameter and matter field same as before. For gauge field we solve:

$$\delta_{\{\alpha\}} A_n = \partial_n \Lambda_{\{\alpha\}} + i[\Lambda_{\{\alpha\}} \stackrel{\star}{,} A_n]$$

$$\delta_{\{\alpha\}} A_j = \partial_j \Lambda_{\{\alpha\}} + i\Lambda_{\{\alpha\}} \star A_j - iA_j \star e^{-ia\partial_n} \Lambda_{\{\alpha\}}$$

and obtain

$$A_j = A_j^0 - \frac{ia}{2}\partial_n A_j^0 - \frac{a}{4}\{A_n^0, A_j^0\} + \frac{1}{4}C_\lambda^{\rho\sigma} x^\lambda \Big(\{F_{\rho i}^0, A_\sigma^0\} - \{A_\rho^0, \partial_\sigma A_j^0\} \Big)$$

$$A_n = A_n^0 + \frac{1}{4}C_\lambda^{\rho\sigma} x^\lambda \Big(\{F_{\rho n}^0, A_\sigma^0\} - \{A_\rho^0, \partial_\sigma A_n^0\} \Big)$$

The action is defined as:

$$S = \int F \wedge_* \tilde{F}$$

,

However, we don't have Hodge dual $\tilde{F}$! Simple guess

$$\tilde{F} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \star dx^{\mu} \wedge dx^{nu}$$

is not transforming covariantly, so we construct one using SW map.

The gauge part of the action is the same in first order in expansion.

The matter part of action is (Castellani talk)

$$S_m = \int ((D\psi)\bar{\psi} - \psi(\bar{D\psi})) \wedge_* V \wedge_* V \wedge_* V\gamma_5$$

After expansion we obtain

$$S_m = \int d^4x \left\{ \bar{\psi}^0(i\gamma^{\mu}\mathcal{D}^0_{\mu} - m)\psi^0 - \frac{1}{4}C^{\rho\sigma}_{\lambda}x^{\lambda}\bar{\psi}^0 F^0_{\rho\sigma}(i\gamma^{\mu}(\mathcal{D}^0_{\mu}\psi^0) - m\psi^0) - \right.$$
$$\left. - \frac{i}{2}C^{\rho\sigma}_{\lambda}x^{\lambda}\bar{\psi}^0\gamma^{\mu}F^0_{\mu\rho}(\mathcal{D}^0_{\sigma}\psi^0) \underline{- \frac{3}{2}a\bar{\psi}^0\gamma^{\mu}\mathcal{D}^0_n\mathcal{D}^0_{\mu}\psi + \frac{1}{2}a\bar{\psi}^0\gamma^{j}\mathcal{D}^0_n\mathcal{D}^0_j\psi} \right\}$$

Compare with previous:

$$\ldots - \frac{1}{4}C^{\rho\sigma}_{\lambda}\left(\bar{\psi}^0\gamma_{\rho}\mathcal{D}^0_{\sigma}\mathcal{D}^{0\lambda}\psi^0 + \overline{\mathcal{D}^0_{\sigma}\mathcal{D}^{0\lambda}\psi^0}\gamma_{\rho}\psi^0\right)\ldots$$

CONCLUDING REMARKS

- Geometric picture still incomplete.
- Disentangling space-time and internal symmetries?
- Higher-dimensional differential calculus - gauge theory after reduction?