Tube dislocations in gravity

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The problem: $g_{\mu\nu}(x)$ - metric

$$\Gamma_{\mu\nu\rho}=\partial_\mu g_{\nu\rho}+\partial_\nu g_{\mu\rho}-\partial_\rho g_{\mu\nu}$$ - Christoffel's symbols

$$R_{\mu\nu\rho}{}^{\sigma}=\partial_\mu\Gamma_{\nu\rho}{}^{\sigma}-\partial_\nu\Gamma_{\mu\rho}{}^{\sigma}-\Gamma_{\mu\rho}{}^{\varsigma}\Gamma_{\nu\varsigma}{}^{\sigma}+\Gamma_{\nu\rho}{}^{\varsigma}\Gamma_{\mu\varsigma}{}^{\sigma}$$ - curvature

$$R_{\mu\nu}=R_{\mu\rho\nu}{}^{\rho}$$ - Ricci tensor

$$R=g^{\mu\nu}R_{\mu\nu}$$ - scalar curvature

Einstein's equations:

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=-\frac{1}{2}T_{\mu\nu}$$

$T_{\mu\nu}$ - energy-momentum tensor

$$g\in\theta(x)=\begin{cases}0, & x\le 0\\ 1, & x>0\end{cases}$$ - step function

$\Gamma\sim\delta(x)$ $R\sim\delta^2(x)$ - is not defined

Tube dislocation in the linear elasticity theory

$\mathbb{R}^3, \quad x^i, y^i \quad i = 1, 2, 3$ - Euclidean space with Cartesian coordinates (infinite homogeneous and isotropic elastic media or eather in general relativity)

$\delta_{ij} = \mathrm{diag}(+++)$ - Euclidean metric

$u^i(x)$ - displacement vector field

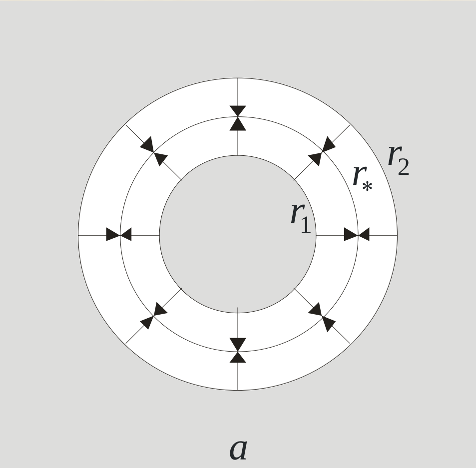
$$\boxed{(1-2\sigma)\Delta u_i + \partial_i \partial_j u^j = 0}$$

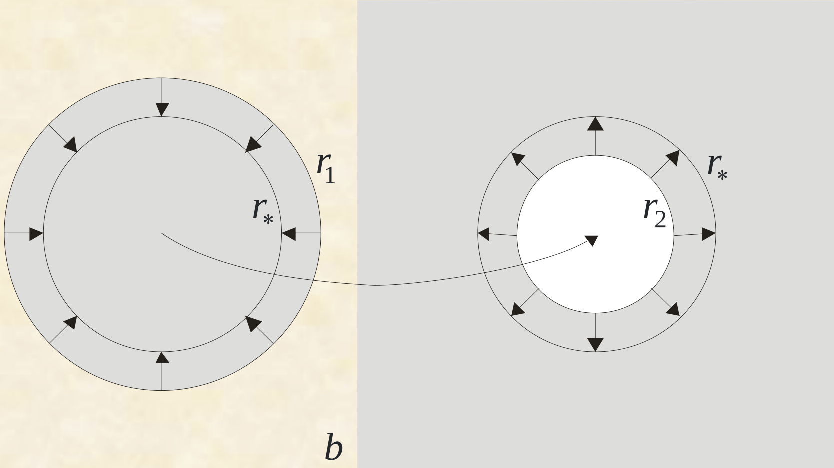
-equations for the equilibrium state (Newton’s and Hook’s laws)

$f^i(x) = 0$ - external forces

$\sigma = \dfrac{\lambda}{2(\lambda + \mu)}$ - the Poisson ratio $\quad -1 \le \sigma \le \dfrac{1}{2}$

λ, μ - Lame coefficients





$y^i \to x^i(y) = y^i + u^i(x)$ - definition of the displacement vector field

r, φ, z - cylindrical coordinates

u^r, u^φ, u^z - displacement vector field

$$u^\varphi = 0, \;\; u^z = 0 \qquad u^r = u = \begin{cases} u_{\text{in}} \\ u_{\text{ex}} \end{cases}$$

$$u\,|_{r=0} = 0, \;\; u\,|_{r=\infty} = 0, \;\; \left.\frac{du_{\text{in}}}{dr}\right|_{r=r^*} = \left.\frac{du_{\text{ex}}}{dr}\right|_{r=r^*}$$ - boundary conditions

$$\boxed{\partial_r \left[\frac{1}{r} \partial_r (ru) \right] = 0}$$ - equation of equilibrium

$$l = r_2 - r_1 \qquad r_* = \frac{r_2 + r_1}{2}$$

$$u = ar - \frac{b}{r} = \begin{cases} u_{\text{in}} = ar, & a > 0, \\ u_{\text{ex}} = -\dfrac{b}{r}, & b > 0. \end{cases}$$ - solution

$$a = \frac{l}{2r_*}, \quad b = \frac{lr_*}{2}$$

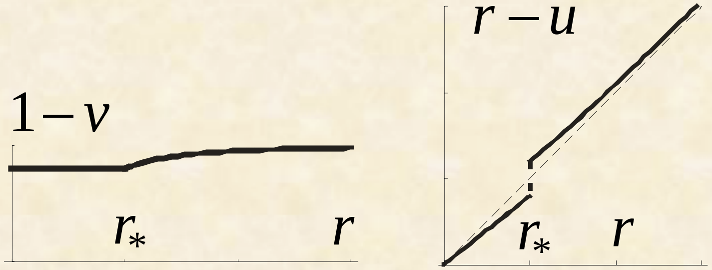
The geometry of the tube dislocation

$$g_{\mu\nu} = \frac{\partial y^{\rho}}{\partial x^{\mu}} \frac{\partial y^{\sigma}}{\partial x^{\nu}} \hat{g}_{\rho\sigma}$$ - the induced metric

$\hat{g}_{\rho\sigma}$ -Euclidean metric in cylindrical coordinates

$$u(r) = \begin{cases} ar, & r < r_* \\ -\dfrac{b}{r}, & r > r_* \end{cases} \qquad v(r) = \begin{cases} a, & r < r_* \\ \dfrac{b}{r^2}, & r > r_* \end{cases} \qquad u' = v - l\delta(r - r_*)$$

$$ds^2 = (1-v)^2 dr^2 + (r-u)^2 d\varphi^2 + dz^2$$ - the induced metric



Scalar curvature: $$\tilde{R} = \frac{2l}{(r-u)(1-v)^2} \left[\delta'(r-r_*) + \frac{v'}{1-v} \delta(r-r_*) \right]$$

Einstein's equations

$$\sqrt{g} \left(\tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R} \right) = -\frac{1}{2} T_{\mu\nu}$$

Energy momentum tensor

$$T_{zz} = \frac{4lr_*}{2r_* - l} \delta'(r - r_*)$$

Summary of the geometric theory of defects

(physical interpretation of Riemann-Cartan geometry)

Media with dislocations and disclinations $=$

$= \mathbb{R}^3$ with a given Riemann-Cartan geometry

Independent variables $\begin{cases} e_\mu{}^i & \text{- triad field} \\ \omega_\mu{}^{ij} & \text{- SO(3)-connection} \end{cases}$

$$T_{\mu\nu}{}^i = \partial_\mu e_\nu{}^i - \omega_\mu{}^{ij} e_{\nu j} - (\mu \leftrightarrow \nu)$$

- torsion (surface density of the Burgers vector)

$$R_{\mu\nu}{}^{ij} = \partial_\mu \omega_\nu{}^{ij} - \omega_\mu{}^{ik} \omega_{\nu k}{}^j - (\mu \leftrightarrow \nu)$$

- curvature (surface density of the Frank vector)

Elastic deformations: $R_{\mu\nu}{}^{ij} = 0, \qquad T_{\mu\nu}{}^i = 0$

Dislocations: $R_{\mu\nu}{}^{ij} = 0, \qquad T_{\mu\nu}{}^i \neq 0$

Disclinations: $R_{\mu\nu}{}^{ij} \neq 0, \qquad T_{\mu\nu}{}^i = 0$

Dislocations and disclinations: $R_{\mu\nu}{}^{ij} \neq 0, \qquad T_{\mu\nu}{}^i \neq 0$

The free energy

$$S = \int d^3 x \, e L, \quad e = \det e_\mu{}^i$$

$$\boxed{L = \kappa \tilde{R} - \gamma R_{[ij]} R^{[ij]}}$$

$\tilde{R}(e)$ - the Hilbert-Einstein action

$R_{[ij]}(e,\omega)$ - antisymmetric part of Ricci tensor

equations of equilibrium admit solutions

$$\begin{cases} R = 0, \quad T \neq 0 & \text{- only dislocations} \\ R \neq 0, \quad T = 0 & \text{- only disclinations} \\ R = 0, \quad T = 0 & \text{- elastic deformations} \end{cases}$$

The action is invariant with respect to general coordinate transformations and local SO(3) rotations. Therefore we must fix the gauge.

Elastic gauge

$$(1-2\sigma)\Delta u_i + \partial_i \partial_j u^j = 0$$ - the elasticity equation

$$\sigma = \frac{\lambda}{2(\lambda+\mu)}$$ - Poisson ratio

$$e_{\mu i} \approx \delta_{\mu i} - \partial_\mu u_i$$ - the linear approximation

$$\boxed{(1-2\sigma)\partial^\mu e_{\mu i} + \partial_i e_\mu{}^\mu = 0}$$ - the elastic gauge (fixes diffeomorphisms)

Lorentz gauge

$$\boxed{\partial^\mu \omega_\mu{}^{ij} = 0}$$ - the Lorenz gauge (fixes SO(3)-invariance)

If there are no disclinations $R_{\mu\nu}{}^{ij} = 0$, then $\omega_{\mu i}{}^j = l_{\mu i}{}^j = (\partial_\mu S^{-1})_i{}^k S_k{}^j$

pure gauge

$$\partial^\mu l_\mu{}^{ij} = 0$$ - principal chiral SO(3)-model

Tube dislocation in the geometric theory of defects

No disclinations: $R_{\mu\nu i}{}^{j}=0 \quad \Rightarrow \omega_\mu{}^{ij}=0$

$$\sqrt{|g|}\left(\tilde{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\tilde{R}\right)=-\frac{1}{2}T_{\mu\nu}$$ - Einstein's equations

$$(1-2\sigma)\partial^\mu e_{\mu i}+\partial_i e_\mu{}^\mu=0$$ - the elastic gauge

$$T_{zz}=L\delta'(r-r_*)$$

$$L=\text{const}$$

$$ds^2=A^2dr^2+B^2d\varphi^2+dz^2$$ - ansatz for metric

$A(r),\ B(r)$ - unknown functions

Asymptotically flat boundary conditions: $B_{\text{in}}\big|_{r=0}=0,\quad A_{\text{in}}\big|_{r=0}<\infty,\quad B_{\text{ex}}\big|_{r\to\infty}=r$

Matching condition: $A_{\text{in}}\big|_{r=r_*}=A_{\text{ex}}\big|_{r=r_*}$

$$ds^2_{\text{in}}=(1-a)^2dr^2+(1-a)^2r^2d\varphi^2+dz^2$$

$$ds^2_{\text{ex}}=\left(1-\frac{b}{r}\right)^2dr^2+\left(r+\frac{b}{r}\right)^2d\varphi^2+dz^2$$

- the exact solution coincides with the solution in elasticity theory

$$a=\frac{L}{L+4r_*},\quad b=\frac{Lr_*^2}{L+4r_*}$$

Tube dislocation in general relativity

Einstein's equations:

$$\boxed{\sqrt{g}\left(\tilde{R}_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}\tilde{R}\right)=-\frac{1}{2}T_{\alpha\beta},}\qquad \alpha,\beta=0,1,2,3$$

$$T_{00}=-T_{zz}=L\delta'(r-r_*)$$ - energy-momentum tensor

$$L=\mathrm{const}$$

$$\boxed{(1-2\sigma)\partial^{\mu}e_{\mu i}+\partial_i e_{\mu}{}^{\mu}=0}$$ - the elastic gauge

The exact solution for the tube dislocation

$$\boxed{\begin{aligned} ds^2_{\mathrm{in}}&=dt^2-(1-a)^2dr^2-(1-a)^2r^2d\varphi^2-dz^2\\ ds^2_{\mathrm{ex}}&=dt^2-\left(1-\frac{b}{r}\right)^2dr^2-\left(r+\frac{b}{r}\right)^2d\varphi^2-dz^2\end{aligned}}$$

$$a=\frac{L}{L+4r_*},\quad b=\frac{Lr_*^2}{L+4r_*}$$

Conical tube dislocation

Einstein's equations: $$\boxed{\sqrt{g}\left(\tilde{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\tilde{R}\right)=-\frac{1}{2}T_{\mu\nu}}\qquad \mu,\nu=1,2,3$$

$T_{zz}=\Theta\delta(r-r_*)$ - the source $\Theta=\text{const}$

$$\boxed{(1-2\sigma)\partial^{\mu}e_{\mu i}+\partial_i e_{\mu}{}^{\mu}=0}$$

- the elastic gauge

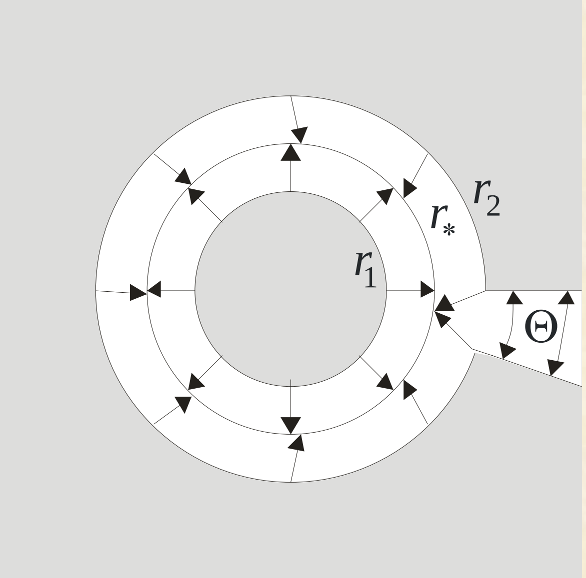
r,φ,z - cylindrical coordinates $0\le r\le R$

$ds^2=A^2dr^2+B^2d\varphi^2+dz^2$ - ansatz for the metric

Boundary conditions: $B_{\text{in}}\big|_{r=0}=0,\quad A_{\text{in}}\big|_{r=0}<\infty,\quad A_{\text{ex}}\big|_{r=R}=1$

Matching condition: $A_{\text{in}}\big|_{r=r_*}=A_{\text{ex}}\big|_{r=r_*}$

Solution for the conical tube dislocation



$$ds^2 = A^2dr^2 + B^2d\varphi^2 + dz^2$$ - the exact solution

$$A = \begin{cases} D, & 0 \le r \le r_* \\ \dfrac{1}{\alpha}\left(E_1\gamma_1 r^{\gamma_1 - 1} + E_2\gamma_2 r^{\gamma_2}\right), & r_* \le r \le R \end{cases}$$

$$B = \begin{cases} Dr, & 0 \le r \le r_* \\ E_1 r^{\gamma_1} + E_2 r^{\gamma_2}, & r_* \le r \le R \end{cases}$$

$\alpha = 1 + \Theta,\quad \Theta$ - the deficit angle

$\gamma_1 > 0,\quad \gamma_2 < 0$ are the roots of the equation $\gamma^2 + \dfrac{\Theta\sigma}{1-\sigma}\gamma - \alpha = 0$

$$D = \frac{\alpha}{\gamma_1 \dfrac{\alpha - \gamma_2}{\gamma_1 - \gamma_2}\left(\dfrac{R}{r_*}\right)^{\gamma_1 - 1} - \gamma_2 \dfrac{\alpha - \gamma_1}{\gamma_1 - \gamma_2}\left(\dfrac{R}{r_*}\right)^{\gamma_2 - 1}}$$

$$E_1 = D\frac{\alpha - \gamma_2}{\gamma_1 - \gamma_2} r_*^{-\gamma_1 + 1},\qquad E_2 = -D\frac{\alpha - \gamma_1}{\gamma_1 - \gamma_2} r_*^{-\gamma_2 + 1}$$

Asymptotically flat wedge dislocation

Einstein's equations: $$\boxed{\sqrt{g}\left(\tilde{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\tilde{R}\right)=-\frac{1}{2}T_{\mu\nu}}\qquad \mu,\nu=1,2,3$$

$T_{zz}=-\Theta\delta(r-r_*)$ - the source $\quad \Theta=\text{const}$

$$\boxed{(1-2\sigma)\partial^{\mu}e_{\mu i}+\partial_i e_{\mu}{}^{\mu}=0}$$ - the elastic gauge

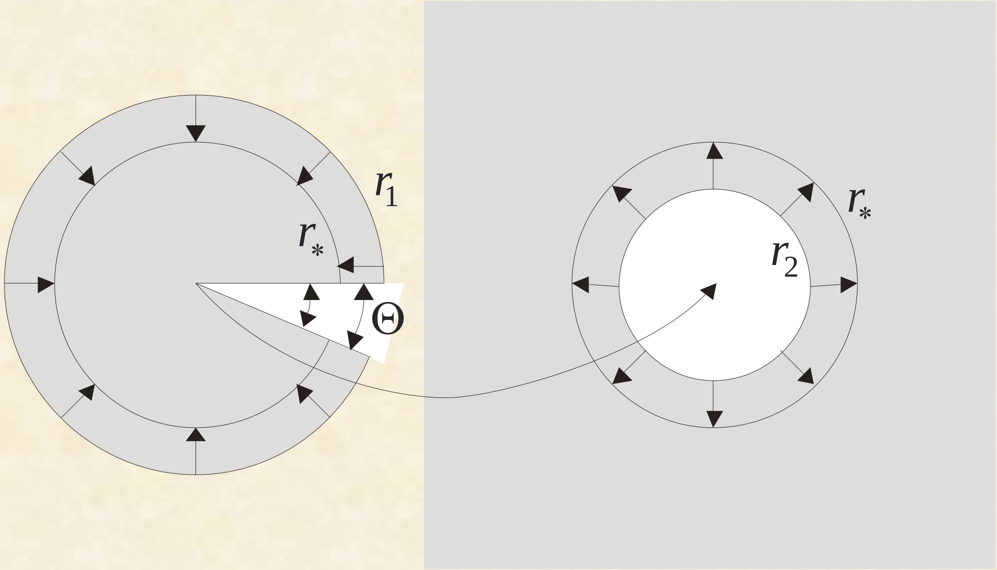
r,φ,z - cylindrical coordinates

Boundary conditions: $B_{\text{in}}\big|_{r=0}=0,\quad \left.\dfrac{rA_{\text{in}}}{B_{\text{in}}}\right|_{r=0}=\dfrac{\gamma_1}{\alpha}$ - conical singularity at the origin

$A_{\text{ex}}\big|_{r=\infty}=1$ - asymptotic flatness

Matching condition: $A_{\text{in}}\big|_{r=r_*}=A_{\text{ex}}\big|_{r=r_*}$

Solution for the asymptotically flat tube dislocation



$\alpha = 1 + \Theta,$ Θ - the deficit angle

$\Theta < 0$ - the wedge is cut out

$2\pi\alpha r_1 = 2\pi r_2$ - the continuity of metric components

The exact solution:

$$ds^2_{\text{in}} = \frac{4\gamma_1^2}{(\alpha+\gamma_1)^2}\left(\frac{r}{r_*}\right)^{2\gamma_1-2}\left(dr^2 + \frac{\alpha^2}{\gamma_1^2}r^2 d\varphi^2\right) + dz^2$$

$$ds^2_{\text{ex}} = \left[1 - \frac{\alpha-\gamma_1}{\alpha+\gamma_1}\left(\frac{r_*}{r}\right)^2\right]^2 dr^2 + \left[1 + \frac{\alpha-\gamma_1}{\alpha+\gamma_1}\left(\frac{r_*}{r}\right)^2\right]^2 r^2 d\varphi^2 + dz^2$$

$\gamma_1 > 0$ is the positive root of the equation $\gamma^2 + \dfrac{\Theta\sigma}{1-\sigma}\gamma - \alpha = 0$

Continuous distribution of tube dislocations

Einstein's equations: $$\boxed{\sqrt{g}\left(\tilde{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\tilde{R}\right)=-\frac{1}{2}T_{\mu\nu}}\qquad \mu,\nu=1,2,3$$

$T_{zz}=2f(r)$ - the source term $f(r)$ - an arbitrary function

$ds^2=A^2dr^2+B^2d\varphi^2+dz^2$ - ansatz for the metric

Einstein's equations: $$\boxed{\left(\frac{B'}{A}\right)'=f(r)}$$

The exact solution: $$B=\int_0^r dsA(s)\int_0^s dtf(t)+c_1\int_0^r dsA(s)+c_2\qquad c_{1,2}=\text{const}$$

The boundary condition: $$B\big|_{r=0}=0\quad\Rightarrow\quad c_2=0$$

$$\frac{B'}{A}\bigg|_{r=0}=1\quad\Rightarrow\quad c_1=1$$

The elastic gauge: $$\frac{B''}{F}-\frac{B'f}{F^2}+\frac{B'}{Fr}-\frac{B}{r^2}+\frac{\sigma}{1-2\sigma}\left(\frac{B''}{F}-\frac{B'f}{F^2}+\frac{B'}{r}-\frac{B}{r^2}\right)=0$$

$$F(r)=\int^r dsf(s)$$

Conclusion

1) A new family of exact solutions of Einstein’s equations is found

2) Physical interpretation of solutions: tube dislocations

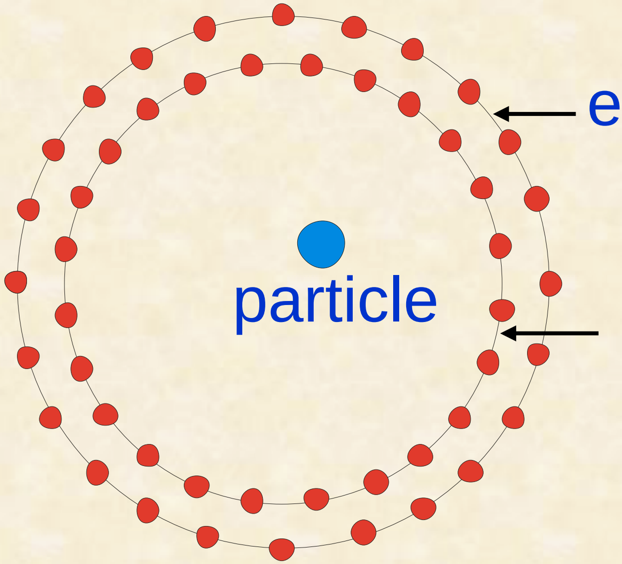
3) Metric components or its derivatives are discontinuous functions

4) All ambiguous terms in Einstein’s equations cancel

5) The source term (energy-momentum tensor) is proportional to $\delta'(r)$ or $\delta(r)$

May tube dislocations have any apllications ?

Double wall nonotube



particle

external wall: 24 atoms

internal wall: 23 atoms

Motion of a quantum particle inside a nanotube

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi \quad\Longrightarrow\quad i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\,g^{\mu\nu}\partial_\nu\psi\right)+\cancel{U\psi}$$