Divergences in QFT on the Noncommutative Minkowski Space with Grosse-Wulkenhaar Potential

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Motivation

- The introduction of the Grosse-Wulkenhaar potential on the noncommutative Euclidean space was a spectacular success (ϕ^4 renormalizable to all orders, asymptotically safe).
- Typically, NCQFTs behave quite differently on spaces with Euclidean and Lorentzian signature.

$\Rightarrow$ What happens if one puts the Grosse-Wulkenhaar potential on Minkowski space? [Fischer & Szabo 09]

Outline

The noncommutative Minkowski space

- The noncommutative Minkowski space is generated by coordinates x_μ that fulfill the canonical commutation relations

$$[x_\mu, x_\nu] = i\sigma_{\mu\nu}.$$

In the 2d case we have $\sigma = \lambda_{\mathrm{nc}}^2 \epsilon$.

- The $\star$-product is defined as

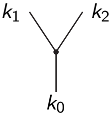
$$(f \star g)\hat{}(k) = (2\pi)^{-d/2} \int \mathrm{d}^d l \; e^{-\frac{i}{2} k\sigma l} \hat{f}(k-l)\hat{g}(l).$$

- One may define the $\star$-product at different points by

$$(f \otimes_\star g)\hat{}(k, \tilde{k}) = e^{-\frac{i}{2} k\sigma\tilde{k}} \hat{f}(k)\hat{g}(\tilde{k}).$$

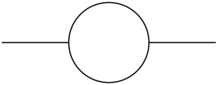
Euclidean NCQFT

In a Euclidean setting, modified Feynman rules can be derived formally in a path integral formalism, i.e. [Filk 96]



$$= e^{-\frac{i}{2} k_1 \sigma k_2} \delta(k_0 + k_1 + k_2).$$

In the planar graph



the phase factors at the vertices cancel. Thus, it is exactly as in the commutative case and has to be renormalized.

Euclidean NCQFT

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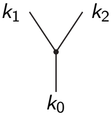
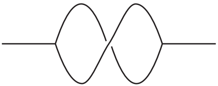


$$= e^{-\frac{i}{2}k_1\sigma k_2}\delta(k_0 + k_1 + k_2).$$

In the nonplanar graph



the phase factors add up, yielding the finite (for $|\sigma k|^{-1} < \infty$) loop integral

$$\int \mathrm{d}^4 l \ \hat{\Delta}_F(k - l)\hat{\Delta}_F(l)e^{-ik\sigma l}.$$

Euclidean NCQFT

- This UV/IR–mixing spoils renormalizability.

⇒ Grosse-Wulkenhaar potential.

The modified Feynman rules are only valid in the Euclidean setting. In the case of space/time noncommutativity $\sigma^{0i} \neq 0$,

- the connection to the Lorentzian metric is unclear,
- their naive application on Minkowski space leads to a violation of unitarity. [Gomis & Mehen 00]

The reason for the violation of unitarity is an inappropriate definition of time-ordering. It may be cured by using the Hamiltonian or the Yang-Feldman formalism. [Bahns et al 02]

The Grosse-Wulkenhaar potential

- In order to improve the IR behaviour, Grosse and Wulkenhaar added a quadratic potential to the Lagrangean:

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi + \Omega^2 \tilde{x}_\mu \phi \tilde{x}^\mu \phi + \mu^2 \phi^2 + g \phi^{\star 4}.$$

  Here

$$\tilde{x}_\mu = 2\sigma^{-1}_{\mu\nu} x^\nu.$$

- The value $\Omega = 1$ is called the self-dual point, as there the theory becomes self-dual in the sense of Langmann and Szabo.
- The $\phi^{\star 4}_4$ model is asymptotically safe, the self-dual point being the UV fixed point. [Grosse & Wulkenhaar 04; Disertori et al 07]
- After a switch to a Lorentzian metric, the quadratic potential will tend to $-\infty$ in some direction.

$\Rightarrow$ Stability?

The Yang-Feldman formalism

Idea: Perturbative recursive construction of the interacting field in terms of the free incoming field.

Example: $\phi^{\star 3}$ model, i.e., $P\phi = g\phi\star\phi$.

Ansatz: $\phi = \sum_{n=0}^{\infty} g^n \phi_n$.

$\Rightarrow$ $P\phi_n = \sum_{k=0}^{n-1} \phi_k \star \phi_{n-1-k}$.

ϕ_0 is the free field. We identify it with the incoming field. Higher order fields are thus obtained by convolution with Δ_{ret}:

$$\phi_1(x) = \int \mathrm{d}^4 y \; \Delta_{\text{ret}}(x,y) \phi_0(y) \star_y \phi_0(y)$$

$$\phi_2(x) = \int \mathrm{d}^4 y \mathrm{d}^4 z \; \Delta_{\text{ret}}(x,y) \Big\{ \phi_0(z) \star_z \phi_0(z) \Delta_{\text{ret}}(y,z) \star_y \phi_0(y) + \phi_0(y) \star_y \Delta_{\text{ret}}(y,z) \phi_0(z) \star_z \phi_0(z) \Big\}$$

The Yang-Feldman formalism

The contractions of two of the free fields ϕ_0 in

$$\phi_2(x) = \int \mathrm{d}^4y\mathrm{d}^4z \; \Delta_{\mathrm{ret}}(x,y)\Big\{\phi_0(z)\star_z\phi_0(z)\Delta_{\mathrm{ret}}(y,z)\star_y\phi_0(y) + \phi_0(y)\star_y\Delta_{\mathrm{ret}}(y,z)\phi_0(z)\star_z\phi_0(z)\Big\}$$

yield

$$\phi_2(x) = \int \mathrm{d}^4y\mathrm{d}^4z \; \Delta_{\mathrm{ret}}(x,y)\phi_0(z) \times \Big\{\Delta_{\mathrm{ret}}(y,z)\star_y\bar{\star}_z\Delta_+(z,y) + \Delta_{\mathrm{ret}}(y,z)\star_y\star_z\Delta_+(z,y) + \Delta_+(y,z)\star_y\bar{\star}_z\Delta_{\mathrm{ret}}(y,z) + \Delta_+(y,z)\star_y\star_z\Delta_{\mathrm{ret}}(y,z)\Big\}$$

plus the uncontracted part. Here $\bar{\star}$ is defined by $f\bar{\star}g = g\star f$, i.e., with σ replaced by $-\sigma$.

The Yang-Feldman formalism

In terms of the planar and the nonplanar product on $\mathbb{R}^{2d}$,

$$(f \star_{\mathrm{pl}} g)(y,z) = f(y,z) \star_y \bar{\star}_z g(y,z),$$
$$(f \star_{\mathrm{np}} g)(y,z) = f(y,z) \star_y \star_z g(y,z),$$

we thus find the one-loop self-energy

$$\Sigma(y,z) = \Delta_{\mathrm{ret}}(y,z) \star_{\mathrm{pl}} \Delta_+(z,y) + \Delta_{\mathrm{ret}}(y,z) \star_{\mathrm{np}} \Delta_+(z,y) + \Delta_+(y,z) \star_{\mathrm{pl}} \Delta_{\mathrm{ret}}(y,z) + \Delta_+(y,z) \star_{\mathrm{np}} \Delta_{\mathrm{ret}}(y,z).$$

The planar and nonplanar products may also be defined at different points (with $k, \tilde{k} \in \mathbb{R}^{2d}$):

$$(f \otimes_{\star_{\mathrm{pl}}} g)\hat{}(k; \tilde{k}) = e^{-\frac{i}{2} k \sigma_{\mathrm{pl}} \tilde{k}} \hat{f}(k) \hat{g}(\tilde{k})$$

The retarded propagator

We consider the massless case and use light cone coordinates

$$u = x_0 - x_1, \quad v = x_0 + x_1.$$

The retarded propagator solves ($\lambda = \Omega^{-\frac{1}{2}} \lambda_{nc}$)

$$\left(4\partial_{u_1}\partial_{v_1} + 4\lambda^4 u_1 v_1\right) \Delta_{ret}(u_1, v_1; u_2, v_2) = 2\delta(u_1 - u_2)\delta(v_1 - v_2).$$

Proposition

The retarded propagator is given by

$$\Delta_{ret}(u_1, v_1; u_2, v_2) = \tfrac{1}{2} H(u_1 - u_2) H(v_1 - v_2) \times \sum_{n=0}^{\infty} (-1)^n \frac{\left(u_1^2 - u_2^2\right)^n}{2^n \lambda^{2n} n!} \frac{\left(v_1^2 - v_2^2\right)^n}{2^n \lambda^{2n} n!}.$$

The retarded propagator

In terms of the coordinates

$$u_s = u_1 + u_2, \qquad u_t = u_1 - u_2,$$
$$v_s = v_1 + v_2, \qquad v_t = v_1 - v_2,$$

we obtain

$$\Delta_{\mathrm{ret}}(u_s, v_s, u_t, v_t) = \tfrac{1}{2} H(u_t) H(v_t) J_0(\lambda^{-2} \sqrt{u_t u_s v_t v_s}).$$

Remark

In a massive theory (without quadratic potential), we have

$$\Delta_{ret}(u_s, v_s, u_t, v_t) = \tfrac{1}{2} H(u_t) H(v_t) J_0(\mu \sqrt{u_t v_t}).$$

Thus, we deal with a position dependent mass $\mu = \lambda^{-2} \sqrt{u_s v_s}$.

The retarded propagator

In terms of the coordinates

$$u_s = u_1 + u_2, \qquad u_t = u_1 - u_2,$$
$$v_s = v_1 + v_2, \qquad v_t = v_1 - v_2,$$

we obtain

$$\Delta_{\text{ret}}(u_s, v_s, u_t, v_t) = \tfrac{1}{2} H(u_t) H(v_t) J_0(\lambda^{-2} \sqrt{u_t u_s v_t v_s}).$$

For $u_t, v_t > 0$ and $u_s v_s < 0$ this diverges as

$$e^{\lambda^{-2}\sqrt{-u_t u_s v_t v_s}} \leq e^{\frac{1}{4\lambda^2}(u_t^2 + u_s^2 + v_t^2 + v_s^2)}.$$

Thus, the retarded propagator is no tempered distribution.

Gelfand-Shilov spaces

Definition

The Gelfand-Shilov space $\mathcal{S}_{\alpha,A}(\mathbb{R}^4)$ is the space of Schwartz functions that fulfill the bound (with $a_i < \alpha_i e^{-1} A_i^{-1/\alpha_i}$)

$$|\partial_\beta f(z)| \leq C_\beta e^{-\sum_{i=1}^4 a_i |z_i|^{1/\alpha_i}}.$$

We interpret the retarded propagator as a distribution on $\mathcal{S}_{\alpha,A}(\mathbb{R}^4)$, with $\alpha = \frac{1}{2}$ and $A = \sqrt{2e^{-1}}(\lambda - \varepsilon)$. The Fourier transforms of such test functions are entire function which fulfill, for arbitrary $\delta > 0$,

$$|\hat{f}(k + ip)| \leq C e^{\left((\lambda-\varepsilon)^2+\delta\right)|p|^2}.$$

$\star$-products of distributions

Given two distributions $F, G \in \mathcal{S}'_{\alpha,A}(\mathbb{R}^4)$, we want to define their (planar) $\star$-product at different points via duality, i.e.,

$$\langle F \otimes_{\star_{\mathrm{pl}}} G, f \otimes g \rangle = \langle F \otimes G, f \otimes_{\star_{\mathrm{pl}}} g \rangle.$$

In momentum space, we have

$$(f \otimes_{\star_{\mathrm{pl}}} g)\hat{}(k; \tilde{k}) = e^{-2i\lambda_{\mathrm{nc}}^2 k\sigma_{\mathrm{pl}}\tilde{k}} \hat{f}(k)\hat{g}(\tilde{k}) \qquad (1)$$

with

$$k\sigma_{\mathrm{pl}}\tilde{k} = k_{u_s}\tilde{k}_{v_t} + k_{u_t}\tilde{k}_{v_s} - k_{v_s}\tilde{k}_{u_t} - k_{v_t}\tilde{k}_{u_s}.$$

Proposition

For $\lambda_{nc} \geq \lambda$, $\alpha = \frac{1}{2}$ and $A = \sqrt{2e^{-1}}(\lambda - \varepsilon)$, there are no nontrivial $f, g \in \mathcal{S}_{\alpha,A}(\mathbb{R}^4)$, s.t. (1) is the F.T. of an element of $\mathcal{S}_{\alpha,A}(\mathbb{R}^8)$.

Proof

It suffices to show that the bounds (with $b = (\lambda - \epsilon)^2 + \delta$)

$$|\hat{f}(k + ip)| \leq c e^{b|p|^2}, \quad |\hat{g}(k + ip)| \leq c' e^{b|p|^2}, \tag{2}$$

$$|(f \otimes_{\star_{\text{pl}}} g)\hat{}(k + ip; \tilde{k} + i\tilde{p})| \leq C e^{b(|p|^2 + |\tilde{p}|^2)}, \tag{3}$$

can not be fulfilled simultaneously for $\lambda_{\text{nc}}^2 \geq \lambda^2 > b$. With $l = k + ip$ we obtain from (3)

$$|\hat{f}(l)\hat{g}(i\sigma_{\text{pl}}^{-1} l)| \leq C e^{-(2\lambda_{\text{nc}}^2 - b)|k|^2 + (2\lambda_{\text{nc}}^2 + b)|p|^2}.$$

But (2),

$$|\hat{f}(l)\hat{g}(i\sigma_{\text{pl}}^{-1} l)| \leq C' e^{b(|k|^2 + |p|^2)}.$$

Thus, for $\lambda_{\text{nc}}^2 > b$, the entire function $F(l) = e^{bl^2} \hat{f}(l)\hat{g}(i\sigma_{\text{pl}}^{-1} l)$ is bounded on the real and imaginary axis and grows with order 2 in between. By the Phragmén-Lindelöf principle, it vanishes. □

Discussion

- We have shown that at and above the self-dual point the (planar) $\star$-product at different points can not be defined via duality on elements of $\mathcal{S}'_{\alpha,A}(\mathbb{R}^4)$.
- As Δ_{ret} lies in that distribution space, we expect Δ_+ to also lie (in a subset of) that space.

$\Rightarrow$ We have shown that the products of distributions appearing in the Yang-Feldman series do not exist, not even at different points.

- It is thus no UV-divergence.
- Also a formal direct calculation of these products fails at and above the self-dual point. One finds (as the coefficient of the $n = 0$ term) the geometric series

$$\sum_{m=0}^{\infty} \Omega^{4m}.$$

Planar vs. nonplanar

- The above argument is also valid for nonplanar graphs.
- A formal calculation shows no problems in nonplanar graphs.
- Reason: The choice of $\mathcal{S}_{\alpha,A}(\mathbb{R}^4)$ as test function space is too restrictive. It suffices to restrict to test functions that fall off stronger than

$$e^{-\frac{1}{4\lambda^2}x^2}$$

in the two directions

$$(u_s, v_s, u_t, v_t) = xe_1 = \tfrac{x}{2}(1,-1,1,1) \text{ and } xe_2 = \tfrac{x}{2}(-1,1,1,1).$$

However, for these directions we have

$$[e_1, e_2]_{\text{pl}} = 4i\lambda_{\text{nc}}^2, \quad [e_1, e_2]_{\text{np}} = 0.$$

$\Rightarrow$ In the planar case, the joint localization is impossible for $\lambda_{\text{nc}} \geq \lambda$.

Summary

- We found a peculiar kind of divergence in planar graphs in the Grosse Wulkenhaar model on the 2d noncommutative Minkowski space.
- The divergence is present at and above the self-dual point and is not UV.
- The same problems seem to present in the 4d case.
- Also in terms of a suitable eigenfunction basis one finds a divergence at the self-dual point. Reason: Continuous set of generalized eigenfunctions.

Outlook

- Use larger class of test functions.
- Look at 4d.
- Renormalization?
- Different potential?

Eigenfunction basis

- At the self-dual point, we may write

$$(-\Box + \tilde{x}^2 + \mu^2)\phi = H\star\phi + \phi\star H + \mu^2\phi$$

with

$$H = \tfrac{1}{2}\tilde{x}_\mu \tilde{x}^\mu .$$

- If we find a set of orthonormal eigenvectors $|ks\rangle$,

$$H|ks\rangle = k|ks\rangle ,$$

with degeneracy index s, then, for χ^{st}_{kl} the Weyl symbol of $|ks\rangle\langle lt|$, we have

$$(-\Box + \tilde{x}^2 + \mu^2)\chi^{st}_{kl} = (k + l + \mu^2)\chi^{st}_{kl} .$$

Eigenfunctions

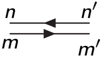
- In the case of the 2d Euclidean space, we have

$$H = 2\lambda_{\mathrm{nc}}^{-4}\left(x_0^2 + x_1^2\right).$$

In the canonical representation, this is the Hamiltonian of the harmonic oscillator. Thus, there is no degeneracy, and

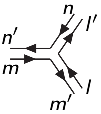
$$k = 4\lambda_{\mathrm{nc}}^{-2}(n + \tfrac{1}{2}).$$

- For the propagator, we assume



$$= \delta_{mm'}\delta_{nn'}\frac{-\lambda_{\mathrm{nc}}^2}{4(m+n+1)+\lambda_{\mathrm{nc}}^2\mu^2}$$

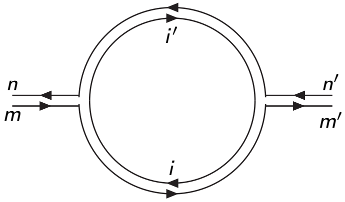
- The vertex is given by



$$= g\delta_{mm'}\delta_{nn'}\delta_{ll'}.$$

A planar graph

For the planar fish graph



one obtains the expression

$$ig^2\lambda_{\mathrm{nc}}^4\delta_{mm'}\delta_{nn'}\sum_{i,i'}\frac{1}{4(m+i+1)+\lambda_{\mathrm{nc}}^2\mu^2}\,\frac{1}{4(i'+n+1)+\lambda_{\mathrm{nc}}^2\mu^2}\,[\delta_{ii'}]^2\,.$$

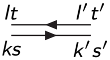
The eigenfunctions

- In the Minkowski case, the Hamiltonian

$$H = 2\lambda_{\rm nc}^{-4}\left(x_0^2 - x_1^2\right)$$

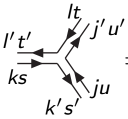
corresponds to an inverted harmonic oscillator. It has spectrum $\mathbb{R}$ with two-fold degeneracy ($s, t = \pm$).

- For the propagator, we assume



$$= \delta^{ss'}\delta^{tt'}\delta(k-k')\delta(l-l')\frac{-\lambda_{\rm nc}^2}{k+l-\lambda_{\rm nc}^2\mu^2+i\epsilon}$$

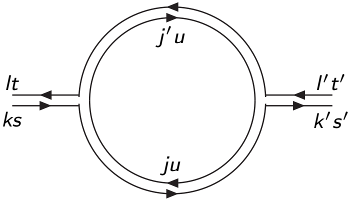
- The vertex is given by



$$= g\delta^{ss'}\delta^{tt'}\delta^{uu'}\delta(k-k')\delta(l-l')\delta(j-j').$$

The planar divergence

For the planar fish graph



one obtains the expression

$$ig^2 \lambda_{\text{nc}}^4 \delta(k-k')\delta(l-l')\delta^{ss'}\delta^{tt'}$$
$$\times \sum_u \int \mathrm{d}j \mathrm{d}j' \; \frac{1}{k+j-\lambda_{\text{nc}}^2\mu^2+i\epsilon} \frac{1}{j'+l-\lambda_{\text{nc}}^2\mu^2+i\epsilon} \left[\delta(j-j')\right]^2 .$$

This diverges even before evaluating the loop integral.

The planar divergence

In a certain sense, this result is generic:

- It only relies on the conservation of the generalized momentum k by the propagator and at the vertices, and will thus also appear in the Yang-Feldman formalism.
- The choice of a ϕ^3 vertex was only a matter of convenience.

However, the discussion in terms of the eigenfunctions has several shortcomings

- It is limited to the self-dual point.
- It uses a basis that is not well understood (e.g. in terms of localization).
- The propagator is not completely specified.

To overcome these, we switch to position space. There, we construct the retarded propagator, which is uniquely defined.

The eigenfunctions

In light cone coordinates, the Hamiltonian is a multiple of

$$2i(u\partial_u + \partial_u u) = 4iu\partial_u + 2i.$$

Normalised generalised eigenfunctions are

$$\psi_k^{\pm}(u) = \frac{1}{2\sqrt{2\pi}} u_{\pm}^{-i\frac{k}{4}-\frac{1}{2}} = \begin{cases} \frac{1}{2\sqrt{2\pi}} |u|^{-i\frac{k}{4}-\frac{1}{2}} \text{ for } u \gtrless 0 \\ 0 \text{ otherwise} \end{cases}$$

with $k \in \mathbb{R}$.