Realization of Ginsparg-Wilson type Chiral Symmetry in ERG

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Motivation

♢ One of the most important subjects in ERG:

How to realize gauge symmetry, naively incompatible with

blocking with momentum cutoff.

It does not mean that the symmetry is lost.

The symmetry can exist even though suffering from deformation due to blocking.

♢ For Wilson action $S[\Phi]$, the presence of exact symmetry and its properties described by

· Ward-Takahashi (WT) identiy:

$$\Sigma[\Phi] = \frac{\partial S}{\partial \Phi}\delta\Phi - \frac{\partial}{\partial \Phi}\delta\Phi = 0,$$

lifted to

· Quantum Master Equation (QME) in antifield formalism:

$$\Sigma[\Phi, \Phi^*] \quad = \quad \frac{\partial \bar{S}}{\partial \Phi} \frac{\partial \bar{S}}{\partial \Phi^*} - \frac{\partial^2 \bar{S}}{\partial \Phi \partial \Phi^*} = 0$$

♢ In order to discuss gauge symmetry, we need to solve WT identity (or QME) , which has non-trivial contributions induced by blocking.

♢ To get some insights into non-perturbative solutions, reconsider prototype of symmetry

whose standard realization is incompatible with reg. scheme:

chiral symmetry on the lattice

- Here, Ginsparg-Wilson (GW) relation for Dirac op. D_0 can be used to define

  new object $\hat{\gamma}_5 = \gamma_5(1 - 2aD_0)$ as

$$\gamma_5 D_0 + D_0\gamma_5 - 2aD_0\gamma_5 D_0 = 0 \quad \Rightarrow \gamma_5 D_0 + D_0\hat{\gamma}_5 = 0$$

  so that action $\bar{\Psi}D_0\Psi$ on the lattice with spacing a is invariant under chiral tr.

$$\delta\bar{\Psi} = \bar{\Psi}i\varepsilon\gamma_5, \quad \delta\Psi = i\varepsilon\hat{\gamma}_5\Psi.$$

- GW relation, $\gamma_5 D_0 + D_0 \hat{\gamma}_5 = 0$, can be identified with WT identity $\Sigma = 0$

  for free-field theory. (YI-Itoh-So '01)

What we wish to discuss here is

to solve WT identity with Yukawa couplings.

Extension of work (YI-So-Ukita '02) on the lattice to continiuum theory.

♢ Outline

[1] Derivation of WT identity for Wilson action ($\Sigma_\Phi = 0$).

[2] WT identity for GW realization of chiral symmetry

[3] Solutions to WT identity for interacting theory

[4] Transformation to "$\hat{\gamma}_5$ prescription"

[5] Summary and outlook

Derivation of Ward-Takahashi (WT) identity for Wilson action

◇ Consider generic theory described by

$$\mathcal{Z}_\phi[J] = \int \mathcal{D}\phi \exp\left(-\mathcal{S}[\phi] + J\cdot\phi\right), \qquad J\cdot\phi = J_A\phi^A \qquad (1)$$

$$\mathcal{S}[\phi] = \frac{1}{2}\phi\cdot D_0\cdot\phi + \mathcal{S}_I[\phi], \qquad \phi\cdot D\cdot\phi = \phi^A (D_0)_{AB}\phi^B.$$

Using momentum cutoff function

$$K(p) \approx \begin{cases} 1 & \text{for } p^2 < \Lambda^2 \\ 0 & \text{for } p^2 > \Lambda^2 \end{cases}$$

and kernel $\alpha(p)$, a function of $K(p)$, rewrite (1) as

$$\mathcal{Z}_\phi[J] = \int \mathcal{D}\phi\mathcal{D}\Phi \exp -\frac{1}{2}\Big(\Phi - K\phi - J\cdot(K\alpha)^{-1}\Big)\cdot\alpha\cdot\Big(\Phi - K\phi - (-)^{\epsilon_J}(K\alpha)^{-1}\cdot J\Big)$$
$$\times \exp\left(-\mathcal{S}[\phi] + J\cdot\phi\right)$$

we obtain

$$
\begin{aligned}
\mathcal{Z}_\phi[J] &= N_J \mathcal{Z}_\Phi[J] = N_J \int \mathcal{D}\Phi \exp\left(-S[\Phi] + JK^{-1}\cdot\Phi\right) \\
S[\Phi] &= \frac{1}{2}\Phi\cdot\left[\alpha - K\alpha(D_0 + K^2\alpha)^{-1}K\alpha\right]\cdot\Phi + S_I[\Phi] \\
N_J &= \exp -\frac{1}{2}\left((-)^J J_A K^{-2}(\alpha^{-1})^{AB} J_B\right) \qquad \left((-)^J\ J = (-)^{\epsilon(A)} J^A\right)
\end{aligned}
$$

where the Wilson effective action is defined by

$$
\exp -S_I[\Phi] \equiv \int \mathcal{D}\chi \exp -\Big(\frac{1}{2}\chi\cdot(D_0 + K^2\alpha)\cdot\chi + \mathcal{S}_I[\chi + (D_0+K^2\alpha)^{-1}K\alpha\Phi]\Big)
$$

♢ Consider symmetry properties of Wilson action, making a change of variables

$$
\phi^A \to \phi'^A = \phi^A + \delta\phi^A\ , \qquad \delta\phi^A = \mathcal{R}^A[\phi]
$$

Since $\mathcal{Z}$ is invariant under the change of variables, we obtain

$$\int \mathcal{D}\phi \Big(J \cdot \delta\phi - \Sigma[\phi] \Big) \exp\left(-\mathcal{S}[\phi] + J \cdot \phi\right) = 0 \tag{2}$$

where $\Sigma[\phi]$ is *the WT operator* given as

$$\Sigma[\phi] \equiv \delta\mathcal{S} + \delta \ln \mathcal{D}\phi = \frac{\partial^r \mathcal{S}}{\partial \phi^A} \delta\phi^A - (-)^{\epsilon_A} \frac{\partial}{\partial \phi^A} \delta\phi^A \ .$$

It is given by sum of the change of the action $\mathcal{S}$ and that of the functional measure $\mathcal{D}\phi$. From (2) we obtain

$$\int \mathcal{D}\phi \ \Sigma[\phi] \exp\left(-\mathcal{S}[\phi] + J \cdot \phi\right) = N_J \left\{ N_J^{-1} \left(J \cdot \mathcal{R}\left[\frac{\partial}{\partial J}\right] \ N_J \right) + J \cdot \mathcal{R}\left[\frac{\partial}{\partial J}\right] \ \right\} \ Z_\Phi[J]$$

$$= N_J \int \mathcal{D}\Phi \ \Sigma[\Phi] \exp\left(-S[\Phi] + J \cdot K^{-1}\Phi\right)$$

We consider below a linear global symmetry described by

$$\begin{aligned}\Sigma[\phi] &= \delta\mathcal{S} = \frac{\partial \mathcal{S}}{\partial \phi^A}\,\delta\phi^A = 0\\ \delta\phi^A &= \mathcal{R}^A{}_B \phi^B\,,\end{aligned}$$

where $\mathcal{R}^A{}_B$ do not depend on the fields.

Then, *the WT operator* for the Wilson action is given by

$$\begin{aligned}\Sigma[\Phi;\Lambda] &= \frac{\partial S}{\partial \Phi^A}\delta\Phi^A - (-)^{\epsilon_A}\frac{\partial}{\partial \Phi^A}\delta\Phi^A\\ \delta\Phi^A &= \mathcal{R}^A{}_B\left\{\Phi^A - \left(\alpha^{-1}\right)^{AB}\frac{\partial S}{\partial \Phi^B}\right\}\,.\end{aligned}$$

Note that symmetry transformation $\delta\Phi$ depends on the Wilson action.

(see also G. Bergner-F. Bruckmann-J.M. Pawlowski '08)

WT identity for GW realization of chiral symmetry

◇ Consider the standard action of σ model

$$
\begin{aligned}
\mathcal{S}[\phi] &= \int_p \left[\bar\psi(-p)\not{p}\psi(p) + \phi^\dagger(-p)p^2\phi\right] + \mathcal{S}_I \\
\mathcal{S}_I &= g\int_{p,q}\left[\bar\psi(-p)(1+\gamma_5)\phi(q)\psi(p-q) + \bar\psi(-p)(1-\gamma_5)\phi^\dagger(q)\psi(p-q)\right] \\
&\quad +h(\phi^\dagger\phi)\,,
\end{aligned}
$$

which is invariant under chiral transformation

$$
\begin{aligned}
\delta\psi(p) &= i\varepsilon\gamma_5\psi(p), \qquad \delta\bar\psi(-p) = \bar\psi(-p)i\varepsilon\gamma_5 \\
\delta\phi(p) &= -2i\varepsilon\phi(p), \qquad \delta\phi^\dagger(-p) = 2i\varepsilon\phi^\dagger(-p)\,.
\end{aligned}
$$

For non-trivial realization of chiral symmetry, we take chirally non-invariant blocking for the Dirac fields

$$\alpha_D(p) = \frac{M}{K(p)(1-K(p))}\,,$$

where M is a constant with mass dimension 1. For the scalar field, take a blocking

$$\alpha_S(p) = \frac{p^2}{K(p)(1-K(p))}\,.$$

The WT identity for GW type chiral symmetry is given by

$$\begin{aligned}
\Sigma[\Phi,\Lambda] &= \int_p \Bigg[\frac{\partial^r S}{\partial\Psi(p)} i\varepsilon\gamma_5 \left\{\Psi(p) - 2\alpha_D^{-1}(p)\frac{\partial^l S}{\partial\bar{\Psi}(-p)}\right\} + \bar{\Psi}(-p) i\varepsilon\gamma_5 \frac{\partial^l S}{\partial\bar{\Psi}(-p)} \\
&\qquad -\frac{\partial^r}{\partial\Psi(p)} 2i\varepsilon\gamma_5\alpha_D^{-1}(p)\frac{\partial^l S}{\partial\bar{\Psi}(-p)} + \frac{\partial S}{\partial\varphi(p)}(-2i)\varphi(p) + \varphi^\dagger(-p)(2i)\frac{\partial S}{\partial\varphi^\dagger(-p)}\Bigg] \\
&= 0\,.
\end{aligned}$$

Free-field Wilson action S_0 of the IR fields $\Phi^A = \{\Psi,\ \bar{\Psi},\ \varphi,\ \varphi^\dagger\}$ is given by

$$\begin{aligned} S_0[\Phi, \Lambda] &= \frac{1}{2}\Phi \cdot \left[D_0(D_0 + f^2\alpha)^{-1}\alpha\right] \cdot \Phi \\ &= \int_p \left[\bar{\Psi}(-p)\mathcal{D}_0(p)\Psi(p) + K^{-1}(p)\varphi^\dagger(-p)\ p^2\varphi(p)\right], \end{aligned}$$

where $D_0^D = \not{p}$ for the Dirac fields and $D_0^S = p^2$ for the scalar fields. The Dirac operator takes the form

$$\mathcal{D}_0(p) = \frac{1}{K(p)}\left\{\frac{M\not{p}}{(1 - K(p))\not{p} + K(p)M}\right\}.$$

Using WT identity, we obtain chiral transformation for free-field theory

$$\begin{aligned} \delta\Psi(p) &= i\varepsilon\gamma_5\left(1 - 2\alpha^{-1}\mathcal{D}_0\right)(p)\Psi(p), \qquad \delta\bar{\Psi}(-p) = \bar{\Psi}(-p)i\varepsilon\gamma_5 \\ \delta\varphi(p) &= -2i\varepsilon\varphi(p), \qquad \delta\varphi^\dagger(-p) = 2i\varepsilon\varphi^\dagger(-p) \qquad (\alpha_D = \alpha). \end{aligned}$$

The WT identity for the free-field action

$$\Sigma[\Phi, \Lambda] = \frac{\partial^r S_0}{\partial \Phi^A} \delta \Phi^A = 0$$

reduces to GW relation

$$\gamma_5 \mathcal{D}_0 + \mathcal{D}_0 \gamma_5 = \{\gamma_5,\ \mathcal{D}_0\} = 2\mathcal{D}_0 \gamma_5 \alpha^{-1} \mathcal{D}_0 \,.$$

We observe that the Dirac action satisfies the GW relation, and therefore free theory is chiral invariant.

Solutions to WT identity for interacting theory

♢ Consider bilinear action of Dirac fields $\Psi,\ \bar{\Psi}$ coupled to some functional of scalars $\vartheta(\varphi),\ \vartheta^\dagger(\varphi)$:

$$S[\Phi] \quad = \quad S_{\rm scalar}[\vartheta, \vartheta^\dagger] + \int_{p,q} \bar{\Psi}(-p)\mathcal{D}(p,q)\Psi(q)\,.$$

Assume Dirac operator to take the form

$$\mathcal{D}(p,q) \quad = \quad \mathcal{D}_0(p)\delta(p-q) + \{1+\mathcal{L}(D_0(p))\}\,\Theta(p,q)\,\{1+\mathcal{R}(D_0(q))\}$$

$$\Theta(p,q) \quad = \quad \frac{1+\gamma_5}{2}\vartheta(p,q) + \frac{1-\gamma_5}{2}\vartheta^\dagger(p,q)\,,$$

where D_0 is the Dirac operator of free-fields satisfying $\{\gamma_5,\ \mathcal{D}_0\} = 2\mathcal{D}_0\gamma_5\alpha^{-1}\mathcal{D}_0$.

For cutoff removing limit $\Lambda \to \infty$, we expect

$$\mathcal{D}_0(p) \to \not{p}, \quad \alpha^{-1}(p) \to 0, \quad \vartheta(p,q) \to \varphi(p)\delta(p-q)$$

For finite Λ, chiral transformation is given by

$$\delta\bar{\Psi}(-p) = \bar{\Psi}(-p)i\varepsilon\gamma_5, \qquad \delta\Psi(p) = i\varepsilon\gamma_5\left\{\Psi(p) - 2\int_q \alpha^{-1}(p)\mathcal{D}(p,q)\Psi(q)\right\}.$$

WT identity for fermionic sector becomes:

$$\{\gamma_5,\ \mathcal{D}(p,q)\} - 2\int_k \mathcal{D}(p,k)\gamma_5\alpha^{-1}(k)\mathcal{D}(k,q)$$
$$= -(i\varepsilon)^{-1}\left\{1 + \mathcal{L}(D_0(p))\right\}\delta\Theta(p,q)\left\{1 + \mathcal{R}(D_0(q))\right\}$$

lhs is expanded up to quadractic in Θ, so we assume

$$\delta\Theta(p,q) = -2i\varepsilon\gamma_5\left(\Theta(p,q) - \int_k \Theta(p,k)\alpha^{-1}(k)\Theta(k,q)\right),$$

we obtain a set of solutions

$$\mathcal{L}(p) = -\left(\alpha^{-1}\mathcal{D}_0\right)(p) + \left(\frac{1}{1-\gamma_5\alpha^{-1}\mathcal{D}_0\gamma_5}\gamma_5\alpha^{-1}\mathcal{D}_0\gamma_5\alpha^{-1}\mathcal{D}_0\right)(p)$$
$$\mathcal{R}(p) = -\left(\mathcal{D}_0\alpha^{-1}\right)(p).$$

The Dirac operator is given by

$$\begin{aligned}\mathcal{D}(p,q) &= \mathcal{D}_0(p)\delta(p-q) + \gamma_5 \frac{1}{1-\alpha^{-1}(p)\mathcal{D}_0(p)}\gamma_5 \\ &\quad \times\left\{\frac{1+\gamma_5}{2}\vartheta(p,q) + \frac{1-\gamma_5}{2}\vartheta^\dagger(p,q)\right\}(1-\mathcal{D}_0(q)\alpha^{-1}(q)) \ ,\end{aligned}$$

where solution to

$$\delta\vartheta(p,q) = -2i\varepsilon\left(\vartheta(p,q) - \int_k \vartheta(p,k)\alpha^{-1}(k)\vartheta(k,q)\right)$$

is expanded in terms of φ and α^{-1}:

$$\begin{aligned}\vartheta(p,q) &= \int_k \varphi(p-k)\Big[\delta(k-q) - \alpha^{-1}(k)\varphi(k-q) \\ &\quad + \int_l \alpha^{-1}(k)\varphi(k-l)\alpha^{-1}(l)\varphi(l-q) + \cdots\Big] \\ &= \int_k \varphi(p-k)\left[1+\alpha^{-1}\varphi\right]^{-1}(k,q)\,.\end{aligned}$$

• Jacobian factor associated with field dependent chiral transformation in WT identity $\Sigma = 0$

$$\begin{aligned} J &= -2i\varepsilon \int_p \frac{\partial^r}{\partial \Psi(p)} \gamma_5 \alpha^{-1}(p) \frac{\partial^l S}{\partial \bar{\Psi}(-p)} \\ &= -2i\varepsilon \ \mathrm{Tr}(\gamma_5 \alpha^{-1} \mathcal{D}) = -2i\varepsilon \ \mathrm{Tr}(\gamma_5 \alpha^{-1} \Theta)\,, \end{aligned}$$

The counter term which cancels this contribution is given by

$$S_{\mathrm{counter}}[\Theta] = \mathrm{Tr}\log(1 - \alpha^{-1}\Theta)\,,$$

because we find in matrix notation

$$\delta\Theta \ = \ -2i\varepsilon\gamma_5\Theta\left(1 - \alpha^{-1}\Theta\right), \qquad J + \delta S_{\mathrm{counter}}[\Theta] = 0\,.$$

In summary, quantum system

$$\int \mathcal{D}\Phi \exp(-S[\Phi,\Lambda])\,.$$

with total action

$$\begin{aligned} S[\Phi,\Lambda] &= \int_p \left[K^{-1}(p)\varphi^\dagger(-p)\; p^2\varphi(p) + \int_q \bar{\Psi}(-p)\mathcal{D}(p,q)\Psi(q) \right] + S_I(\varphi,\varphi^\dagger) \\ &\quad + S_{\rm counter}[\varphi,\varphi^\dagger]\,, \end{aligned}$$

is chiral invariant.

Transformation to "$\hat{\gamma}_5$ prescription"

Making field redefinition

$$\Psi' = \left(\frac{1}{1-\alpha^{-1}\mathcal{D}_0}\right)(1-\alpha^{-1}\Theta)(1-\alpha^{-1}\mathcal{D}_0)\Psi\,,$$

and using relations

$$\begin{aligned}
\Theta\left(\frac{1}{1-\alpha^{-1}\Theta}\right) &= \left[\left(\frac{1+\gamma_5}{2}\right)\varphi + \left(\frac{1-\gamma_5}{2}\right)\varphi^\dagger\right] \\
\left(\frac{1\pm\gamma_5}{2}\right)(1-\alpha^{-1}\mathcal{D}_0) &= \left(\frac{1\pm\gamma_5}{2}\right)\left(\frac{1\pm\hat{\gamma}_5}{2}\right) \equiv P_\pm \hat{P}_\pm \\
\hat{\gamma}_5 &= \gamma_5(1-2\alpha^{-1}\mathcal{D}_0)\,,
\end{aligned}$$

we obtain continuum analog of chiral invariant Yukawa action

$$
\begin{aligned}
&\int_{p,q} \bar{\Psi}(-p)\mathcal{D}(p,q)\Psi(q) \\
&= \int_{p,q} \bar{\Psi}\left[\mathcal{D}_0(p)\delta(p-q) + P_+\varphi(p-q)\hat{P}_+(q) + P_-\varphi^\dagger(p-q)\hat{P}_-(q)\right]\Psi'(q) \\
&\equiv \int_{p,q} \bar{\Psi}(-p)\mathcal{D}'(p,q)\Psi'(q)
\end{aligned}
$$

Note that the change of Ψ to Ψ' generates a Jacobian factor:

$$
\mathcal{D}\Psi\mathcal{D}\bar{\Psi} = \mathcal{D}\Psi'\mathcal{D}\bar{\Psi}\ \exp\left[\mathrm{Tr}\log(1-\alpha^{-1}\Theta)\right] \equiv \mathcal{D}\Psi'\mathcal{D}\bar{\Psi}\exp\mathcal{J}\,,
$$

where the Jacobian factor $\mathcal{J}$ exactly cancels the counter action

$$
\mathcal{J} - S_{\mathrm{counter}}[\varphi,\varphi^\dagger] = 0
$$

In summary, Wilson action for $\Phi'^A = \{\Psi',\ \bar\Psi,\ \varphi,\ \varphi^\dagger\}$

$$
\begin{aligned}
S'[\Phi',\Lambda] &= \int_p \left[K^{-1}(p)\varphi^\dagger(-p)\ p^2\varphi(p) + \int_q \bar\Psi(-p)\mathcal{D}'(p,q)\Psi'(q)\right] + S_I(\varphi,\varphi^\dagger) \\
\mathcal{D}'(p,q) &= \mathcal{D}_0(p)\delta(p-q) + \frac{1+\gamma_5}{2}\varphi(p-q)\frac{1+\hat\gamma_5(p)}{2} + \frac{1-\gamma_5}{2}\varphi^\dagger(p-q)\frac{1-\hat\gamma_5(p)}{2}
\end{aligned}
$$

is invariant under field independent chiral transformation

$$
\begin{aligned}
\delta\bar\Psi(-p) &= \bar\Psi(-p)i\varepsilon\gamma_5, \qquad \delta\Psi(p) = i\varepsilon\hat\gamma_5(p)\Psi(p) \\
\delta\varphi(p) &= -2i\varepsilon\varphi(p), \qquad \delta\varphi^\dagger(p) = 2i\varepsilon\varphi^\dagger(p)\,.
\end{aligned}
$$

Having chiral projections, we can construct chiral invariant higher dimensional operators.

Summary and outlook

♢ A set of (non-perturbative) solutions to WT identity for GW-type realization of chiral symmetry is constructed.

♢ The well-known representation with $\hat{\gamma}_5$ is obtained via a field redefinition. This corresponds to a canonila tr. which turns the Quantum Master Equation into the Classical Master Equation in antifiled formalism.

♢ Nothing new, but the method discussed here is expected to apply to gauge theories, for which Θ is a non-linear functional of gugae field.