Asymptotic behaviour of gravity in three dimensions

Marc Henneaux

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However, the case with negative cosmological constant has attracted in the last years a lot of interest.

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Three-dimensional gravity with negative cosmological constant presents therefore many of the interesting features of the higher-dimensional versions, but in a much simpler context.

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- To that end, it is necessary to first review the Hamiltonian approach to conserved charges in gravitation theory and understand the possible occurence of central charges.
- We shall first consider pure gravity.
- We shall then consider supersymmetric and higher spin extensions of 3D AdS gravity, using the Chern-Simons formulation.

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It has the advantage of clearly connecting the charges to the generators of the corresponding symmetries.

It can be applied to asymptotic symmetries that are not exact symmetries.

Finally, it gives direct access to the algebra of the charges and the possibility of non-Poissonian actions (“central charges")

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$$\delta H[\xi] = \int d^d x \left(\xi^{\perp} \frac{\partial \mathcal{H}}{\partial g_{ij}} \delta g_{ij} + \xi^{\perp} \frac{\partial \mathcal{H}}{\partial g_{ij,m}} \delta g_{ij,m} + \cdots \right) + \delta Q[\xi]$$

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$$= \int d^d x \left[\left(\xi^\perp \frac{\partial \mathcal{H}}{\partial g_{ij}} - \left(\xi^\perp \frac{\partial \mathcal{H}}{\partial g_{ij,m}} \right)_{,m} + \cdots \right) \delta g_{ij} + \cdots \right]$$

$$+ \underbrace{\int d^{d-1} S_m \left(\xi^\perp \frac{\partial \mathcal{H}}{\partial g_{ij,m}} \delta g_{ij} + \cdots \right)}_{M[\xi]} + \delta Q[\xi]$$

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$$\delta_\xi g_{ij} = \frac{\delta H[\xi]}{\delta \pi^{ij}(x)}, \quad \delta_\xi \pi^{ij} = -\frac{\delta H[\xi]}{\delta g_{ij}(x)}.$$

Boundary conditions

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$$\begin{aligned} M[\xi] &= -\int d^d S_n G^{ijmn} \left(\xi^{\perp} \delta g_{ij|m} - \xi^{\perp}_{,m} \delta g_{ij}\right) \\ &\quad - \int d^d S_n 2 N_m \delta \pi^{mn} \\ &\quad - \int d^d S_n (2 N^k \pi^{mn} - N^n \pi^{km}) \delta g_{km}. \end{aligned}$$

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If one modifies the Lagrangian (by the topological mass term in 3 dimensions, or by matter fields, or in any other (local) way), the explicit expression for $M[\xi]$ changes but can again be worked out.

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Thus, even though the expression for $M[\xi]$ can be worked out without knowing the boundary conditions, verifying explicitly the existence of $Q[\xi]$ requires a specific form of the boundary conditions.

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- This constant is usually taken to be zero for some natural "background" configuration.
- Which boundary conditions to impose depends on the context.

Asymptotic symmetries - Definition and properties

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Two asymptotic symmetries that have the same asymptotic behaviour have the same charge and should be identified.

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The asymptotic symmetries with zero charges are the true gauge symmetries.

Algebra of charges : A useful theorem

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Hence $[H[\xi], H[\eta]]$ and $H\left[[\xi, \eta]\right]$ differ, up to trivial terms, by a c-number $K[\xi, \eta]$ (which fulfills cocycle condition etc...).

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- The asymptotic symmetry group cannot be realized as background isometry group.

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If the background is invariant (strictly and not just asymptotically), $\delta_B Q_A = 0$ and hence, $K_{AB} = 0$.

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- *$N = 2$ supergravity in 4 dimensions with asymptotically flat boundary conditions.*
  The asymptotic symmetry algebra is the $N = 2$ super-Poincaré algebra times $U(1)$ (electric). In the presence of a magnetic pole, this algebra acquires a non-vanishing central charge ($U(1)$ magnetic).

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Theories exist, however, with $K_{AB} \neq 0$:

- *$N = 2$ supergravity in 4 dimensions with asymptotically flat boundary conditions.*
  The asymptotic symmetry algebra is the $N = 2$ super-Poincaré algebra times $U(1)$ (electric). In the presence of a magnetic pole, this algebra acquires a non-vanishing central charge ($U(1)$ magnetic).
- *AdS gravity in 3 dimensions.*
  This theory is discussed below.

Asymptotically anti-de Sitter spaces - Background metric

Want to describe isolated systems in an AdS background

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The background metric is (the universal cover of) AdS,

$$ds^2 = -\left(1+\left(\frac{r}{l}\right)\right)dt^2 + \left(1+\left(\frac{r}{l}\right)\right)^{-1} dr^2 + r^2\, d\omega^2,$$

$$D=3: \ d\omega^2 = d\phi^2, \quad D=4: \ d\omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2, \quad \text{etc}$$

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$D = 3 : O(2,2)$; $D = 4 : O(3,2)$ etc.

Boundary of anti-de Sitter space

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The redefinition of the radial coordinate

$$r = \frac{1}{1-\rho}, \quad dr = r^2 d\rho$$

brings the boundary at infinity at the finite value $\rho = 1$.

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For $D = 3$, the boundary is the cylinder $\mathbb{R} \times S^1$ with metric $-dt^2 + d\phi^2$ and the conformal group at infinity is infinite-dimensional and contains $O(2,2)$, which is a proper finite-dimensional subgroup.

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The question is to explicitly give the precise boundary conditions that implement these general ideas.

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They have been extended to include “slowly decaying" scalar fields in M.H. + C. Martínez + R. Troncoso + J. Zanelli (2004, 2007).

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$$H[\xi] = \int d^d x \left(\xi^\perp \mathscr{H} + \xi^k \mathscr{H}_k\right) + \frac{1}{2}\xi_\infty^{AB} Q_{AB}$$

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One recovers in particular the Abbott-Deser energy (1982).

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one allows (in the first case) deviations $\Delta g_{\alpha\beta}$ from AdS that behave as

$$\begin{aligned}\Delta g_{rr} &= \frac{f_{rr}}{r^4} + O(r^{-5}), \quad \Delta g_{rm} = \frac{f_{rm}}{r^3} + O(r^{-4}), \\ \Delta g_{mn} &= f_{mn} + O(r^{-1}),\end{aligned}$$

with $m = +, -$ and $f_{\alpha\beta}(x^m)$.

$D = 3$ - Asymptotic symmetries

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These boundary conditions are invariant under diffeomorphisms η^α of the form

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\eta^+ &= T^+ + \frac{l^2}{2r^2}\partial_-^2 T^- + \cdots,\\
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The Virasoro charges are given by

$$Q_\pm[T^\pm] = \frac{2}{l}\int d\phi T^\pm f_{\pm\pm}.$$

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and invoking the general theorems, one gets two copies of the Virasoro algebra with central charge

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In terms of Fourier modes,

$$[L_n, L_m] = -i\Big((n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}\Big),$$
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AdS gravity and $sl(2,R) \oplus sl(2,R)$ Chern-Simons theory

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- The parameter k is related to the (2+1)-dimensional Newton constant G as $k = \ell/4G$, where ℓ is the AdS radius of curvature.

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in terms of which one finds indeed

$$S[A,\tilde{A}] = \frac{1}{8\pi G} \int_M d^3x \left(\frac{1}{2} eR + \frac{e}{\ell^2} \right)$$

Boundary conditions in terms of the connection

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- The boundary conditions for asymptotically AdS metrics can be reexpressed in terms of the $sl(2,R)$ connections as

$$A \sim \Big[-\frac{1}{r}\frac{2\pi}{k} L(\phi,t)\, X_{11} + r X_{22} \Big] dx^+ - \Big[\frac{1}{r}\frac{X_{12}}{2} \Big] dr$$

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  with the other chirality sector fulfilling a similar condition. Here, $x^{\pm}$ are chiral coordinates, $x^{\pm} = t \pm \phi$ and L is an arbitrary function of t and ϕ. X_{11}, X_{22}, X_{12} are a Chevalley-Serre basis of $sl(2,R)$.

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- It is convenient to eliminate the leading r-dependence by performing a gauge transformation

$$A_i \to \Delta_i = \Omega \partial_i \Omega^{-1} + \Omega A_i \Omega^{-1}.$$

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- We see that the asymptotic boundary conditions are encoded entirely in the highest-weight component (X_{11}).

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$$\delta\Delta = \Lambda' + [\Delta, \Lambda]$$

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One finds that the asymptotic behaviour is preserved ($\delta\Delta$ proportional to X_{11}) if and only if

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- Following the Regge-Teitelboim method, one finds that the surface term that must be added to the (bulk) integral of the Gauss constraint is
$$\oint d\phi \; (\varepsilon L)\,.$$
(and thus on-shell $G[\Lambda] = \oint d\phi \; (\varepsilon L)$)

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Hence, one must add the surface term

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