Corfu Summer Institute
Standard Model and Beyond

Ateneo della Scienza e della Tecnologia

Dipartimento di Fisica

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Lesson 1 : Field Theory and Perturbative QCD I

1) Preliminaries: Basic quantities in field theory
2) Preliminaries: COLOUR
3) The QCD Lagrangian and Feynman rules
4) Asymptotic freedom from $e^+ e^- \to$ hadrons
5) Deep Inelastic Scattering
6) Other partonic processes

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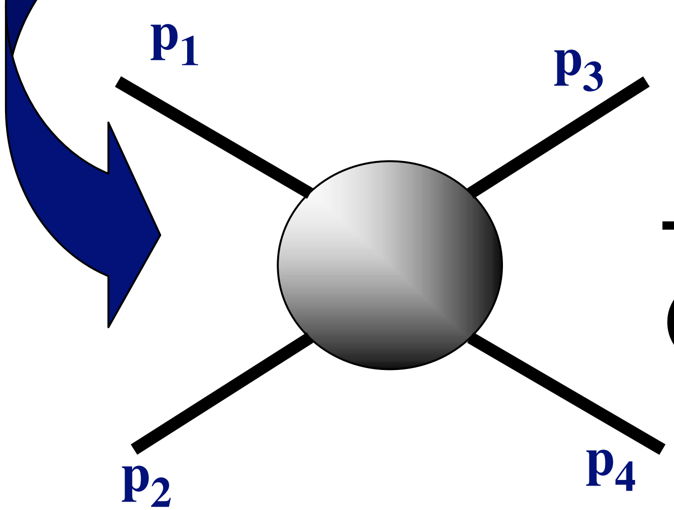
The basic quantities of field theory: the Green Functions

$$\mathcal{L}(\phi) = 1/2\, (\partial_\mu \phi\, (\mathbf{x}))^2 - 1/2\, m_0^2\, \phi^2\, (\mathbf{x}) - 1/4!\, \lambda_0\, \phi^4\, (\mathbf{x})$$

$$\mathcal{S}(\phi) = \int d^4x\, \mathcal{L}(\phi)$$

$$\mathbf{G(x_1, x_2, x_3, x_4) = \langle\, 0 \mid T\, [\phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi(x_4)] \mid 0\, \rangle_c}$$

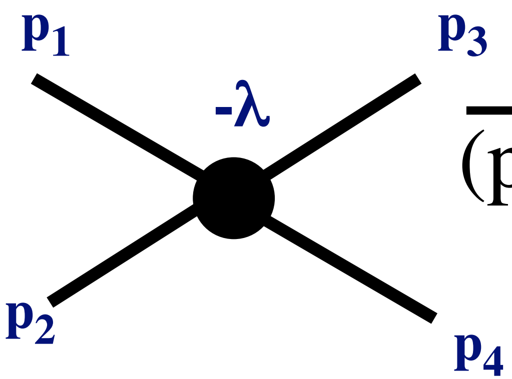


$$\frac{(\mathcal{Z}^{1/2}_{\phi})^{4}\, \mathcal{M}(p_1, p_2, p_3, p_4, m, \lambda)}{(p_1^2 - m^2)\, (p_2^2 - m^2)\, (p_3^2 - m^2)\, (p_4^2 - m^2)} + \ldots .$$

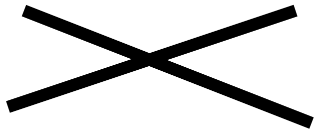
$$\lim_{p^2_{1,2,3,4} \to m^2} \frac{(p_1^2-m^2)\,(p_2^2-m^2)\,G(p_1, p_2, p_3, p_4, m, \lambda)\,(p_3^2-m^2)\,(p_4^2-m^2)}{(\mathcal{Z}^{1/2}_\phi)^4}$$

$\Rightarrow$ $\boxed{\begin{array}{l} S(p_1 + p_2 \to p_3 + p_4) \\ \textbf{S-matrix element} \end{array}}$

At lowest order in perturbation theory:



$$\frac{\lambda}{(p_1^2-m_0^2)\,(p_2^2-m_0^2)\,(p_3^2-m_0^2)\,(p_4^2-m_0^2)}$$

+  + 

The Functional Integral

It can be shown that the Green Functions can be written in terms of Functional Integrals over classical fields

$$G(x_1, x_2, x_3, x_4) = \langle \phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi(x_4) \rangle \equiv$$

$$\boxed{Z^{-1} \int [d\,\phi]\ \phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi(x_4)\ e^{i\,\mathcal{S}(\phi)}}$$

where $Z = \int [d\,\phi]\ e^{i\,\mathcal{S}(\phi)}$

In perturbation theory

$$e^{i\,\mathcal{S}(\phi)} = e^{i\,\mathcal{S}_o(\phi) + i\,\mathcal{S}_i(\phi)} \sim e^{i\,\mathcal{S}_o(\phi)}\,(1 + i\mathcal{S}_i(\phi) - \mathcal{S}^2_i(\phi)/2 + \ldots)$$

$$\mathcal{S}_i(\phi) \sim O(\lambda)$$

$$\langle x^N \rangle = Z^{-1} \int dx \;\; x^N \;\; e^{-S(x)}$$

where $Z(\lambda) = \int dx \;\; e^{-S(x)}$ and $S(x) = x^2 + \lambda\, x^4$

$$Z(\lambda) = \sum_n (-\lambda)^n\, \Gamma(1/2+2n)/n! \approx \sum_n (-\lambda)^n (2n)!/n!$$

- For small values of λ the series expansion is accurate
- The series is however asymptotic (although Borel summable) with zero radius of convergence
- Starting from a certain term, which depends on the value of λ, an increase of the number of terms can only worsen the accuracy

Field theories of interest for physics are not even Borel summable, a nonperturbative method to evaluate the FUNCTIONAL INTEGRAL is thus needed

$$Z^{-1} \int [d\phi] \; \phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi(x_4) \; e^{i S(\phi)}$$

This integral is only a formal definition because of the infrared and ultraviolet divergences. These problems can be cured by introducing an infrared and an ultraviolet cutoff.

1) We introduce an ultraviolet cutoff by defining the fields on a (hypercubic) four dimensional lattice $\phi(x)$ **->** $\phi(a\, n)$
where $n=(n_x , n_y , n_z , n_t)$ and a is the lattice spacing;

$$\partial_\mu \phi\, (\mathbf{x}) \;\mathbf{->}\; \nabla_\mu \phi\, (\mathbf{x}) = (\phi(x+a\, n_{\boldsymbol{\mu}}) - \phi(x)) \,/\, a \quad ;$$

The momentum p is cutoff at the first Brioullin zone, $|p| \le \pi / a$
The cutoff can be in conflict with important symmetries of the theory, as for example Lorentz invariance or chiral invariance
This problem is common to all regularizations like for example Pauli-Villars, dimensional regularization etc.

2) We introduce an infrared cutoff by working in a finite volume, that is
$n_i = 1, 2, \ldots, L$ and $p_i a = 2\pi k_i / L$ with $k_i = 0, 1, \ldots, L - 1$
At finite volume the Green functions are subject to finite size effects

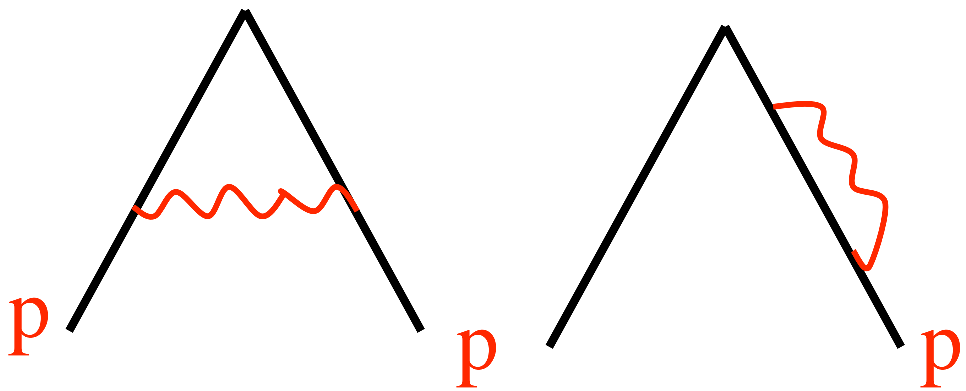
The physical theory is obtained in the limit

$a \to 0$ **renormalizability**

$L \to \infty$ **thermodinamic limit**

Non physical quantities like Green Functions may develop divergencies in this limits; S matrix elements are however finite.

$$Z_\phi(a) = 1 + \lambda \log(p\, a) + \ldots$$

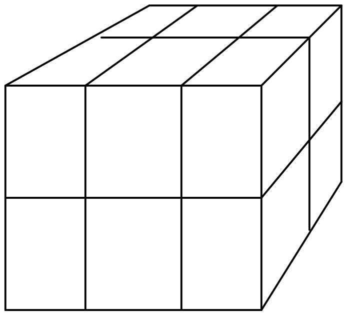


$= g^2 \log(p^2 a^2)$

$$Z^{-1} \int [d\phi]\ \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\ e^{i\,S(\phi)}$$

On a finite volume (**L**) and with a finite lattice spacing (a) this is now an integral on L^4 real variables which can be performed with

Important sampling techniques



$$Z = \sum_{\{\sigma=\pm1\}} e^{J_{ij}\,\sigma_i\,\sigma_j} \quad \text{Ising Model}$$

$2^N = 2^{L^3} \approx 10^{301}$ for $L = 10$!!!

Wick Rotation $t \to i\, t_E$

$$Z^{-1} \int [d\,\phi]\ \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\ e^{i\,\mathcal{S}(\phi)}$$

$$\to\ Z^{-1} \int [d\,\phi]\ \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\ e^{-\,\mathcal{S}(\phi)}$$

This is like a statistical Boltzmann system with

$$\beta\, \mathcal{H} = \mathcal{S}$$

Several important sampling methods can be used, for example the Metropolis technique, to extract the fields with weight $e^{-\,S(\phi)}$

$$<\phi\,\phi\,\phi\,\phi> = Z^{-1} \Sigma_{\{\phi\,(x)\}_n}\ \phi_n(x_1)\ \phi_n(x_2)\ \phi_n(x_3)\ \phi_n(x_4)$$

$$Z = \Sigma_{\{\phi\,(x)\}_n}\, 1 = N$$

QCD

The COLOUR

1) QUARK MODEL $\quad |\Delta^{++}, J_z = 3/2\rangle = |u^\uparrow u^\uparrow u^\uparrow\rangle$

SYMMETRIC IN CONTRAST WITH FERMI STATISTICS

LET US ASSUME:

flavour $\rightarrow$ u_i^α $\leftarrow$ spin

colour

Also: $\quad d_i^\alpha, s_i^\alpha, \ldots$

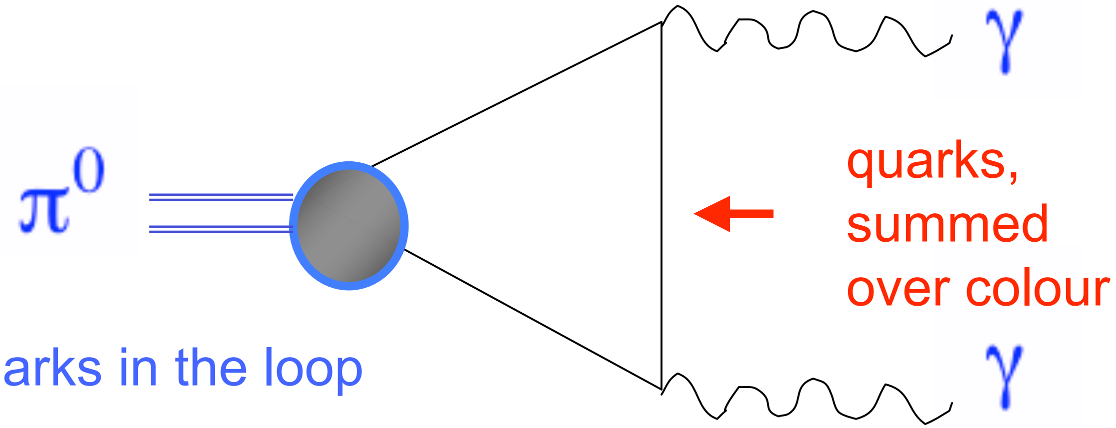
Then $\rightarrow$ $|\Delta^{++}, J_z = 3/2\rangle = \frac{1}{\sqrt{6}} \varepsilon_{ijk} |u_i^\uparrow u_j^\uparrow u_k^\uparrow\rangle$ <u>antisymmetric</u>

<u>only colour singlets exist as asymptotic states</u>

$$M = \frac{1}{3} \sum_i \bar{q}_i q_i \,, \qquad B = \frac{1}{\sqrt{6}} \varepsilon_{i,j,k} q_i q_j q_k$$

The COLOUR

2) $\pi^0 \rightarrow \gamma\,\gamma$



$\mathcal{A}(\pi^0 \to 2\gamma) \propto$ number of quarks in the loop

assuming Q_u = 2/3 and Q_d = - 1/3

N_c =3.06 ± 0.10

$$\Gamma(\pi^0 \to 2\gamma) = \frac{m_\pi^3}{32\pi f_\pi^2}\left(\frac{\alpha}{\pi}\right)^2\left(\frac{N_c}{3}\right)^2$$

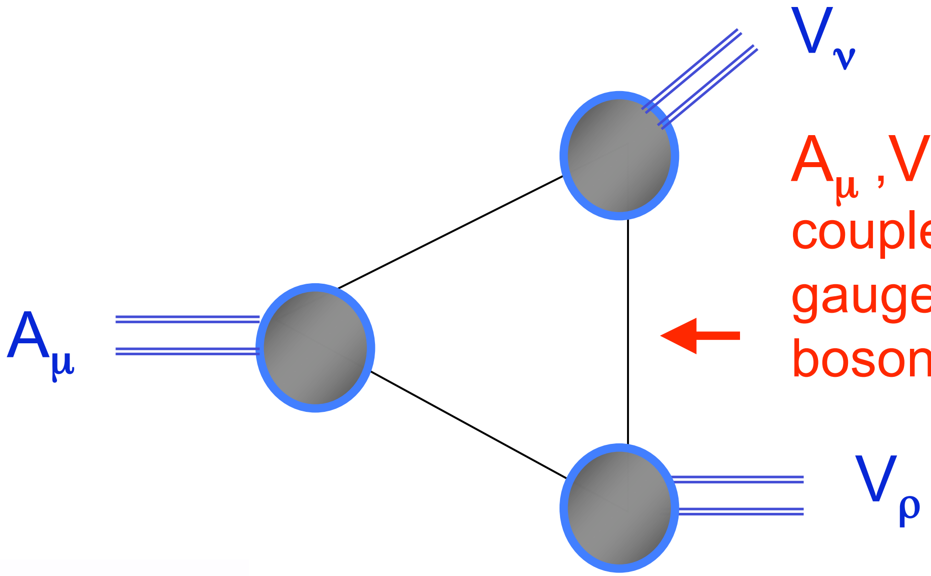
$$\frac{e^2}{\hbar c} = \frac{1}{137....} \quad f_\pi \sim 132\,MeV$$

measured from

$\pi^\pm \rightarrow \mu^\pm \nu_\mu$

The COLOUR

3) gauge anomaly



Renormalizability requires

$$\sum_{leptons} Q_l + \sum_{quarks} Q_q = 0$$

$$(Q_{e^-} + Q_{\nu_e}) + N_c \times (Q_u + Q_d)$$

$$(-1 + 0) + 3 \times (2/3 + (-1/3)) = 0$$

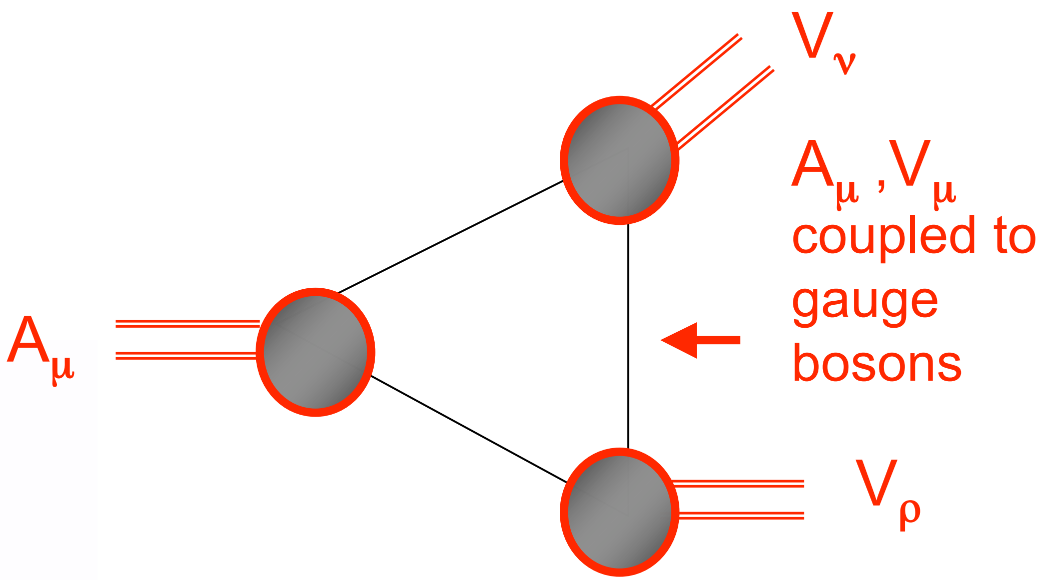
$$(Q_{\mu^-} + Q_{\nu_\mu}) + N_c \times (Q_c +, Q_s) = 0$$

$$(Q_{\tau^-} + Q_{\nu_\tau}) + N_c \times (Q_t +, Q_b) = 0$$

The COLOUR

4) $\boxed{\eta' \quad mass}$

octect mesons



$$M^A = \sum_i \bar{q}^i_f \lambda^A_{fg} q^i_g$$

$$q_1 = u\,, q_2 = d\,, q_3 = s\,;$$

$$i = colour\,, \quad (f,g) = flavour\,,$$

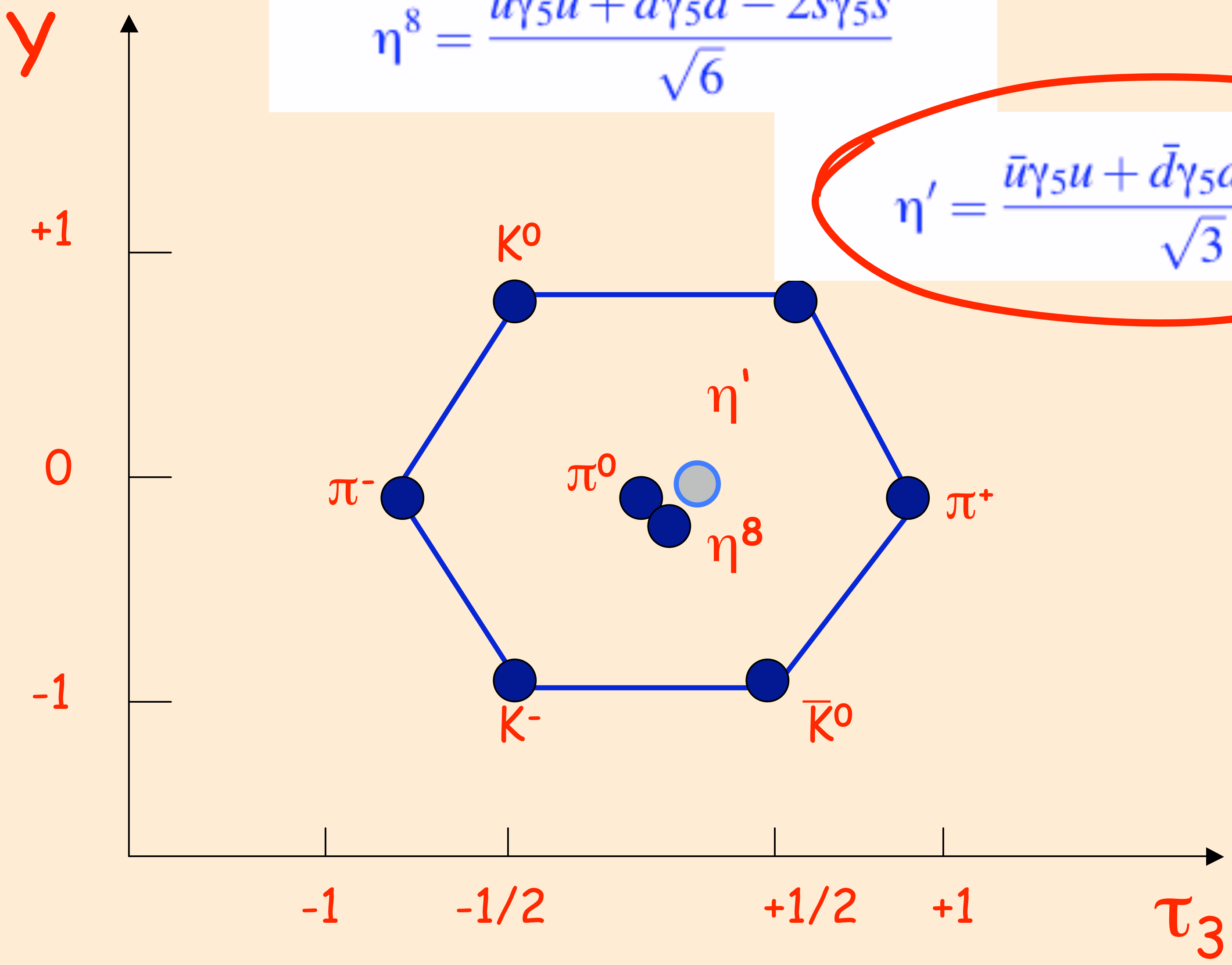
$$\lambda^A_{fg} = Gell - Mann \;\; matrices$$

$$\pi^+ = \bar{d}^i \gamma_5 u^i \quad \pi^- = \bar{u}^i \gamma_5 d^i \quad \pi^0 = \frac{\bar{u}^i \gamma_5 u^i - \bar{d}^i \gamma_5 d^i}{\sqrt{2}}$$

In the quark model $\quad m_{\eta'} \leq \sqrt{3} m_\pi$

$$\eta^8 = \frac{\bar{u}\gamma_5 u + \bar{d}\gamma_5 d - 2\bar{s}\gamma_5 s}{\sqrt{6}}$$

$$\eta' = \frac{\bar{u}\gamma_5 u + \bar{d}\gamma_5 d + \bar{s}\gamma_5 s}{\sqrt{3}}$$



$$\boxed{A_\mu^{\eta'} = \bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d + \bar{s}\gamma_\mu\gamma_5 s}$$

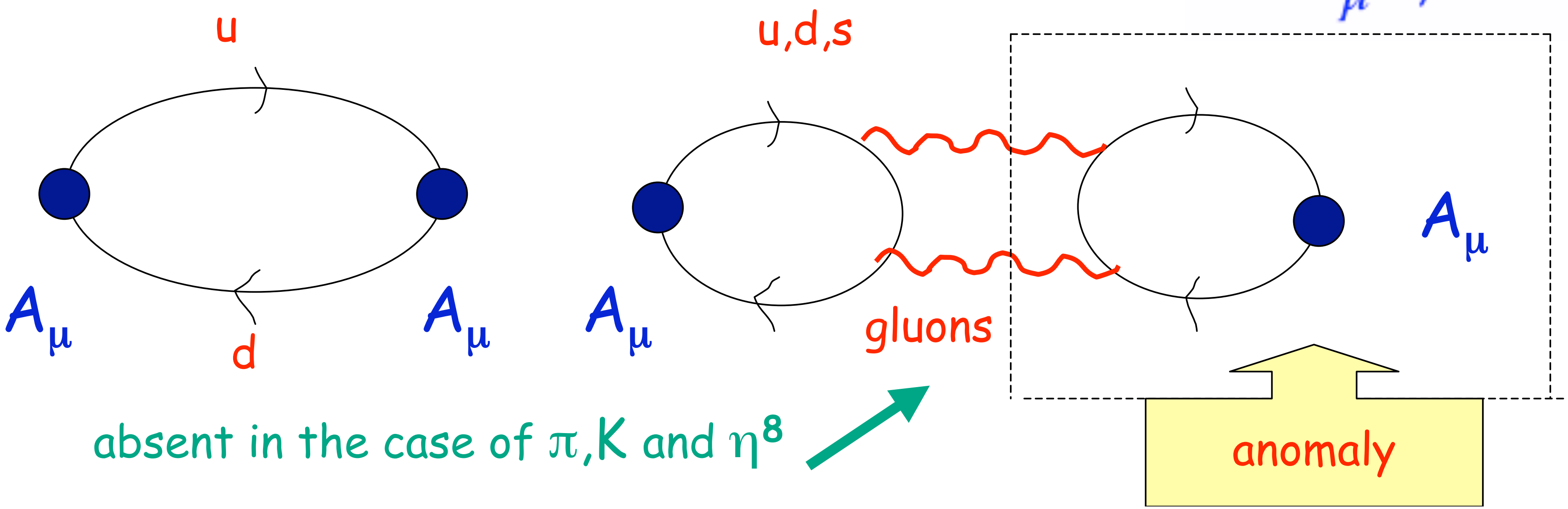
$$\partial^\mu A_\mu^{\eta'} = \frac{g^2}{32\pi^2} G_{\mu\nu}\tilde{G}^{\mu\nu} + 2m_q P^{\eta'}$$

$$P^{\eta'} = \bar{u}\gamma_5 u + \bar{d}\gamma_5 d + \bar{s}\gamma_5 s \quad \tilde{G}^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}$$

In the chiral limit $m_q \to 0$

$$\partial^\mu A_\mu^{\pi} = 0$$

$$\partial^\mu A_\mu^{\eta'} \neq 0$$



THE QCD LAGRANGIAN

Let us start with a simple example:

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi - V(\phi^\dagger \phi)$$

This Lagrangian is invariant under the global symmetry transformation

where $\lambda^A, A = 1, \ldots N_c^2 - 1$

are the generators of the tranformation

in the representation to which ϕ belongs

$$[\lambda^A, \lambda^B] = i f^{ABC} \lambda^C$$

$$Tr[\lambda^A \lambda^B] = \frac{\delta^{AB}}{2}$$

$$\phi'(x) = e^{i\lambda \cdot \alpha} \phi(x) = U(\alpha)\phi(x), \quad \phi^\dagger(x) \to \phi^\dagger(x) U^\dagger(\alpha)$$

For a global transformation $\partial^\mu \phi' = U(\alpha) \partial^\mu \phi$

<u>**This is not true for a local transformation $\alpha = \alpha(x)$**</u>

THE QCD LAGRANGIAN

We look for a ``covariant" derivative such that $[D^\mu \phi]' = U(\alpha) D^\mu \phi$

$(D_\mu = \partial_\mu - i g G_\mu)$

The trasformation rules are

$$G'_\mu = U(\alpha) G_\mu U^\dagger(\alpha) - \frac{i}{g} \partial_\mu U(\alpha) U^\dagger(\alpha)$$

By writing

$$G_\mu = \sum_A G^A_\mu \lambda^A$$

For an infinitesimal transf. we have

$$G^{A\prime}_\mu = G^A_\mu + \frac{1}{g} \alpha^A - f^{ABC} \alpha^B G^B_\mu$$

By defining

$$G^A_{\mu\nu} = \partial_\mu G^A_\nu - \partial_\nu G^A_\mu + g_0 f^{ABC} G^B_\mu G^C_\nu$$

$$G_{\mu\nu} \to U(\alpha) G_{\mu\nu} U^\dagger(\alpha)$$

Invariant under a gauge transf.

$$-\frac{1}{2} Tr[G_{\mu\nu} G^{\mu\nu}]$$

THE QCD LAGRANGIAN

$$\mathcal{L} = -\frac{1}{2} Tr[G_{\mu\nu} G^{\mu\nu}] + \sum_f \bar{q}_f \left(\not{D} - m_f \right) q_f$$

$$+\theta \, Tr[G_{\mu\nu} \tilde{G}^{\mu\nu}]$$

Gluons and quarks
(vector fields and spinors)

The QCD Lagrangian :

$$\mathcal{L}(\phi) = -1/4 \; G^{A}{}_{\mu\nu} G_{A}{}^{\mu\nu}$$

GLUONS

$$+ \sum_{f=\text{flavour}} \bar{q}_f \, (i \, \gamma_\mu D_\mu - m_f) \, q_f$$

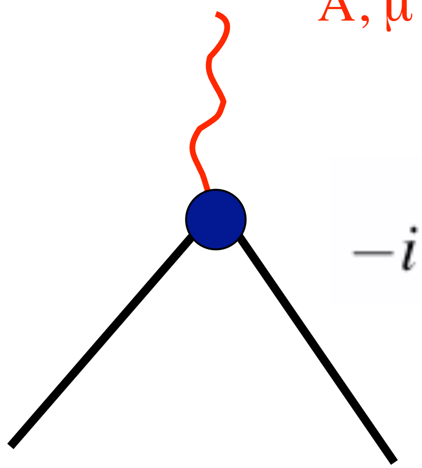
QUARKS (& **GLUONS**)

$$G^{A}{}_{\mu\nu} = \partial_\mu G^{A}{}_{\nu} - \partial_{\nu} G^{A}{}_{\mu} - g_0 \; f^{ABC} G^{B}{}_{\mu} G^{C}{}_{\nu}$$

$$q_f \equiv q_f{}^{\mathbf{a}}{}_{\alpha}(x) \qquad \gamma_\mu \equiv (\gamma_\mu)^{\alpha\beta} \qquad D_\mu \equiv \partial_\mu \mathbf{I} + i \, g_0 \, t^{A}{}_{ab} \, G^{A}{}_{\mu}$$

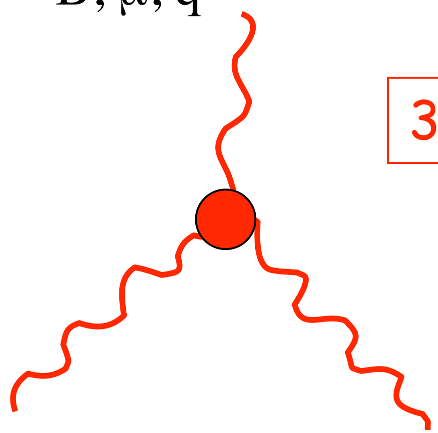
Feynman Rules

$$G^A_{\mu\nu} = \partial_\mu G^A_\nu - \partial_\nu G^A_\mu + g_0 f^{ABC} G^B_\mu G^C_\nu$$



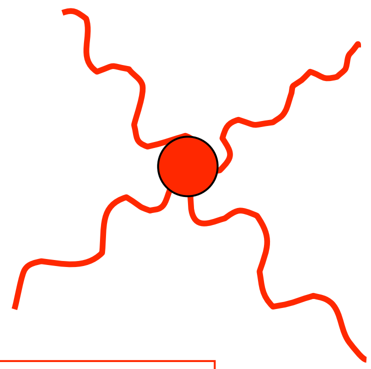
$$-i g_0 t^A_{ab} \gamma_\mu$$

quark-gluon

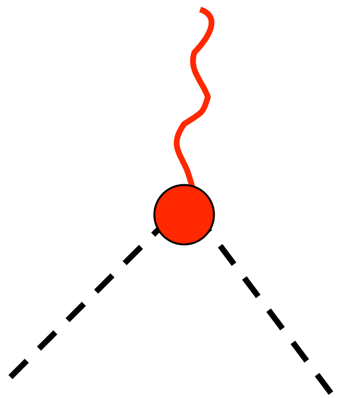


3-gluons

$$g_0 f^{ABC} \left[(p-q)_\nu g_{\lambda\mu} + (q-r)_\lambda g_{\mu\nu} + (r-p)_\mu g_{\nu\lambda} \right]$$



4-gluons



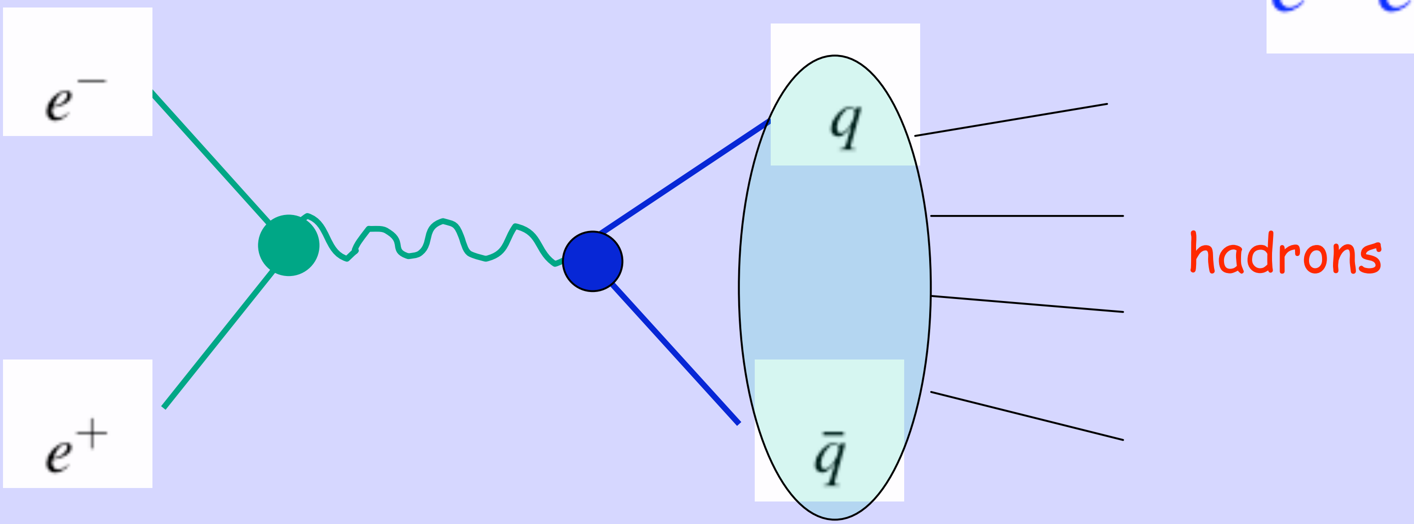
ghost-gluon

SU(2)	$t^A{}_{ab}$	=	1/2	$\sigma^A{}_{ab}$
SU(3)	$t^A{}_{ab}$	=	1/2	$\lambda^A{}_{ab}$

$[t^A, t^B] = i f^{ABC} t^C$

$R_{e+e-}(S)$

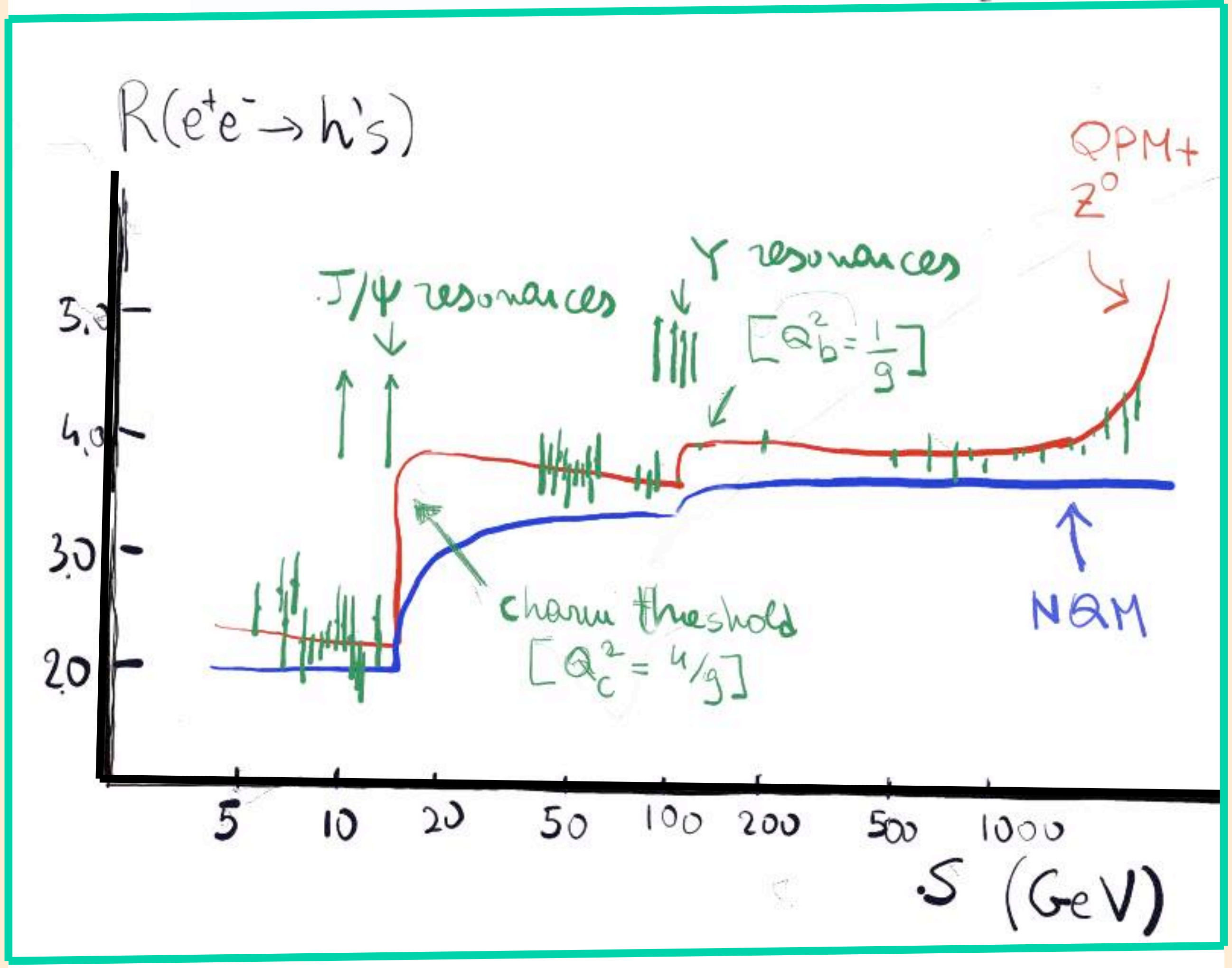
$$e^+e^- \rightarrow \bar{q}q$$



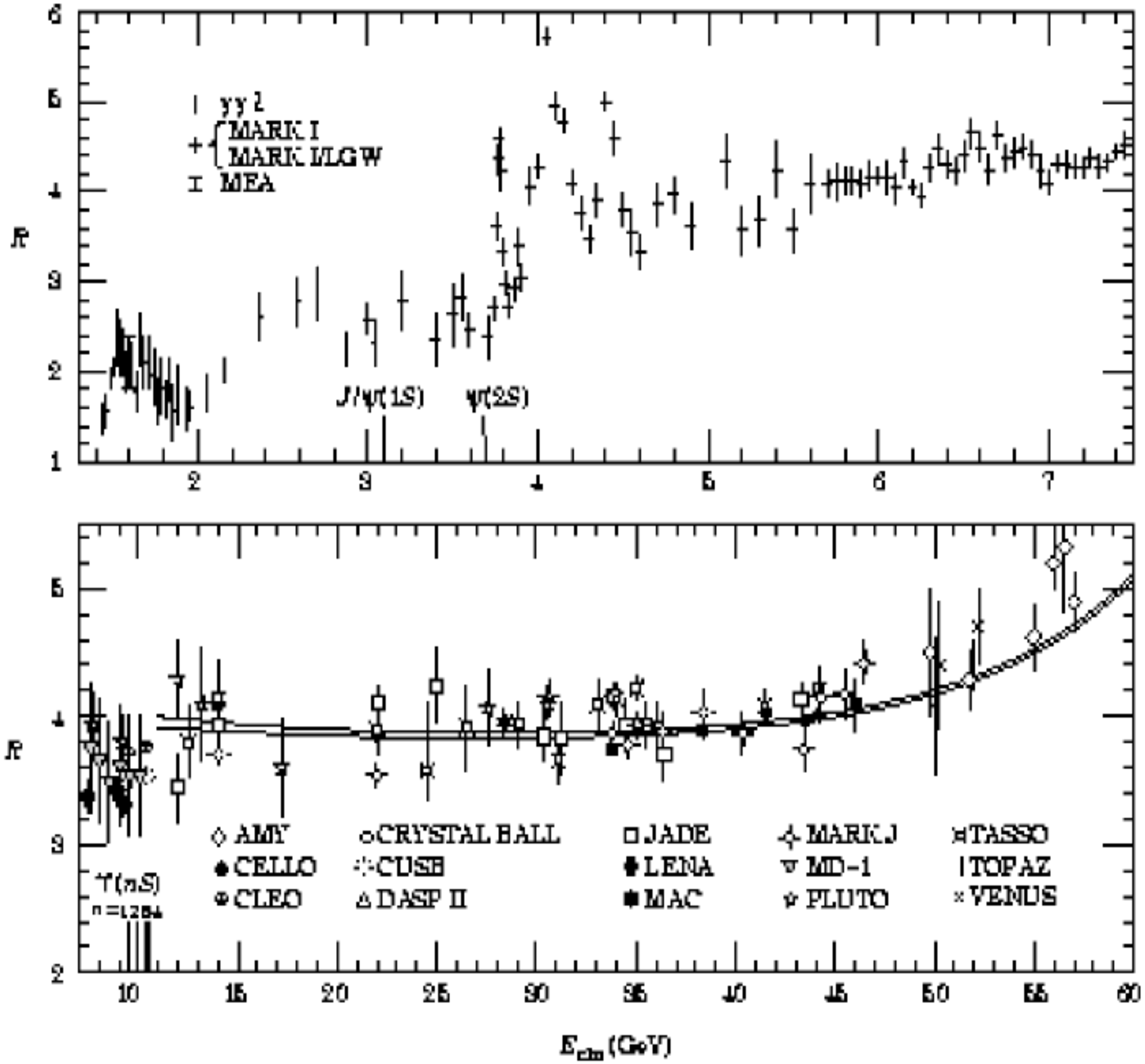
$$R_{e^+e^-}(S) = \frac{\sigma(e^+e^- \rightarrow hadrons)}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$$

The COLOUR

$$\sigma[P + P(\bar{P}) \rightarrow \gamma(Q^2) + X] \propto \frac{1}{N_c}$$



$R_{e+e-}(S)$



The data clearly support the existence of colour

Large Momenta and Energies

The QCD Lagrangian depends only on the quark masses and on the dimensionless couplig g_0

At large scales $\sqrt{s} \to \infty$ the theory can be studied as in the massless _case -> Any dimensionless quantity as $R_{e^+e^-}(s)$ which depends only on $\sqrt{s}$ is naively expected to become a constant

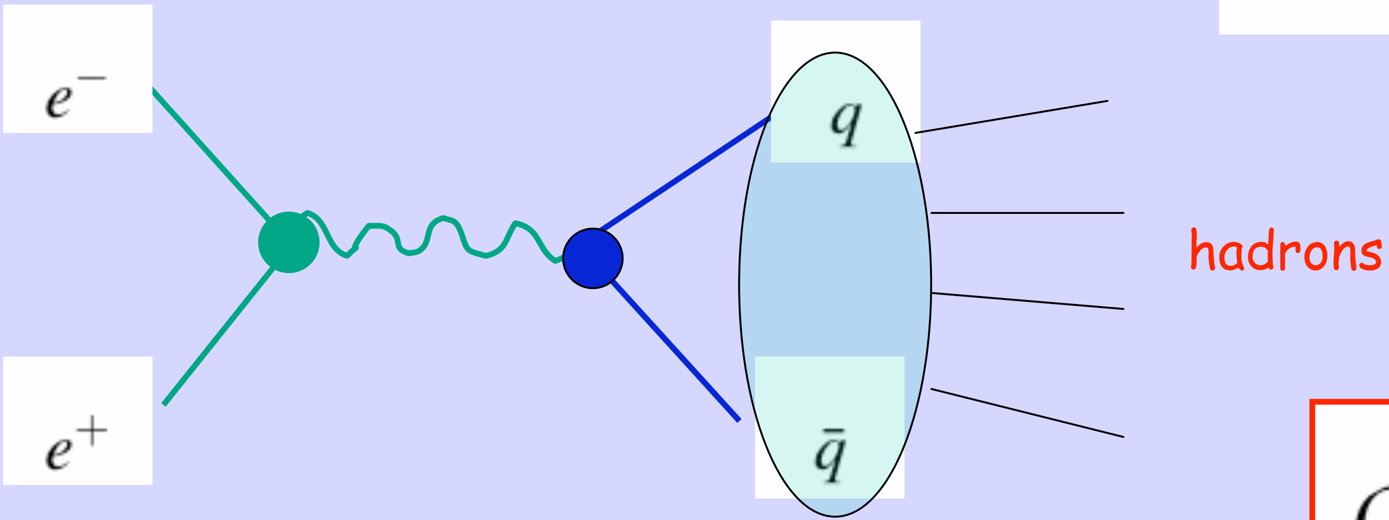
$$\boxed{R_{e^+e^-} = N_c \sum_f Q_f^2 + O\left(\frac{m_f}{\sqrt{s}}\right)}$$

at the quantum level this expectation is wrong

Asymptotic Freedom from $R_{e+e-}(S)$

Let us compute Re+e- in perturbation theory:

$$e^+ e^- \to \bar{q} q$$



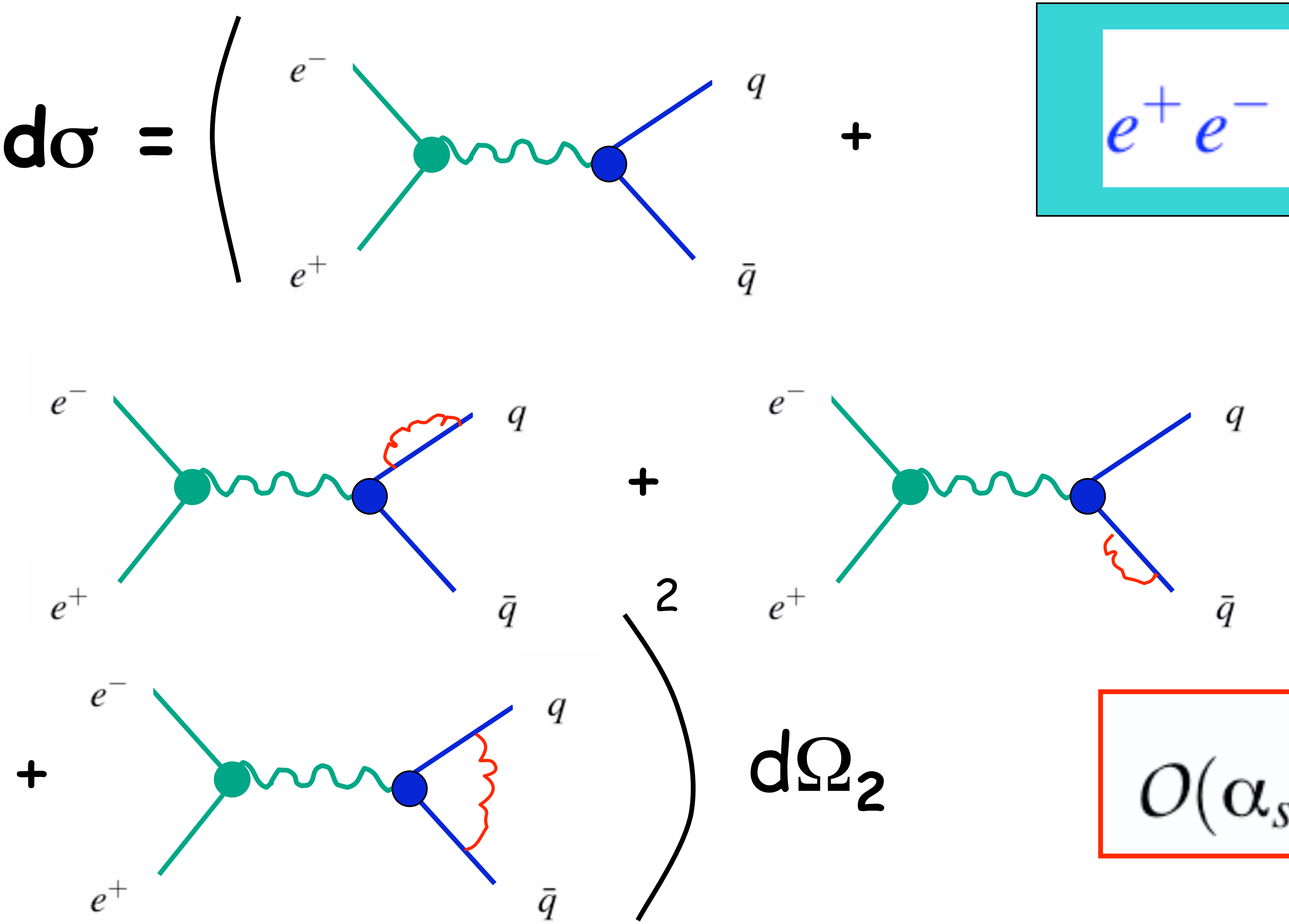
$$\boxed{O(\alpha_s^0)}$$

$$\sigma(e^+ e^- \to hadrons) = \frac{4\pi\alpha_{em}^2}{3S} N_c \sum_f Q_f^2$$

$$R_{e^+e^-}(S) = \frac{\sigma(e^+ e^- \to hadrons)}{\sigma(e^+ e^- \to \mu^+ \mu^-)} = N_c \sum_f Q_f^2$$

Asymptotic Freedom from $R_{e+e-}(S)$

$d\sigma =$

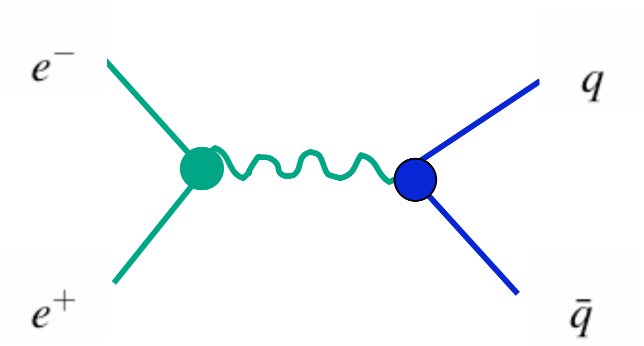


$d\Omega_2$

$$e^+ e^- \to \bar{q}\, q$$

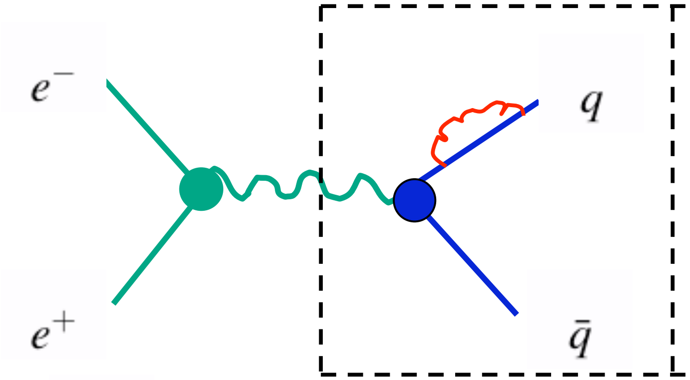
$$\boxed{O(\alpha_s)}$$

Asymptotic Freedom from $R_{e+e-}(S)$



+

$$e^+e^- \to \bar{q}q$$



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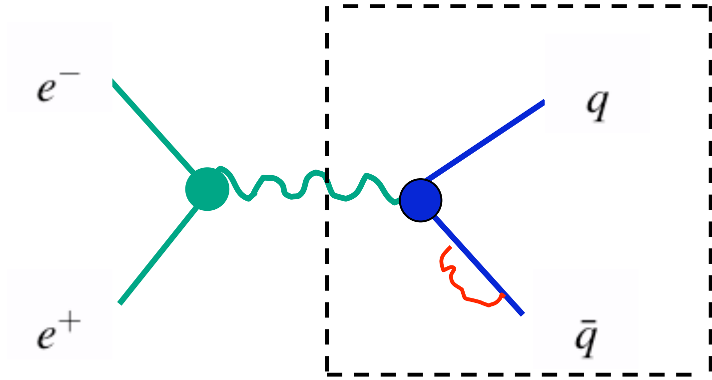

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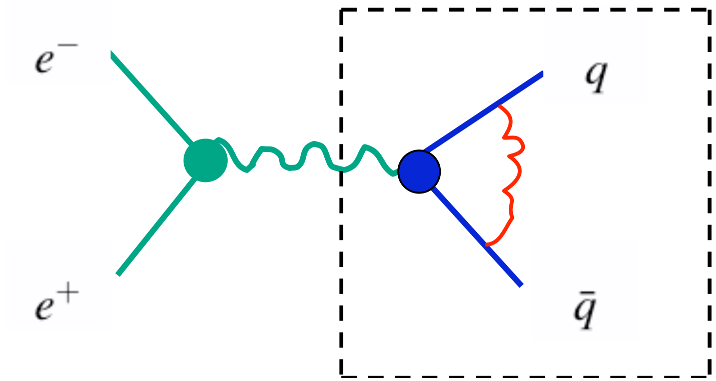


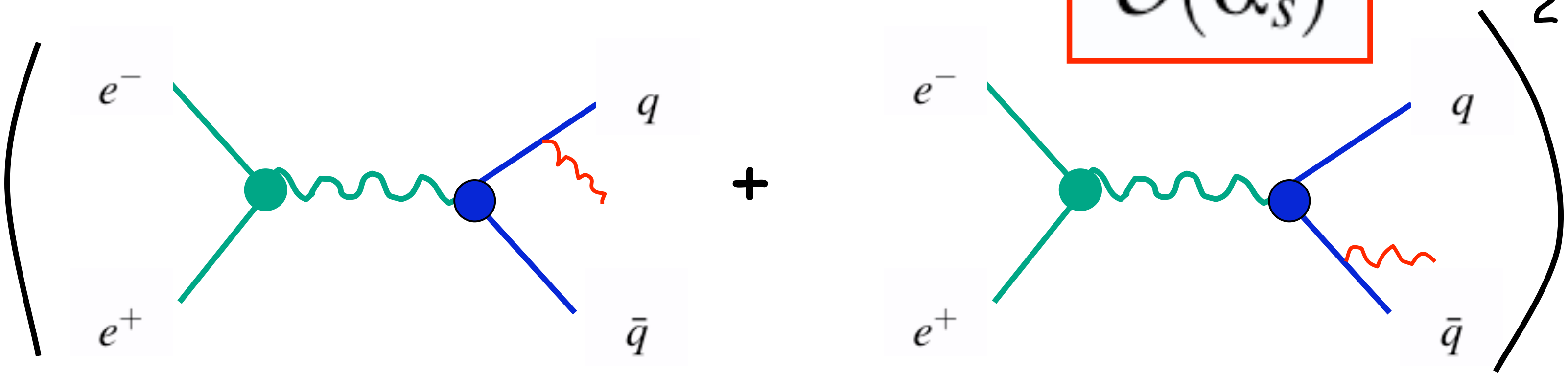
The sum of the diagrams in the dashed boxes is ultraviolet finite because of the electromagnetic current conservation

Asymptotic Freedom from $R_{e+e-}(S)$

The sum of the virtual and real emission diagrams is infrared finite because it corresponds to the physical inclusive cross-section

$e^+e^- \rightarrow \bar{q}qg$

$O(\alpha_s)$



$d\Omega_3$

$$R_{e^+e^-}(S) = N_c \sum_f Q_f^2 \left(1 + \frac{\alpha_s}{\pi} + \dots\right)$$

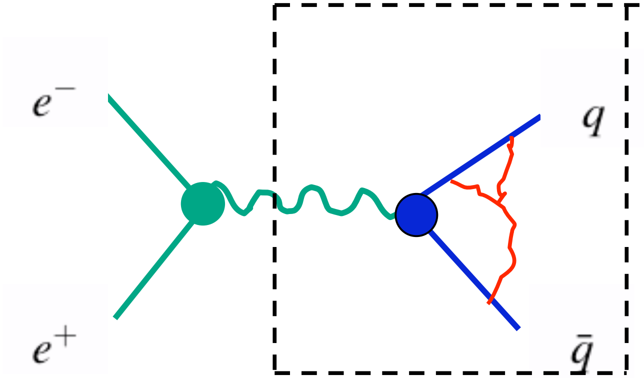
VIRTUAL + REAL CONTRIBUTIONS

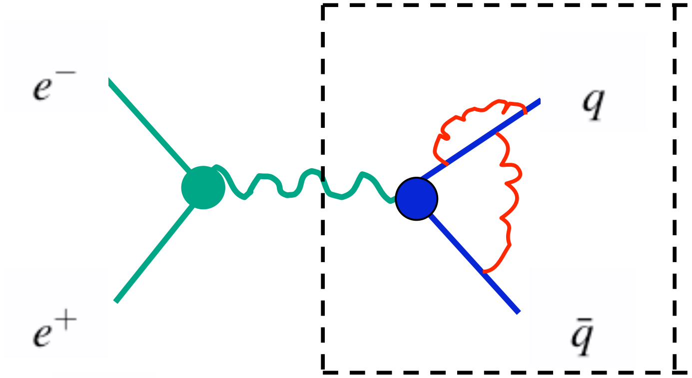
Asymptotic Freedom from $R_{e+e-}(S)$

$O(\alpha_s^2)$

$e^+e^- \rightarrow \bar{q}q$

... +





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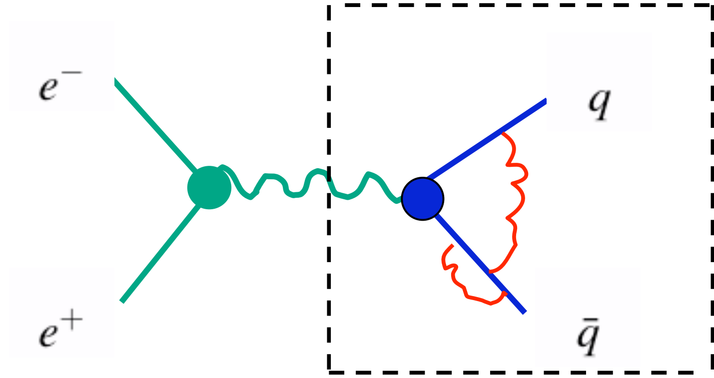

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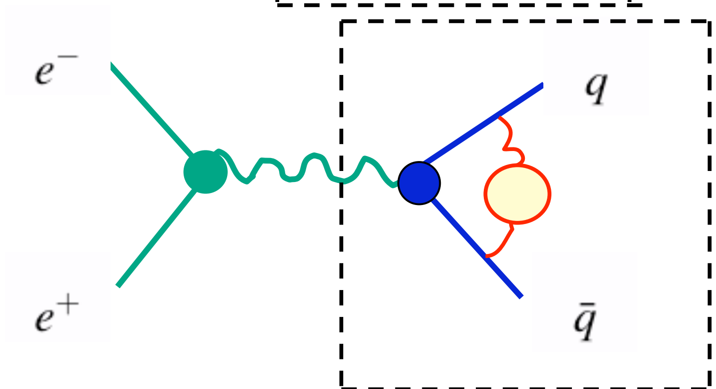
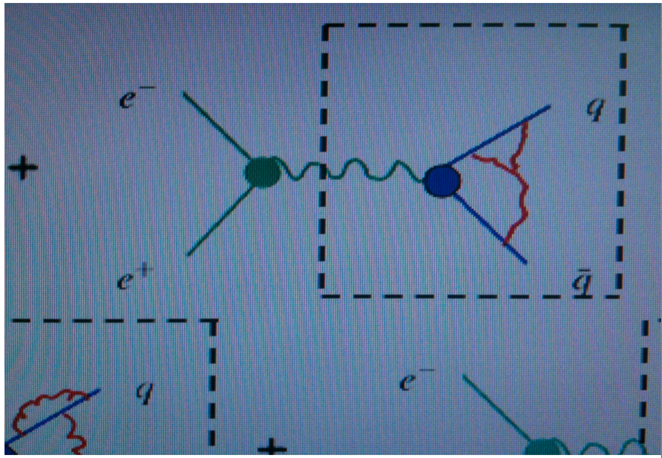
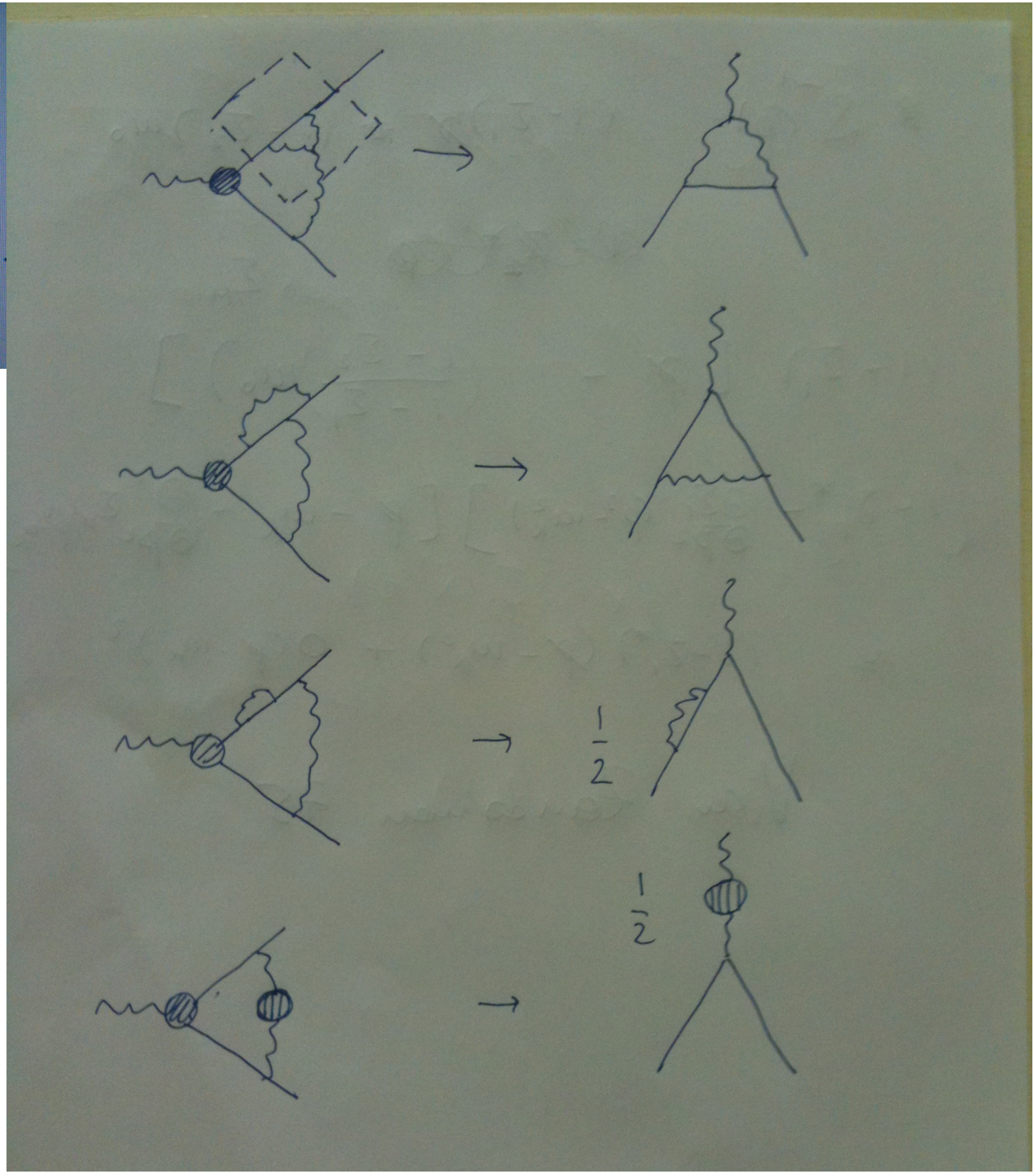


The sum of these diagrams is ultraviolet divergent and renormalizes the strong coupling constant, similarly for the real emission diagrams





This is true also for
the vertices of emission
Of real gluons in

$$e^{+} + e^{-} \rightarrow q + \bar{q} + g$$

Asymptotic Freedom from $R_{e+e-}(S)$

WE HAVE TO INTRODUCE A CUTOFF OVER THE MOMENTA IN THE LOOPS IN ORDER TO REGULARIZE THE ULTRAVIOLET DIVERGENCES

$$R_{e^+e^-}(S) = N_c \sum_f Q_f^2 \left[1 + \frac{\alpha_0}{\pi} - b\left(\frac{\alpha_0}{\pi}\right)^2 \ln\left(\frac{S}{\Lambda_{uv}^2}\right) + O\left(\frac{\alpha_0}{\pi}\right)^2\right]$$

WE THEN DEFINE THE ``RENORMALIZED COUPLING" α_S AT THE SCALE S

$$\frac{\alpha_s(S)}{\pi} = \frac{\alpha_0}{\pi} - b\left(\frac{\alpha_0}{\pi}\right)^2 \ln\left(\frac{S}{\Lambda_{uv}^2}\right)$$

$$\boxed{R_{e^+e^-}(S) = N_c \sum_f Q_f^2 \left[1 + \frac{\alpha_s(S)}{\pi}\right]}$$

We choose $\alpha_S(S)$ in such a way that at the scale S_0

$$\boxed{R^{exp}_{e^+e^-}(S_0) = R^{th}_{e^+e^-}(S_0) = N_c \sum_f Q_f^2 \left[1 + \frac{\alpha_s(S_0)}{\pi}\right]}$$

We cannot predict Re+e-(S_0), which is used to define $\alpha_S(S_0)$, as much as in QED α_{em} is fixed from g-2 or Thomson scattering

Asymptotic Freedom from $R_{e+e-}(S)$

With the previous definition of $\alpha_s(S_0)$

$$R_{e^+e^-}(S) = N_c \sum_f Q_f^2 \left[1 + \frac{\alpha_s(S_0)}{\pi} - b \left(\frac{\alpha_s(S_0)}{\pi} \right)^2 \ln \left(\frac{S}{S_0} \right) + \ldots \right]$$

- $R_{e^+e^-}(S)$ is also independent of the ultraviolet cutoff Λ_{uv}
- $R_{e^+e^-}(S)$ is only expressed in terms of measurable quantities

$$\boxed{S,\, S_0,\, \alpha_s(S_0),\, R^{exp}_{e^+e^-}(S_0)}$$

or:

$$R_{e^+e^-}(S) = N_c \sum_f Q_f^2 \left[1 + \frac{\alpha_s(S)}{\pi} \right]$$

where

$$\frac{\alpha_s(S)}{\pi} = \frac{\alpha_s(S_0)}{\pi} - b \left(\frac{\alpha_s(S_0)}{\pi} \right)^2 \ln \left(\frac{S}{S_0} \right)$$

Asymptotic Freedom from $R_{e+e-}(S)$

$$\frac{\alpha_s(S)}{\pi} = \frac{\alpha_s(S_0)}{\pi} - b\left(\frac{\alpha_s(S_0)}{\pi}\right)^2 \ln\left(\frac{S}{S_0}\right)$$
$$\sim \frac{\alpha_s(S_0)/\pi}{1 + b\,(\alpha_s(S_0)/\pi)\ln(S/S_0)} \equiv \frac{1}{b\ln\left(S/\Lambda^2_{QCD}\right)}$$

when $\ln\left(\frac{S}{S_0}\right) \gg 1$

$$\Lambda^2_{QCD} = S\,e^{-\pi/b\alpha_S(S)}$$

is a non-perturbative constant, independent of S, like the proton mass (it depends however on the renormalization scheme and on the order computed in perturbation theory)

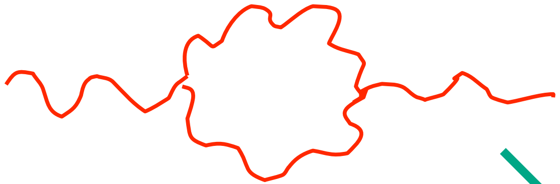
Renormalization Group Equations

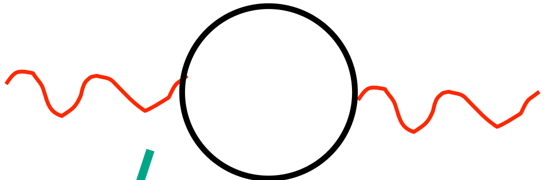
$$\alpha_s(S) = \frac{1}{\beta_0 \ln\left(S/\Lambda^2_{QCD}\right)} \quad \rightarrow \quad \frac{d\alpha_s(S)}{d\ln S} = -\beta_0 \alpha_s^2(S)$$

IN GENERAL

$$\boxed{\frac{d\alpha_s(S)}{d\ln S} = \beta\left(\alpha_s(S)\right) = -\beta_0\alpha_s^2(S) - \beta_1\alpha_s^3(S) + \ldots}$$

NOTE THAT $\beta_0 > 0$ UNLIKE IN QED (also $\beta_1 > 0$)





$$\beta_0 = \frac{1}{4\pi}\left(\frac{11}{3}N_c - \frac{2}{3}n_f\right)$$

Renormalization Group Equations

$$\left(s\frac{\partial}{\partial s}+\beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right) R_{e^+e^-}(S)=0$$

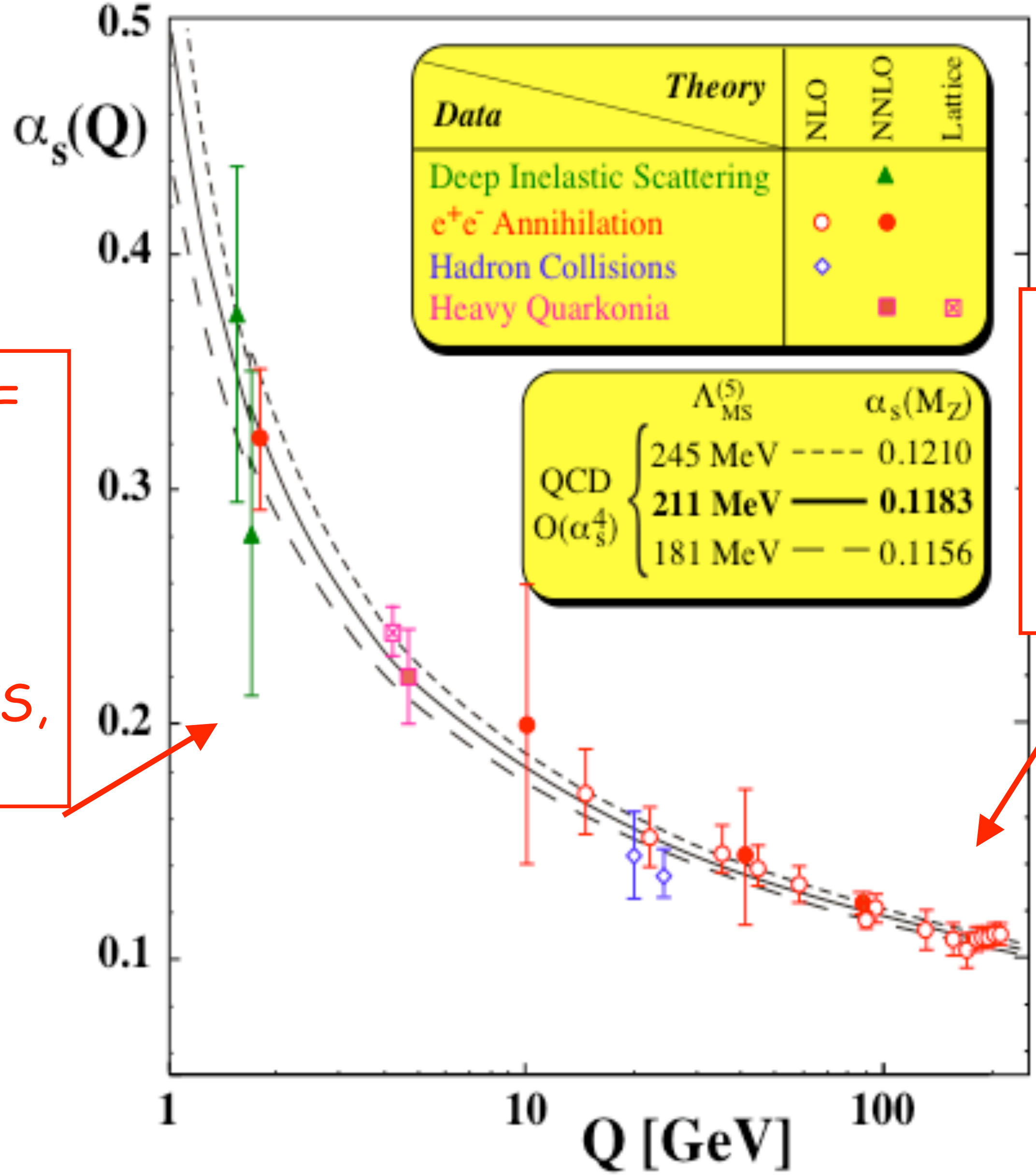
$R_{e^+e^-}(S)$ depends on the scale only because of α_s

$$R_{e^+e^-}(S)=N_c\sum_f Q_f^2\left[1+\frac{\alpha_s(S)}{\pi}+C_2\left(\frac{\alpha_s(S)}{\pi}\right)^2+C_3\left(\frac{\alpha_s(S)}{\pi}\right)^3+\ldots\right]$$

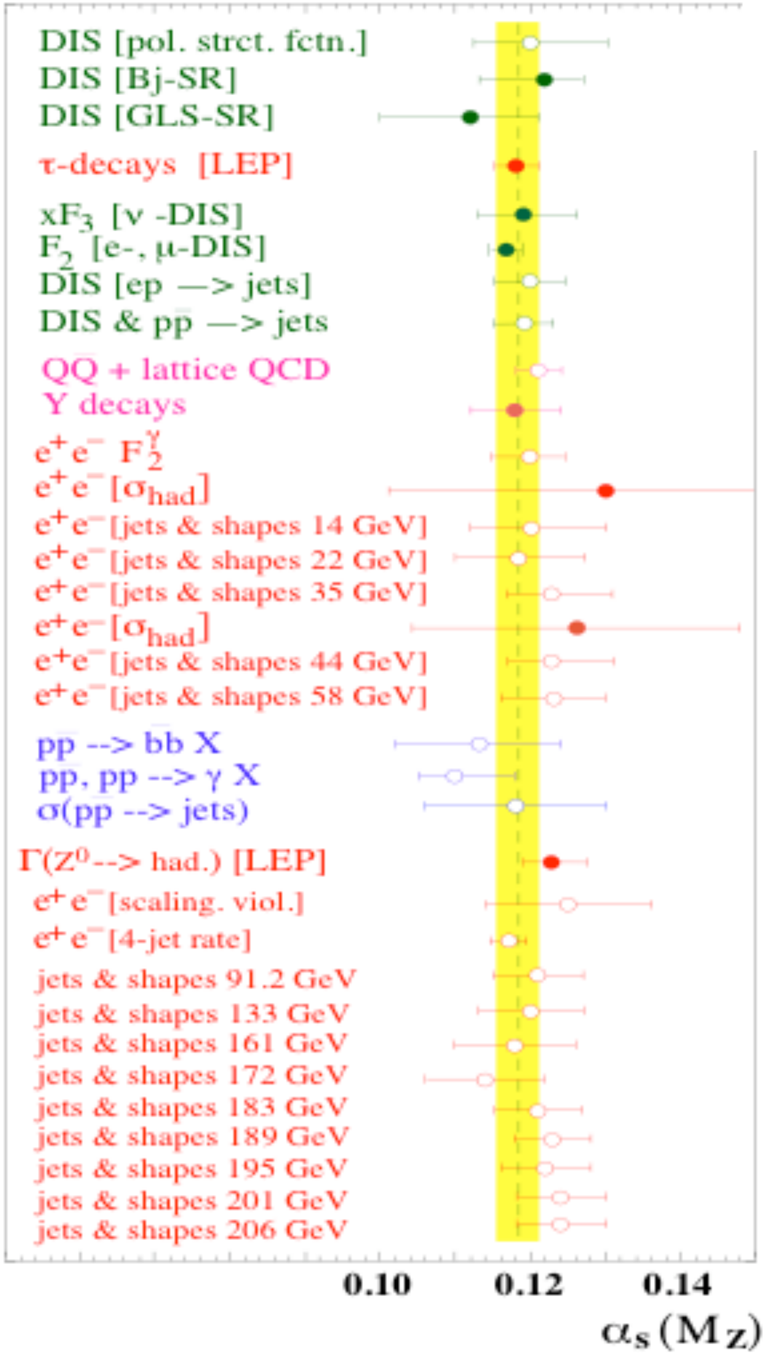
$$\Gamma[Z\to h's]=\Gamma^0[Z\to h's]\,R_{QCD}$$
$$=\Gamma^0[Z\to h's]\left[1+\frac{\alpha_s(S)}{\pi}+1.411\left(\frac{\alpha_s(S)}{\pi}\right)^2-12.8\left(\frac{\alpha_s(S)}{\pi}\right)^3+\ldots\right]$$
$$R_{QCD}\sim 1.04 \quad \delta\Gamma\sim 70\,MeV$$

scheme dependence $\longrightarrow$ $\alpha_s'(S)=\alpha_s(S)+\kappa_2\alpha_s^2(S)+\ldots$

SMALL SCALES = LARGE NON PERTURBATIVE EFFECTS (MASS TERMS, HIGHER TWISTS, ETC.)



LARGE SCALES= SMALL SENSITIVITY TO Λ_{QCD}



$$\alpha_s(M_z) = 0.119 \pm 0.003$$

$$\alpha_S(M_Z) = 0.1189 \pm 0.0010$$

Bethke hep-ex/0606035

The QCD Beta Function (B. Webber)

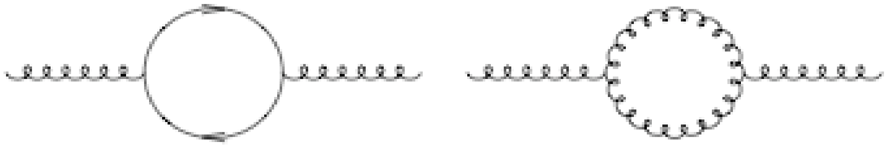
- Running of of the QCD coupling α_S is determined by the β function, which has the expansion

$$\beta(\alpha_S) = -b\alpha_S^2(1 + b'\alpha_S) + \mathcal{O}(\alpha_S^4)$$

$$b = \frac{(11C_A - 2n_f)}{12\pi}, \quad b' = \frac{(17C_A^2 - 5C_A n_f - 3C_F n_f)}{2\pi(11C_A - 2n_f)},$$

where n_f is number of "active" light flavours. Terms up to $\mathcal{O}(\alpha_S^5)$ are known.

- Roughly speaking, quark loop "vacuum polarisation" diagram (a) contributes negative n_f term in b, while gluon loop (b) gives positive C_A contribution, which makes β function negative overall.



a) b)

- QED β function is

$$\beta_{QED}(\alpha) = \frac{1}{3\pi}\alpha^2 + \ldots$$

Thus b coefficients in QED and QCD have opposite signs.

- From previous section,

$$\frac{\partial \alpha_S(Q)}{\partial \tau} = -b\alpha_S^2(Q)\left[1 + b'\alpha_S(Q)\right] + \mathcal{O}(\alpha_S^4).$$

Neglecting b' and higher coefficients gives

$$\alpha_S(Q) = \frac{\alpha_S(\mu)}{1 + \alpha_S(\mu)b\tau}\,, \quad \tau = \ln\left(\frac{Q^2}{\mu^2}\right).$$

- As Q becomes large, $\alpha_S(Q)$ decreases to zero: this is asymptotic freedom. Notice that sign of b is crucial. In QED, $b < 0$ and coupling *increases* at large Q.
Including next coefficient b' gives implicit equation for $\alpha_S(Q)$:

$$b\tau = \frac{1}{\alpha_S(Q)} - \frac{1}{\alpha_S(\mu)} + b' \ln\left(\frac{\alpha_S(Q)}{1 + b'\alpha_S(Q)}\right) - b' \ln\left(\frac{\alpha_S(\mu)}{1 + b'\alpha_S(\mu)}\right)$$

History of Asymptotic Freedom

1954 Yang & Mills study vector field theory with non-Abelian gauge invariance.

1965 Vanyashin & Terentyev compute vacuum polarization due to a massive charged vector field. In our notation, they found

$$b = \frac{1}{12\pi}\left(\frac{21}{2} = 11 - \frac{1}{2}\right)$$

The $\frac{1}{2}$ comes from longitudinal polarization states (absent for massless gluons)

- They concluded that this result ". . . seems extremely undesirable"

1969 Khriplovich correctly computes the one-loop β-function in SU(2) Yang-Mills theory using the Coulomb ($\nabla \cdot A = 0$) gauge

$$b = \frac{C_A}{12\pi}(12 - 1 = 11)$$

In Coulomb gauge the anti-screening (12) is due to an instantaneous Coulomb interaction

- He did not make a connection with strong interactions

1971 't Hooft computes the one-loop β-function for SU(3) gauge theory but does not publish it.

- He wrote it on the blackboard at a conference
- His supervisor (Veltman) told him it wasn't interesting
- 't Hooft & Veltman received the 1999 Nobel Prize for proving the *renormalizability* of QCD (and the whole Standard Model).

1972 Fritzsch & Gell-Mann propose that the strong interaction is an SU(3) gauge theory, later named QCD by Gell-Mann

1973 Gross & Wilczek, and independently Politzer, compute and publish the 1-loop β-function for QCD:

$$b = \frac{1}{12\pi}(11C_A - 2n_f)$$

$\Rightarrow$2004 Nobel Prize (now that 't Hooft has one anyway . . .)

1974 Caswell† and Jones compute the 2-loop β-function for QCD.

1980 Tarasov, Vladimirov & Zharkov compute the 3-loop β-function for QCD.

1997 van Ritbergen, Vermaseren & Larin compute the 4-loop β-function for QCD ($\sim 50,000$ Feynman diagrams):

"... We obtained in this way the following result for the 4-loop beta function in the $\overline{\mathrm{MS}}$-scheme:

$$q^2\frac{\partial a_s}{\partial q^2} = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \beta_3 a_s^5 + O(a_s^6)$$

where $a_s = \alpha_S/4\pi$ and . . .

$$
\begin{aligned}
\beta_0 &= \frac{11}{3}C_A - \frac{4}{3}T_F n_f\ , \quad \beta_1 = \frac{34}{3}C_A^2 - 4C_F T_F n_f - \frac{20}{3}C_A T_F n_f \\
\beta_2 &= \frac{2857}{54}C_A^3 + 2C_F^2 T_F n_f - \frac{205}{9}C_F C_A T_F n_f \\
&\quad - \frac{1415}{27}C_A^2 T_F n_f + \frac{44}{9}C_F T_F^2 n_f^2 + \frac{158}{27}C_A T_F^2 n_f^2 \\
\beta_3 &= C_A^4\left(\frac{150653}{486} - \frac{44}{9}\zeta_3\right) + C_A^3 T_F n_f\left(-\frac{39143}{81} + \frac{136}{3}\zeta_3\right) \\
&\quad + C_A^2 C_F T_F n_f\left(\frac{7073}{243} - \frac{656}{9}\zeta_3\right) + C_A C_F^2 T_F n_f\left(-\frac{4204}{27} + \frac{352}{9}\zeta_3\right) \\
&\quad + 46 C_F^3 T_F n_f + C_A^2 T_F^2 n_f^2\left(\frac{7930}{81} + \frac{224}{9}\zeta_3\right) + C_F^2 T_F^2 n_f^2\left(\frac{1352}{27} - \frac{704}{9}\zeta_3\right) \\
&\quad + C_A C_F T_F^2 n_f^2\left(\frac{17152}{243} + \frac{448}{9}\zeta_3\right) + \frac{424}{243}C_A T_F^3 n_f^3 + \frac{1232}{243}C_F T_F^3 n_f^3 \\
&\quad + \frac{d_A^{abcd}d_A^{abcd}}{N_A}\left(-\frac{80}{9} + \frac{704}{3}\zeta_3\right) + n_f\frac{d_F^{abcd}d_A^{abcd}}{N_A}\left(\frac{512}{9} - \frac{1664}{3}\zeta_3\right) \\
&\quad + n_f^2\frac{d_F^{abcd}d_F^{abcd}}{N_A}\left(-\frac{704}{9} + \frac{512}{3}\zeta_3\right)
\end{aligned}
$$

Here ζ is the Riemann zeta-function ($\zeta_3 = 1.202\cdots$) and the colour factors for SU(N) are

$$T_F = \frac{1}{2}, \quad C_A = N, \quad C_F = \frac{N^2-1}{2N}, \quad \frac{d_A^{abcd}d_A^{abcd}}{N_A} = \frac{N^2(N^2+36)}{24},$$
$$\frac{d_F^{abcd}d_A^{abcd}}{N_A} = \frac{N(N^2+6)}{48}, \quad \frac{d_F^{abcd}d_F^{abcd}}{N_A} = \frac{N^4-6N^2+18}{96N^2}$$

- Substitution of these colour factors for $N = 3$ yields the following numerical results for QCD:

$$\begin{aligned}
\beta_0 &\approx 11 - 0.66667 n_f \\
\beta_1 &\approx 102 - 12.6667 n_f \\
\beta_2 &\approx 1428.50 - 279.611 n_f + 6.01852 n_f^2 \\
\beta_3 &\approx 29243.0 - 6946.30 n_f + 405.089 n_f^2 + 1.49931 n_f^3
\end{aligned}$$

- Expansion parameter $a_s = \alpha_S/4\pi \approx 0.01 \Rightarrow$ good convergence.