Spectral Action from Anomalies

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What if we got it wrong and instead of the familiar

Fiat Lux

we had to use:

Fiat Materia!

We start from some matter fields described by fermion fields Ψ transforming under some reducible representation of a gauge group (such as $SU(3) \times SU(2) \times U(1)$)

The fermions belong to a Hilbert space of spinors $\mathcal{H}$, which I take to have a left and a right component

$$\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R$$

Once you have created fermions and symmetries you need to make them move, so you create a background for them and a classical action

$$\langle\Psi|\, D\, |\Psi\rangle$$

where D is an operator on $\mathcal{H}$ which I will call the Dirac operator

This operator is made of two parts acting on spinors $\begin{pmatrix}\Psi_L \\ \Psi_R\end{pmatrix}$:

$$D = D_0 + A$$

$$D_0 = \begin{pmatrix} \gamma^\mu \partial_\mu & M \\ M^\dagger & \gamma^\mu \partial_\mu \end{pmatrix}$$

Where M contains all masses (and mixings) of the fermions and the γ's are those relative to a possibly curved spacetime

The matrix A is a fixed background gauge field.

I have implicitly introduced a (Euclidean) spacetime. And therefore at this stage I have the notion of the algebra $\boxed{\mathcal{A}}$ of continuous functions of this space time, which in this case is really a noncommutative algebra of matrix valued functions acting as operators on $\boxed{\mathcal{H}}$

I could have gone the other way around, defining from the start the spectral triple $\boxed{\mathcal{A}, \mathcal{H}, D}$ which contains all information about the space and its metric properties

This is Connes programme of translation of all geometry into into algebraic terms, based on the spectral triple

In fact the model is ready for generalizations to genuinely non-commutative spaces, a step I will not undertake now.

I am deliberately vague as to the detail of the model at this stage, and I am not discussing important elements of the theory, like chirality or charge conjugation. What I will be discussing next is rather solid and does not depend crucially on these

So far I have a classical theory of matter fields moving in a fixed background

The objects involved in the writing of the action have physical dimension, for example an unit of length $\boxed{\ell}$, so that I can measure volumes as $\boxed{\ell^{-4}}$, masses and the Dirac operator in general as $\boxed{\ell^{-1}}$ etc. I take the speed of light to be 1

The classical action is invariant under a change of this scale, which can also be local, (Weyl original gauge theory). Obviously the classical theory is invariant under such change.

We therefore have a scale transformation symmetry:

$$\boxed{x^\mu \to \mathrm{e}^{\phi} x^\mu \ , \psi \to \mathrm{e}^{-\frac{3}{2}\phi}\psi \ , D \to \mathrm{e}^{-\frac{1}{2}\phi} D \mathrm{e}^{-\frac{1}{2}\phi}}$$

But if the classical theory is invariant, the measure in the quantum path integral is not. We have an anomaly!

We start from the partition function

$$\boxed{Z(D) = \int [\mathrm{d}\psi][\mathrm{d}\bar{\psi}] e^{-S_\psi} = \det(D)}$$

where the last equality is formal because the expression is divergent and needs regularizing. In fact we need two regulators:

- An infrared cutoff $\boxed{\mu}$ in order to have a discrete spectrum

- An ultraviolet cutoff $\boxed{\Lambda}$ in order to tame the short distance infinities

We will regularize the theory in the ultraviolet using a procedure introduced by A. Andrianov and Bonora (see also V. Andrianov, Novozhilov^2, Vassilevich)

The energy cutoff is enforced by considering only the first N eigenvalues of D

Consider the projector $P_N = \sum_{n=0}^{N} |\lambda_n\rangle\langle\lambda_n|$ with λ_n and $|\lambda_n\rangle$ the eigenvalues and eigenvectors of D

The integer N is a function of the cutoff and is defined as

$$N = \max n \text{ such that } \lambda_n \leq \Lambda$$

We effectively use the N^{th} eigenvalue as cutoff

In the framework of noncommutative geometry this is the most natural cutoff procedure, although it was introduced before Connes-Lott-Chamseddine work on the standard model

Actually $\boxed{N}$ depends also on the infrared cutoff, and the number of dimensions, fact it goes as $\boxed{\sim \left(\frac{\Lambda}{\mu}\right)^d}$, but I will not discuss infrared issues in this talk

The fermionic action is still invariant under the scale transformation

However the measure of the partition function is not invariant. This means that we have an anomaly.

The anomaly can however be cancelled adding another term to the action which will correct the measure. This term contains the bosonic degrees of freedom

Thus the cancellation of the anomaly forces the addition of the bosonic degrees of freedom to the fermionic one.

This new effective action is product of quantization, but in the end we will consider the full action to be the sum of two terms.

Define the regularized partition function

$$Z_\Lambda(D) = \prod_{n=0}^{N} \lambda_n = \det\left(\mathbb{1} - P_N + P_N \frac{D}{\Lambda} P_N\right)$$

Z_Λ has a well defined meaning setting $\psi = \sum a_n |\lambda_n\rangle \;\; , \bar{\psi} = \sum b_n |\lambda_n\rangle$ with a_n, b_n anticommuting (Grassman) quantities, we have

$$Z_\Lambda(D) = \int \prod_{n=0}^{N} \mathrm{d}a_n \mathrm{d}b_n \mathrm{e}^{-\sum_{n=0}^{N} b_n \frac{\lambda_n}{\Lambda} a_n} = \det(D_N)$$

where we defined $D_N = 1 - P_N + P_N \frac{D}{\Lambda} P_N$ which corresponds to set to 1 all eigenvalues larger than 1.

D_N is dimensionless and depends on Λ both explicitly and intrinsically via the dependence of N and P_N

The compensating term, the effective action, is

$$Z_{\mathrm{inv}\Lambda}(D) = Z_{\Lambda}(D) \int \mathrm{d}\phi \, \mathrm{e}^{-S_{\mathrm{anom}}}$$

The calculation is standard and not difficult:

Define

$$Z^{-1}_{\mathrm{inv}\Lambda}(D) = \int \mathrm{d}\phi Z^{-1}_{\Lambda}(\mathrm{e}^{-\frac{1}{2}\phi} D \mathrm{e}^{-\frac{1}{2}\phi})$$

therefore

$$S_{\mathrm{anom}} = \log Z_{\Lambda}(D) Z^{-1}_{\mathrm{inv}\,N}(D)$$

Let us indicate

$$Z_t = Z_{\Lambda}(\mathrm{e}^{-\frac{t}{2}\phi} D \mathrm{e}^{-\frac{t}{2}\phi})$$

therefore $Z_0 = Z_{\Lambda}(D)$ and

$$Z_{\Lambda}(D) Z^{-1}_{\mathrm{inv}\,N}(D) = \int \mathrm{d}\phi \frac{Z_0}{Z_1}$$

and hence

$$S_{\mathrm{eff}} = \int_0^1 \mathrm{d}t \partial_t \log Z_t = \int_0^1 \mathrm{d}t \frac{\partial_t Z_t}{Z_t}$$

We have the following relation that can easily proven:

$$D_N^{-1} = (1 - P_N + P_N D P_N)^{-1} = 1 - P_N + P_N D^{-1} P_N$$

and

$$\begin{aligned}
\partial_t Z_t &= \partial_t \det(\mathrm{e}^{-\frac{t}{2}\phi} D \mathrm{e}^{-\frac{t}{2}\phi})_N \\
&= \partial_t \mathrm{e}^{\mathrm{tr}\,\log(1 - P_N + \mathrm{e}^{-\frac{t}{2}\phi} D_N \mathrm{e}^{-\frac{t}{2}\phi})} \\
&= \mathrm{Tr}\,(\partial_t \log(1 - P_N + \mathrm{e}^{-\frac{t}{2}} D_N \mathrm{e}^{-\frac{t}{2}\phi})) Z_t \\
&= \mathrm{Tr}\,((1 - P_N + \mathrm{e}^{-\frac{t}{2}\phi} D_N \mathrm{e}^{-\frac{t}{2}\phi})^{-1} \phi \mathrm{e}^{-\frac{t}{2}\phi} D_N \mathrm{e}^{-\frac{t}{2}\phi}) Z_t \\
&= \phi Z_t \,\mathrm{tr}\, P_N
\end{aligned}$$

In the end

$$S_{\text{anom}} = \int_0^1 \mathrm{d}t\, \phi\, \mathrm{tr}\, P_N$$

The quantity $\mathrm{tr}\, P_N = N$ is the number of eigenvalues of D smaller than Λ. It depends on D_0 as well as its fluctuations in both the gravitational and gauge sectors, in fact we can express it as

$$\mathrm{Tr}\, P_N = N = \mathrm{Tr}\, \chi \left(\frac{D_\phi^2}{\Lambda^2} \right)$$

Where χ is the characteristic function of the interval $[0, 1]$ and $D_\phi = \mathrm{e}^{-\frac{1}{2}\phi} D \mathrm{e}^{-\frac{1}{2}\phi}$

For $\boxed{\phi = 0}$ this is the Chamseddine-Connes **Spectral Action** introduced to describe the bosonic degree of freedom of the standard model coupled with gravity

We have obtained it as an additional term to the fermionic action upon quantizing it and demanding freedom from anomalies

The fact that the two terms of the spectral action must be on the same footing has been advocated already by Sitarz

To obtain the standard model take as algebra the product of the algebra of functions on spacetime times a finite dimensional matrix algebra

$$\boxed{\mathcal{A} = C(\mathbb{R}^4) \otimes \mathcal{A}_F}$$

Likewise the Hilbert space is the product of fermions times a finite dimensional space which contains all matter degrees of freedom, and also the Dirac operator contains a continuous part and a discrete one

$$\mathcal{H} = \mathrm{Sp}(\mathbb{R}^4) \otimes \mathcal{H}_F$$

$$D_0 = \gamma^\mu \partial_\mu \otimes \mathbb{I} + \gamma \otimes D_F$$

In its most recent form (Chamseddine-Connes-Marcolli) a crucial role is played by the mathematical requirements that the non-commutative algebra satisfies the requirements to be a manifold

Then the internal algebra, is almost uniquely derived to be

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

The bosonic spectral action can be evaluated using Vassilevich's manual on heath kernel techniques and the final result gives the action of the standard model coupled with gravity.

The fascinating aspect of this theory is that the Higgs appears naturally as the "vector" boson of the internal noncommutative degrees of freedom Connes, Lott, Dubois-Violette, Madore, Kerner . . . . In the process of writing the action all masses and coupling are used as inputs, but one saves one parameter.

The Higgs mass is predicted, in the present form of the model, to be $\boxed{\sim 170\text{GeV}}$. A value too small and experimentally disfavoured. Nevertheless I still find it fascinating that a theory without so little input finds a Higgs mass relatively close to the expected value

Technically the bosonic spectral action is a sum of residues and can be expanded in a power series in terms of Λ^{-1} as

$$S_B = \sum_n f_n \, a_n(D^2/\Lambda^2)$$

where the f_n are the momenta of χ

$$\begin{aligned} f_0 &= \int_0^\infty \mathrm{d}x \, x\chi(x) \\ f_2 &= \int_0^\infty \mathrm{d}x \, \chi(x) \\ f_{2n+4} &= (-1)^n \partial_x^n \chi(x) \bigg|_{x=0} \quad n \geq 0 \end{aligned}$$

the a_n are the Seeley-de Witt coefficients which vanish for n odd. For D^2 of the form

$$D^2 = g^{\mu\nu}\partial_\mu\partial_\nu \mathbb{1} + \alpha^\mu \partial_\mu + \beta$$

defining

$$\begin{aligned}
\omega_\mu &= \frac{1}{2} g_{\mu\nu} \left(\alpha^\nu + g^{\sigma\rho} \Gamma^\nu_{\sigma\rho} \mathbb{1} \right) \\
\Omega_{\mu\nu} &= \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu] \\
E &= \beta - g^{\mu\nu} \left(\partial_\mu \omega_\nu + \omega_\mu \omega_\nu - \Gamma^\rho_{\mu\nu} \omega_\rho \right)
\end{aligned}$$

then

$$\begin{aligned}
a_0 &= \frac{\Lambda^4}{16\pi^2} \int \mathrm{d}x^4 \sqrt{g}\, \mathrm{tr}\, \mathbb{1}_F \\
a_2 &= \frac{\Lambda^2}{16\pi^2} \int \mathrm{d}x^4 \sqrt{g}\, \mathrm{tr} \left(-\frac{R}{6} + E \right) \\
a_4 &= \frac{1}{16\pi^2} \frac{1}{360} \int \mathrm{d}x^4 \sqrt{g}\, \mathrm{tr} \left(-12 \nabla^\mu \nabla_\mu R + 5R^2 - 2R_{\mu\nu} R^{\mu\nu} \right. \\
&\left. + 2R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} - 60RE + 180E^2 + 60\nabla^\mu \nabla_\mu E + 30\Omega_{\mu\nu}\Omega^{\mu\nu} \right)
\end{aligned}$$

tr is the trace over the inner indices of the finite algebra $\mathcal{A}_F$ and in Ω and E are contained the gauge degrees of freedom including the gauge stress energy tensors and the Higgs, which is given by the inner fluctuations of D

In our case for ϕ constant, after performing hte integration we find

$$S_{\text{anom}} = \int_0^{\phi} \mathrm{d}t' \sum_n \mathrm{e}^{(4-n)t'} a_n f_n = \frac{1}{8}(e^{4\phi} - 1)a_0 + \frac{1}{2}(e^{2\phi} - 1)a_2 + \phi a_4.$$

There are just some numerical corrections to the first two Seeley-de Witt coefficients due to the integration in $t\phi$

I will not discuss in detail the case of $\boxed{\phi}$ nonconstant, in this case we have a dilaton in the theory which will couple with the other fields

The spectral action in this case involves

$$\boxed{D_\phi = \mathrm{e}^{-\phi}\gamma^\mu \left(\partial_\mu + A_\mu - \frac{1}{2}\partial_\mu\phi\right)}$$

where with A_μ I have generically indicated the gauge and spin connections. The calculation (still in progress) can be done again using Vassilevich manual, and I will just make some considerations on the interplay between the Higgs potential and the effective potential of the dilaton

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On the seventh he rested, while the universe was inflating