The large N limit and the Functional Renormalisation Group

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Motivations:

$\rightarrow$ The expansion in $\frac{1}{N}$ is a powerful technique in QFT and Statistical Physics

$\rightarrow$ This approach allows to study theory with large coupling

$\rightarrow$ It has been used in the context of QCD ($N =$ number of colors), Scalar theory ($N =$ number of fields), etc...

Tasks:

$\rightarrow$ How this approximation works in the context of the Functional Renormalisation Group ?

$\rightarrow$ What results can we get from it ?

(Wetterich and Tetradis 94', Litim and Tetradis 95')

$\rightarrow$ Precisely we are interested in a scalar theory with an $\mathcal{O}(N)$ symmetry

$\rightarrow$ Starting point : Flow equation for an average action

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\Big\{\big[\Gamma_k^{(2)}[\phi] + R_k\big]^{-1}\partial_t R_k\Big\}$$

$\rightarrow$ Ansatz for Γ_k : the Local Potential Approximation $(\bar{\rho} = \phi^2/2)$

$$\Gamma_k[\phi] = \int d^d x \,\Big\{U_k(\bar{\rho}) + \frac{1}{2}(\partial_\mu \phi)^2\Big\}$$

We introduce dimensionless variables:

$$\rho = k^{2-d}\bar{\rho} \ , \quad u_k(\rho) = k^{-d} U_k(\bar{\rho})$$

and take the large N limit (we neglect radial mode)

$$\partial_t u - (d-2)\rho u' + du = 2v_d N \int_0^\infty dy y^{\frac{d}{2}+1} \frac{r'(y)}{[y(1+r)+u']}$$

where $y = q^2/k^2$, $R_k(q^2) = q^2 r(y)$ ($2v_d N$ can be absorbed)

NEW : We interchange momenta and scale integration to solve the flow equation $(P^2 = y(1+r))$

$$\partial_t u' - (d-2)\rho u'' + \frac{2}{d}\,\mathcal{I}\left[\frac{P^{2+d}}{(P^2+u')^2}\right]u'' + 2u' = 0$$

We introduce the operator

$$\mathcal{I}[f(y)] = -\frac{d}{2}\int_0^\infty \frac{r'(y)\ dy}{[1+r(y)]^{\frac{d}{2}+1}}\ f(y)$$

$\rightarrow$ Important property : $\mathcal{I}[1] = 1$

$\rightarrow$ The integrand is local in momenta

Using the method of characteristics we find

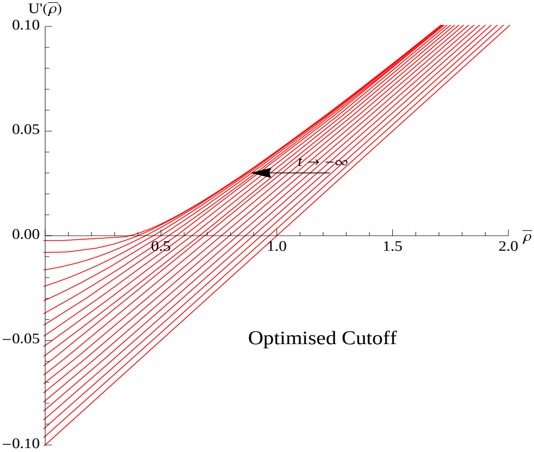
$$\rho \cdot |u'|^{1-d/2} - \mathcal{G}_P(u') = F(u' e^{2t})$$

with

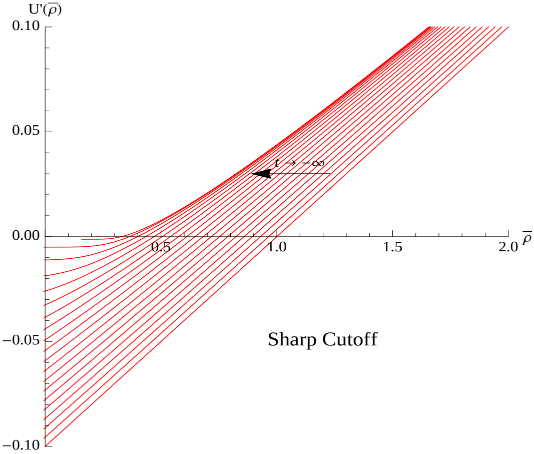
$$\mathcal{G}_P(u') \;=\; \frac{2}{d(2+d)}\,\mathcal{I}\Bigg[\frac{P^{d+2}}{(P^2+u')^{1+d/2}} \times {}_2F_1\left(\frac{d}{2}, 1+\frac{d}{2}, 2+\frac{d}{2}, \frac{P^2}{P^2+u'}\right)\Bigg]$$

$\rightarrow$ Defined for $-1 < \frac{u'}{P^2}$

$\rightarrow$ Full solution for $d > 2$ and arbitrary regulator $r(y)$



Effective potential for $-\infty < t < 0$ and $d = 3$ in dimensionful units ($\Lambda = 1$). We used the initial conditions $u'_{t=0} = \lambda_\Lambda(\rho - \kappa_\Lambda)$ with $\lambda_\Lambda = 0.2$ and $\kappa_\Lambda = 1$


U'($\bar{\rho}$)
0.10
0.05
0.00
−0.05
−0.10
0.5
1.0
1.5
2.0
$\bar{\rho}$
$t \to -\infty$
Sharp Cutoff

Solution in 3d: (we imposed initial conditions)

$$\rho = \frac{|u'|}{\lambda_\Lambda}\, e^t + \frac{\kappa_\Lambda}{e^t} + |u'|^{1/2} \left(\mathcal{G}_P(u') - \mathcal{G}_P(u' e^{2t}) \right)$$

When $t \to -\infty$, for specific initial conditions, there exist a non gaussian (Wilson-Fisher) fixed point solution

$$\rho = \frac{2}{3}\, \mathcal{I}\Big[P\ {}_2F_1\Big(2, -\frac{1}{2}, \frac{1}{2}, -\frac{u'_\star}{P^2}\Big)\Big]$$

(we still have a pole at $u' = -\frac{u'}{P^2}$)

$\rightarrow$ This solution separates the symmetric phase ($\rho_0 = 0$) and the phase with a spontaneously broken symmetry ($\rho_0 \neq 0$)

$\rightarrow$ We can use this solution to compute critical exponent

Expansion in amplitude

For large u' (f depend on the regulator)

$$\rho(u'_\star) = \frac{\pi}{2}\,\sqrt{u'_\star} + \mathcal{O}\Big(f(u'^{-1}_\star)\Big)$$

For small u'

$$\rho(u'_\star) = \sum_{n=0}^{\infty} \alpha_n (u'_\star)^n$$

with

$$\alpha_n = (-1)^{n+1}\,\frac{2}{3}\,\left(\frac{n+1}{2n-1}\right)\,\mathcal{I}[P^{1-2n}]$$

→ Clear dependence of the convergence on the regulator

→ Convergence Radius: $r_c(sharp) = \frac{1}{2}$, $r_c(opt) = 1$

Expansion in the field

→ Implicit solution in ρ

→ Different convergence properties for expansion in ρ

By inverting the expansion of $\rho(u'_\star)$, we compute

$$u'_\star(\rho) = u'_{\star 0} + \sum_{n=1}^{\infty} \gamma_n (\rho - \rho_0)^n .$$

→ Two examples: the sharp and the optimised cutoff

We evaluate numerically the radius of convergence:

	Sharp	Optimised
$\rho = 0$	2.17845 ...	3.21630 ...
$\rho = \rho_0$	2.66189 ...	3.21036 ...

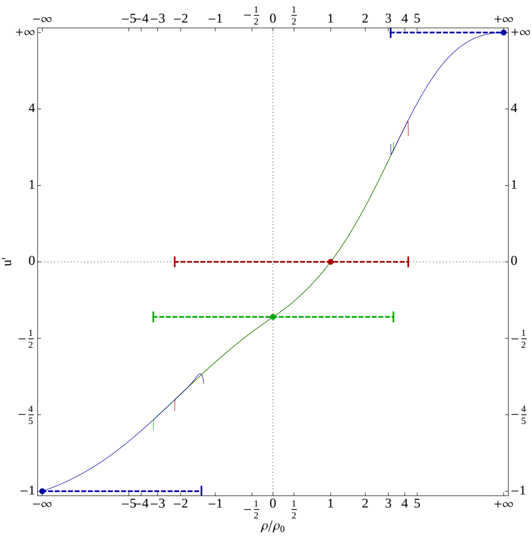
About $\rho = +\infty \to$ similar behavior for both cutoff

$$u'_\star(\rho) \quad = \quad a_0\rho^2 + \frac{a_1}{\rho} + \sum_{n=2}^{\infty} \frac{a_n}{\rho^{n+1}}$$

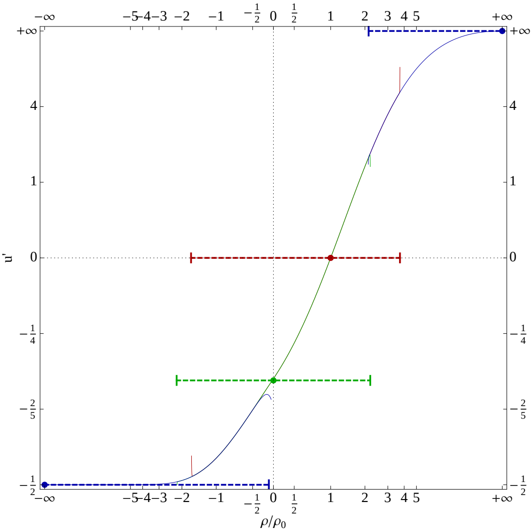
$\to$ Can be anticipated form large u'

$\to$ The first coefficient is regulator independent

$\to$ a_1, a_3 and a_5 vanished with the sharp cutoff



Optimised cutoff



Sharp cutoff

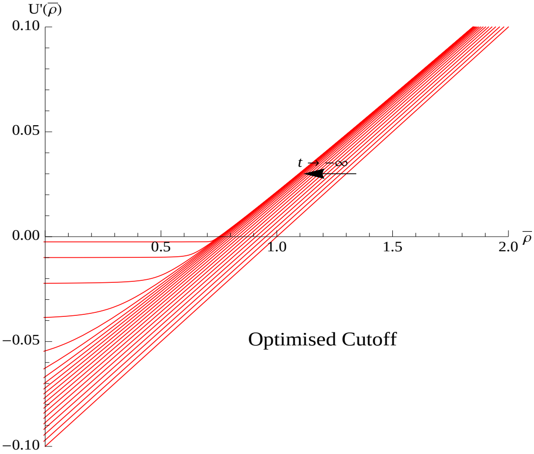
Solution in 4d:

$$\rho = \frac{u'}{\lambda_\Lambda} + \frac{\kappa_\Lambda}{e^{2t}} + u' \left(\mathcal{G}_P(u') - \mathcal{G}_P(u' e^{2t}) \right)$$

But when $t \to -\infty$, the asymptotic behavior of $\mathcal{G}_P$ leads to the gaussian (trivial) fixed point solution $u'_\star = 0$.

$\to$ We can still choose to be slightly way form criticality and find a vanishing quartic coupling (as $\Lambda \gg U'_{k=0}$)

$$U'' = \frac{\bar{\lambda}_\Lambda}{1 - \frac{\bar{\lambda}_\Lambda}{2}\, \mathcal{I}\Big[\ln\Big(\frac{U'}{\Lambda^2}\frac{e^{3/2}}{P^2}\Big)\Big]} \to 0$$



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$t \to -\infty$
Optimised Cutoff

Summary:

$\rightarrow$ Use large N technique in the context of FRG

$\rightarrow$ Full analytical solution without specifying the cutoff by interchanging momenta and scale integration

$\rightarrow$ Detailed study of the fixed point solution in $3d$ by means of local expansion

$\rightarrow$ Next step : finite N (inclusion of radial mode)

The operator $\mathcal{I}$

→ Measure :

$$d\mu(y) = -\frac{d}{2}\frac{r'(y)\ dy}{[1+r(y)]^{\frac{d}{2}+1}} \to d\mu(r) = -\frac{d}{2}\frac{dr}{(1+r)^{\frac{d}{2}+1}} \qquad (1)$$

→ Link with threshold function :

$$\ell(\omega) = \mathcal{I}\left[\frac{P^{2+d}}{P^2+\omega}\right] \qquad (2)$$

with

$$\ell(\omega) = \frac{1}{2}\int_0^\infty dy y^{\frac{d}{2}} \frac{\partial_t r(y)}{[y(1+r)+\omega]} \qquad (3)$$

Behavior at $t \to -\infty$ in $4d$

$\to$ Expression for $\mathcal{G}_p(z)$ about $z = 0$:

$$\begin{aligned} \mathcal{G}_P(z) &= \frac{1}{12}\,\mathcal{I}\left[\left(\frac{P^2}{z}\right)^3 \;{}_2F_1\left(2,3,4,-\frac{P^2}{z}\right)\right] \\ &= \frac{1}{4}\frac{\mathcal{I}[P^2]}{z} + \frac{1}{4} + \frac{1}{2}\,\mathcal{I}\left[\log\left(\frac{|z|}{P^2}\right)\right] + \mathcal{I}\left[\mathcal{O}\left(\frac{z}{P^2}\right)\right] \end{aligned}$$

$\to$ Then the solution becomes

$$\rho = \frac{u'}{\lambda_\Lambda} - u' t + \frac{\mathcal{I}[P^2]}{4}$$