Non-perturbative RG to lattice models

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Refs: N. Dupuis & K. Sengupta, EPJB 66, 271 (2008)
T. Machado & N. Dupuis, arXiv:1004.3651

Outline

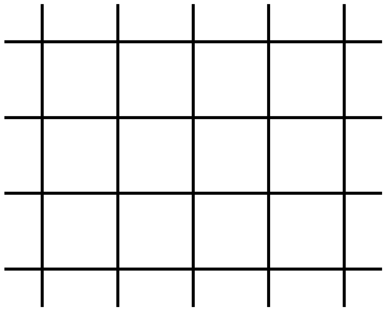
- Why does the lattice matter?
- “Standard” NPRG scheme vs “lattice” NPRG scheme

  (kinetic energy>>interaction vs kinetic energy<<interaction)
- Ising model and classical spin models in the LPA
- Beyond the LPA
- Mott transition in the Bose-Hubbard model
- Conclusion

Why should we care about the lattice?

- Lattice does not matter for the long-distance (critical) properties
- But non-universal quantities (Tc, magnetization, spectral functions, etc.) depend on short-distance fluctuations and therefore on the lattice.
- Lattice gives rise to new physics

Ex.: Localization (Mott) transition of particles hopping on a lattice



$$H = -t \sum_{\langle i,j \rangle} \psi_i^+ \psi_j + \frac{U}{2} \sum_i \psi_i^+ \psi_i^+ \psi_i \psi_i$$

(Hubbard model)

t: hopping amplitude

U: local (on-site) repulsion

Lattice field theory

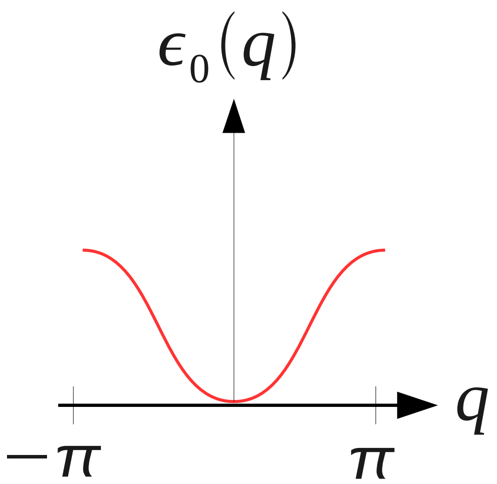
(d-dimensional hypercubic lattice)

- Action:

$$S[\varphi] = \frac{1}{2}\sum_{\boldsymbol{q}} \varphi_{-\boldsymbol{q}}\epsilon_0(\boldsymbol{q})\varphi_{\boldsymbol{q}} + \sum_{\boldsymbol{r}} U_0(\varphi_{\boldsymbol{r}})$$

$$\epsilon_0(\boldsymbol{q}) = 2\epsilon_0 \sum_{\nu=1}^{d}(1-\cos q_\nu)$$

$$\boldsymbol{q} \in [-\pi, \pi]^d$$



- Regulator: $\Delta S_k[\varphi] = \frac{1}{2}\sum_{\boldsymbol{q}} \varphi_{-\boldsymbol{q}} R_k(\boldsymbol{q})\varphi_{\boldsymbol{q}}$

- Average effective action: $\Gamma_k[\phi] = -\ln Z_k[h] + \sum_{\boldsymbol{r}} h_{\boldsymbol{r}}\phi_{\boldsymbol{r}} - \Delta S_k[\phi]$

(+ exact flow equation)

Standard NPRG scheme for a lattice field theory

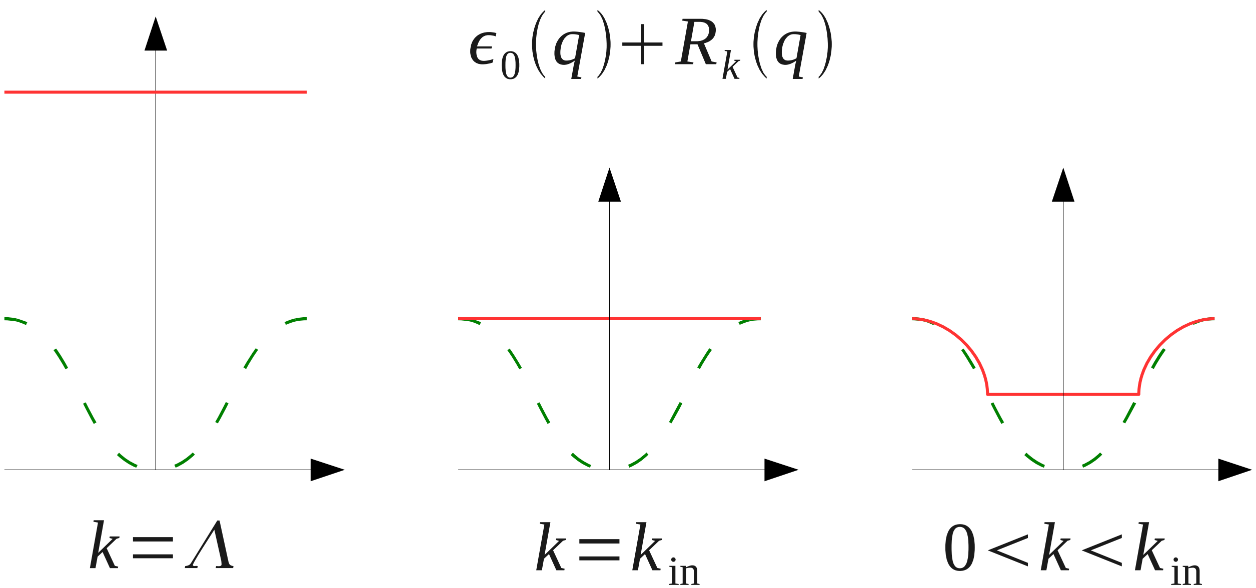
N. Dupuis & K. Sengupta, EPJB 66, 271 (2008)

- Litim's regulator: $R_k(\boldsymbol{q}) = (\epsilon_k - \epsilon_0(\boldsymbol{q}))\,\theta(\epsilon_k - \epsilon_0(\boldsymbol{q}))$ with $\epsilon_k = \epsilon_0 k^2$

- Effective dispersion: $$\epsilon_0(\boldsymbol{q}) + R_k(\boldsymbol{q}) = \begin{cases} \epsilon_k & \text{if} \quad \epsilon_0(\boldsymbol{q}) < \epsilon_k \\ \epsilon_0(\boldsymbol{q}) & \text{if} \quad \epsilon_0(\boldsymbol{q}) > \epsilon_k \end{cases}$$

- Initial condition: if $\epsilon_\Lambda \gg$ all characteristic energy scales, then all fluctuations are frozen and mean-field theory becomes exact:

$$\Gamma_\Lambda[\phi] = S[\phi]$$



- The renormalization from $k=\Lambda$ to $k=k_{\text{in}}$ is a purely local (single-site) problem since all modes are dispersionless: $\epsilon_0(q)+R_k(q)=\epsilon_k$
- Lattice NPRG: starts flow from k_{in}. The initial condition is obtained by solving a single-site problem [T. Machado & ND, arXiv:1004.3651].
- $k\to 0$ limit similar to standard NPRG (mass $\sim k^2$ for critical modes).
- In quantum models: non-trivial local limit due to quantum (imaginary time) on-site fluctuations.

Ising model

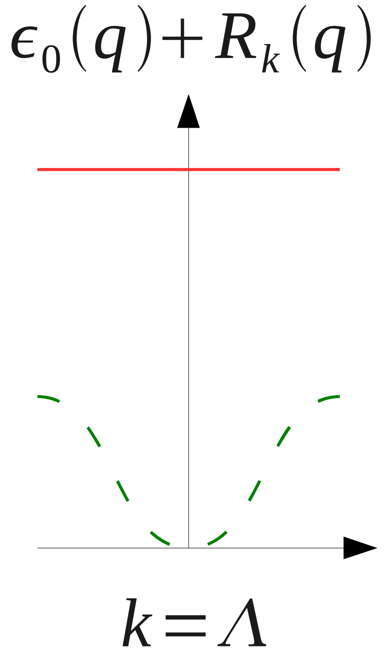
$$H=-\frac{J}{T}\sum_{\langle i,j\rangle}S_iS_j-\frac{\mu}{T}\sum_i S_i^2\equiv-\frac{1}{2}\sum_{\langle i,j\rangle}S_iA_{i,j}^{(\mu)}S_j \qquad (S_i^2=1)$$

For $\mu>Jd$, $A^{(\mu)}$ is a positive matrix:

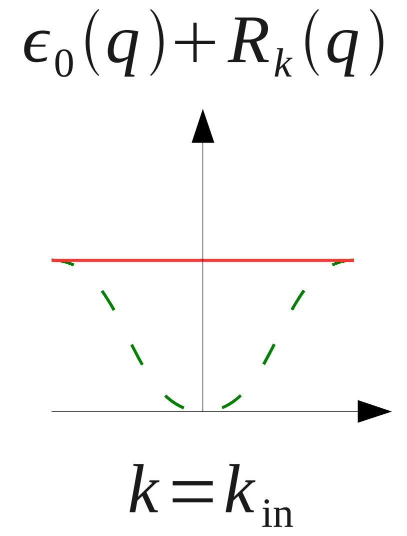
$$Z\propto\sum_{\{S_i=\pm1\}}\int_{-\infty}^{\infty}\prod_i d\varphi_i\exp\left\{-\frac{1}{2}\sum_{i,j}\varphi_iA_{i,j}^{(\mu)-1}\varphi_j+\sum_i\varphi_iS_i\right\}$$

$$\propto\int_{-\infty}^{\infty}\prod_i d\varphi_i\exp\left\{-\frac{1}{2}\sum_{i,j}\varphi_iA_{i,j}^{(\mu)-1}\varphi_j+\sum_i\ln\cosh\varphi_i\right\}$$

$$\equiv\int_{-\infty}^{\infty}\prod_i d\varphi_i\exp\left\{-\frac{1}{2}\sum_{\boldsymbol{q}}\varphi_{-\boldsymbol{q}}\epsilon_0(\boldsymbol{q})\varphi_{\boldsymbol{q}}+\sum_i U_0(\varphi_i)\right\}$$



Standard scheme:
starts from mean-field theory



Lattice scheme:
starts from decoupled sites

$$Z_{k_{\text{in}}}=\prod_i z_{k_{\text{in}}}(h_i)$$

$$z_{k_{\text{in}}}(h)=\int_{-\infty}^{\infty} d\varphi\, e^{-\frac{1}{2}\epsilon_{k_{\text{in}}}\varphi^2-U_0(\varphi)+h\varphi}$$

Local potential approximation

$$\Gamma_k[\phi] = \sum_i U_k(\rho_i) + \frac{1}{2}\sum_{\boldsymbol{q}} \phi_{-\boldsymbol{q}} \epsilon_0(\boldsymbol{q}) \phi_{\boldsymbol{q}} \qquad \rho_i = \phi_i^2/2$$

$$\partial_k U_k(\rho) = \frac{1}{2}\int_{\boldsymbol{q}} \frac{\partial_k R_k(\boldsymbol{q})}{\epsilon_0(\boldsymbol{q}) + R_k(\boldsymbol{q}) + U_k'(\rho) + 2\rho U_k''(\rho)}$$

Both schemes give $T_c^{\mathrm{LPA}} = 0.747\, T_c^{\mathrm{MF}}$ (3D) independent of μ

To be compared with the exact (Monte Carlo) result: $T_c^{\mathrm{exact}} = 0.752\, T_c^{\mathrm{MF}}$

Lattice NPRG approach classical spin models

(without field theory)

$$Z_k[h] = \sum_{\{S_i=\pm1\}} \exp\left\{ \frac{J}{T} \sum_{\langle i,j\rangle} S_i S_j - \frac{1}{2} \sum_{i,j} S_i R_k(i,j) S_j + \sum_i h_i S_i \right\}$$

$$= \sum_{\{S_i \pm 1\}} \exp\left\{ -\frac{1}{2} \sum_{\boldsymbol{q}} S_{-\boldsymbol{q}} \left[\epsilon_0(\boldsymbol{q}) + R_k(\boldsymbol{q}) \right] S(\boldsymbol{q}) + \sum_{\boldsymbol{q}} h_{-\boldsymbol{q}} S_{\boldsymbol{q}} \right\}$$

with $\quad \epsilon_0(\boldsymbol{q}) = 2\frac{J}{T} \sum_{\nu=1}^{d} (1 - \cos q_\nu)$

Magnetization: $\quad m_i = \langle S_i \rangle = \frac{\partial}{\partial h_i} \ln Z_k[h], \qquad m_i \in [-1,1]$

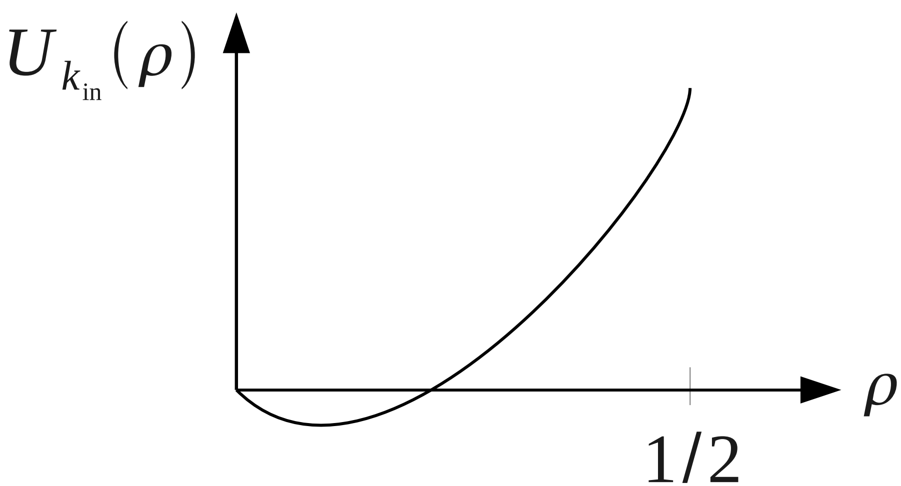
Average effective action: $\quad \Gamma_k[m] = -\ln Z_k[h] + \sum_i h_i m_i - \Delta H_k[m]$

- Standard NPRG scheme cannot be used because of the constraint $S_i^2 = 1$

- Lattice NPRG scheme:

$$Z_{k_{\text{in}}}[h] = \prod_i \sum_{S_i = \pm 1} e^{h_i S_i} = \prod_i 2\cosh(h_i)$$

$$U_{k_{\text{in}}}(\rho) = \frac{1}{2}\ln(1-2\rho) + \sqrt{2\rho}\,\text{atanh}\sqrt{2\rho} - 4\text{d}\frac{J}{T}\rho\,, \qquad \rho = \frac{m^2}{2} \in [0, 1/2]$$

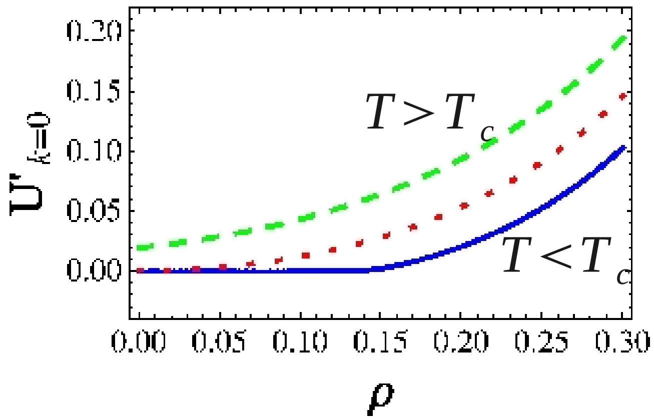


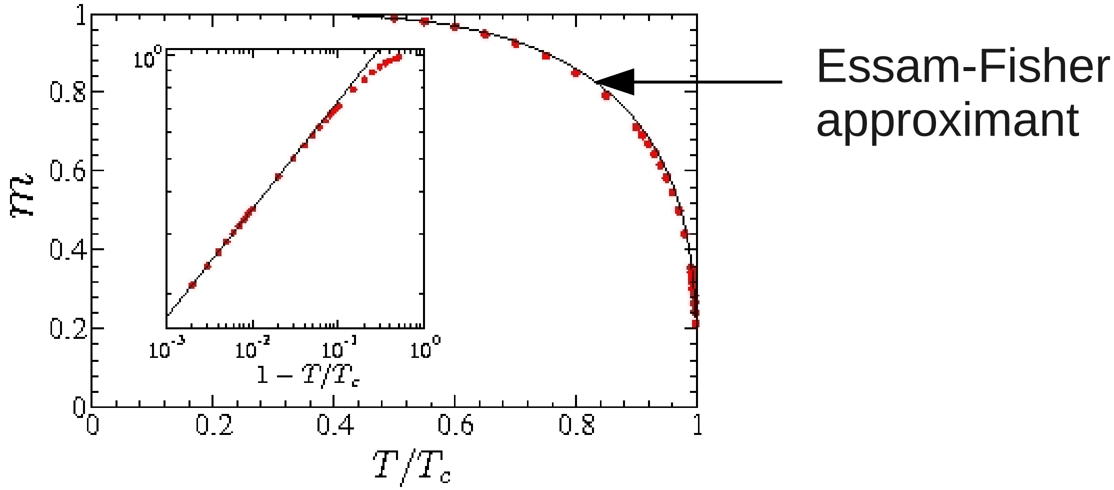
$$U_{k_{\text{in}}}(1/2) = \ln 2 - 2\frac{J}{T}d$$

$$U'_{k_{\text{in}}}(1/2) = \infty$$

Local potential approximation

$$\Gamma_k[m] = \sum_i U_k(\rho_i) + \frac{1}{2}\sum_{\boldsymbol{q}} m_{-\boldsymbol{q}}\epsilon_0(\boldsymbol{q}) m_{\boldsymbol{q}}$$





Tc for 3D spin models:

	T_c^{MF}	T_c^{exact}	T_c^{NPRG}
Ising 3D	6	4.51	4.48
XY 3D	3	2.20	2.18
Heisenberg 3D	2	1.44	1.42

Same critical exponents as in the usual LPA approximation

$\eta = 0$ and $\nu = 2\beta = \gamma/2 \simeq 0.64 - 0.65$

Beyond the LPA

- Renormalization of the spectrum

$$\Gamma_k[m] = \sum_i U_k(\rho_i) + \frac{1}{2}\sum_{\boldsymbol{q}} m_{-\boldsymbol{q}} m_{\boldsymbol{q}} \left[A_k \epsilon_0(\boldsymbol{q}) + \text{higher harmonics} \right]$$

- LPA'

$$\Gamma_k[m] = \sum_i U_k(\rho_i) + \frac{1}{2}\sum_{\boldsymbol{q}} m_{-\boldsymbol{q}} m_{\boldsymbol{q}} Z_k \epsilon_0(\boldsymbol{q})$$

$$\text{with} \quad Z_k = \lim_{\boldsymbol{q}\to 0} \frac{\partial}{\partial \boldsymbol{q}^2} \Gamma_k^{(2)}(\boldsymbol{q}; \rho_{0,k})$$

[T. Machado & ND, arXiv:1004.3651]

- BMW?

2D spin models

- 2D Ising model: $T_c^{\text{LPA}} \simeq 0.48 \;\; T_c^{\text{MF}}$
  $T_c^{\text{exact}} \simeq 0.567 \;\; T_c^{\text{MF}}$

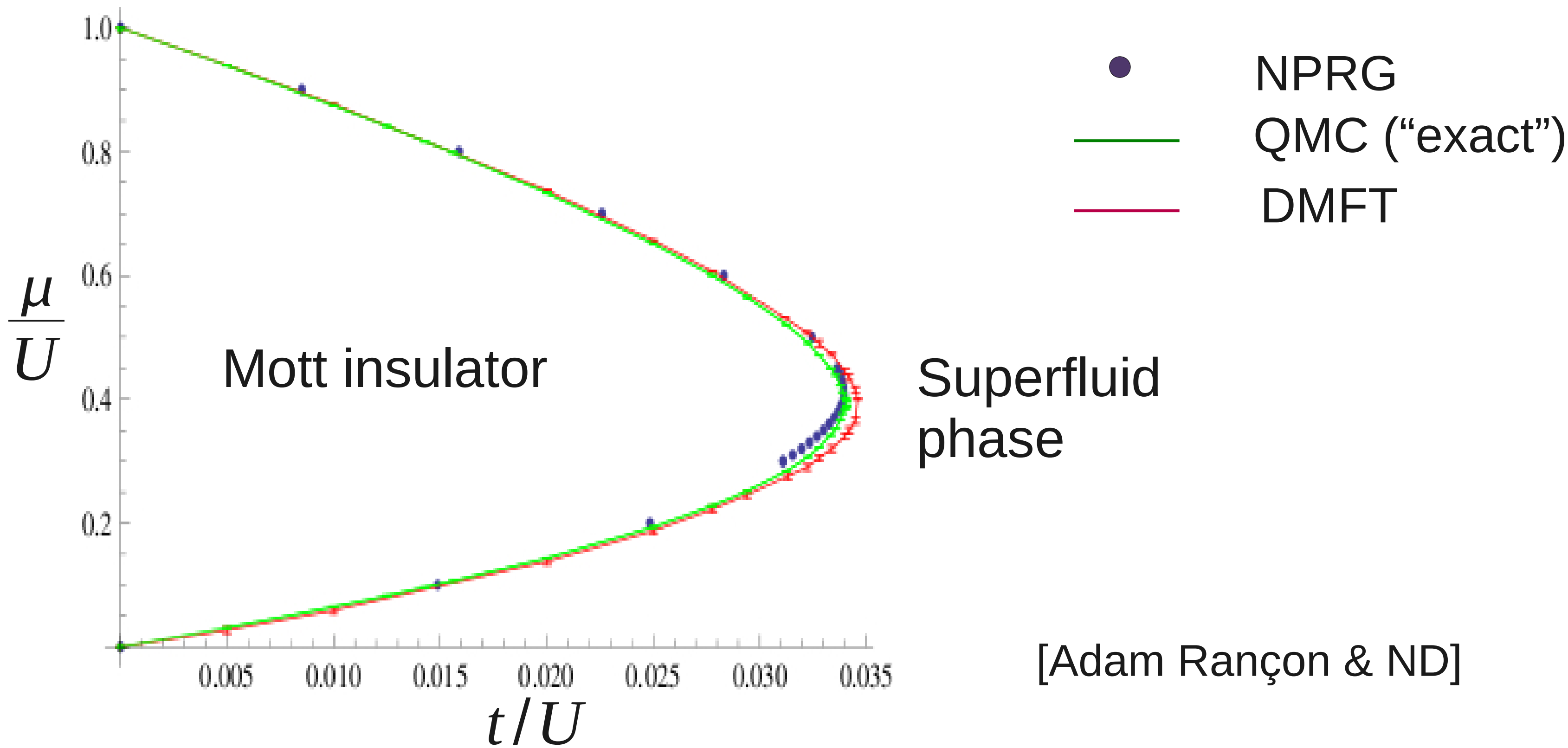
- 2D XY model (BKT transition): $T_c^{\text{LPA'}} \simeq 0.9 - 1 \;\; J$
  $T_c^{\text{exact}} \simeq 0.89 \;\; J$

(T. Machado & ND, arXiv:1004.3651)

Bose-Hubbard model

(interacting bosons on a lattice)

$$H = -t \sum_{\langle i,j \rangle} \psi_i^+ \psi_j + \frac{U}{2} \sum_i \psi_i^+ \psi_i^+ \psi_i \psi_i - \mu \sum_i \psi_i^+ \psi_i$$



[Adam Rançon & ND]

See talk on Saturday at 5.20 pm (parallel session II) by Adam Rançon

Summary

How to solve a lattice model with the NPRG?

$S+\Delta S_\Lambda$	$S+\Delta S_k$ →	S
Reference system	RG equation: $\partial_k \Gamma_k[\phi]$ (to be solved approximately)	System of interest

Reference system must be solved exactly

- Freeze all fluctuations: mean-field theory is exact ("standard" scheme)
- Decouple the sites: solve a single-site problem ("lattice" scheme)

Conclusion

The lattice NPRG

- is a new implementation of the NPRG to lattice models
- captures both local and critical fluctuations in a non-trivial way (equivalent to standard NPRG for critical properties)
- 3D classical spin models in the LPA: non-universal quantities (Tc, magnetization) within 1 percent.
- Superfluid-Mott transition in the Bose-Hubbard model (see Adam Rançon's talk on Saturday at 5.20 pm)
- The idea to include short-range fluctuations in the initial condition of the RG is not new: see the Hierarchical Reference Theory of fluids [Parola & Reatto, PRL'1984, Adv. in Phys.'1995, Ionescu et al., PRE'2007].