An Introduction to Non Perturbative RG or ERG or FRG or eRG

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Some basic notions on Statistical Field Theory

Ising (or Heisenberg) model and the ϕ^4 theory:

– on the lattice:

$$\mathcal{Z}[B] = \sum_{\{S_i=\pm 1\}} \exp\Big\{ -\beta J \sum_{<ij>} S_i S_j + \beta \sum_i B_i S_i \Big\}$$

– in the continuum (keeping only the leading derivative term):

$$\mathcal{Z}[J] = \int D\phi \;\exp\Big\{ -\mathcal{S}[\phi] + \int_x J(x)\phi(x) \Big\}$$

with (a = lattice spacing)

$$\mathcal{S}[\phi] = \int d^d x \Big\{ \frac{1}{2}(\nabla\phi)^2 + d a^{-2}\phi^2 - a^{-d} \ln\Big(\cosh(\beta \sqrt{\frac{4Jd^2}{\beta a^{2-d}}}\phi) \Big) \Big\}$$

Making a field expansion leads to:

$$\mathcal{S} = \int d^d x \Big\{ \frac{1}{2}(\nabla\phi)^2 + \frac{m_0^2}{2}\phi^2 + \frac{g_0}{4!}\phi^4 + \dots \Big\}$$

The physical quantities we are interested in:

– "thermodynamic quantities":

- magnetization (classical field, vev,...) : $M = \langle\phi\rangle = \frac{1}{Z}\frac{\partial Z}{\partial B}$
- susceptibility (response of M to a change of B) : $\chi = \frac{\partial M}{\partial B}$
- correlation length ξ (renormalized mass)
- critical temperature T_c and phase diagram
- etc

– correlation function(s): $\langle S_i S_j\rangle \sim \langle\phi(x)\phi(y)\rangle = G^{(2)}(x,y)$ and $G^{(n)}(x_1,\ldots,x_n)$

$\Rightarrow$ universal and non-universal quantities.

The theoretical tools and methods:

– The generating functional of correlation functions:

$\mathcal{Z}[J]$, $\mathcal{W}[J] = \ln \mathcal{Z}[J]$ and $\Gamma[M]$ = Legendre transform of $\mathcal{W}[J]$:

$$\Gamma[M] + \mathcal{W}[J] = \int_x J_x M_x$$

$$\Rightarrow \quad M_x = \langle \phi_x \rangle = \frac{\delta \mathcal{W}[J]}{\delta J_x} \quad \text{and} \quad J_x = \frac{\delta \Gamma[M]}{\delta M_x}$$

– The mean field (classical, saddle point, Hartree) approximation:

$$\Gamma[M] \to \Gamma^{\mathrm{MF}}[M] = \mathcal{S}[\phi = M]$$

neglects all fluctuations around the mean field configuration M.

– The loop expansion:

$$\mathcal{S}[\phi] = \mathcal{S}_0[\phi] + V[\phi]$$

$$\mathcal{Z}[J] = \left\{ \sum_n \frac{(-1)^n}{n!} \int_{x_1 \ldots x_n} V[\frac{\delta}{\delta J_{x_1}}] \ldots V[\frac{\delta}{\delta J_{x_n}}] \right\} \mathcal{Z}_0[J]$$

$\longrightarrow$ the series of Feynman diagrams.

$\longrightarrow$ two problems:

(i) expansion of the integrand of the functional integral,

(ii) interchange of the series and the functional integral.

(i) $\Rightarrow$ perturbative renormalization

(ii) $\Rightarrow$ convergence properties of the renormalized series ?

Why should we be interested in NPRG?

Perturbation theory works well when:

- g is small: QED;
- g is not small, the renormalized series is Borel-summable and enough terms are known: ϕ^4 in $d = 3$.

Perturbation theory doesn't work well when:

- g is not small and not enough terms are known;
- g is not small, the series is Borel-summable, convergence to a wrong result: ϕ^4 in $d = 2$: $\eta = 0.145(14)$ instead of 0.25;
- g is not small and the series cannot be resummed: $O(N)$ non-linear-sigma model in $d = 2 + \epsilon$;
- infinitely many couplings have to be taken care: computation of T_c;
- the phenomenon is genuinely non-perturbative: no RG trajectory Gaussian $\rightarrow$ fixed point, convexity of the effective potential.

Wilson RG:

Organize the summation over the fluctuations in a different way.

$\Downarrow$

Block-spins *à la* Kadanoff-Wilson

$\Downarrow$

summation over rapid modes $\rightarrow$ effective hamiltonian for the slow modes

$\Downarrow$

flow equations of functions (or even functionals)

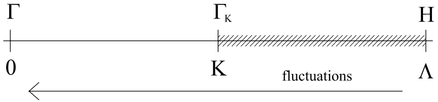
Two main implementations of these ideas:

à la Wilson-Polchinski (flow of actions or of $\mathcal{W}[J]$)
and
à la Parola-Reatto-Wetterich (flow of the Legendre transform $\Gamma[M]$).

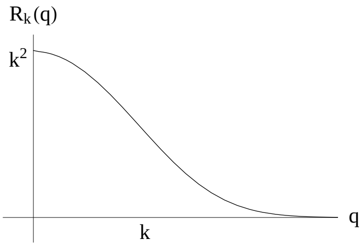
In principle equivalent, but...

Integration over the rapid modes:

hypothesis: the system is close to criticality ($\xi \gg a \Rightarrow m_R \ll \Lambda$)



integrate over the rapid modes only $\longrightarrow$ freeze the slow modes $\longrightarrow$ make them non-critical $\longrightarrow$ give them a large mass



$\rightarrow$ build a one-parameter family of models, indexed by a scale k such that:

$$\mathcal{Z}[J] \rightarrow \mathcal{Z}_k[J] = \int D\phi \; \exp\Big\{ -\mathcal{S}[\phi] - \Delta\mathcal{S}_k[\phi] + \int_x J(x)\phi(x) \Big\}$$

$$\Delta\mathcal{S}_k[\phi] = \frac{1}{2}\int_q R_k(q)\,\phi_q\phi_{-q}$$

• when $k = \Lambda$ all fluctuations are frozen $\Rightarrow$ mean field is exact:
$\forall q, \quad R_{k=\Lambda}(q) \sim \Lambda^2, \quad \Rightarrow \quad \Gamma^{\mathrm{Leg}}_{k=\Lambda} = \mathcal{S} + \Delta\mathcal{S}_{k=\Lambda}$

$$\Rightarrow \quad \text{work with } \; \Gamma_k[M] = \Gamma^{\mathrm{Leg}}_k[M] - \Delta\mathcal{S}_k[M]$$

$$\Rightarrow \quad \Gamma_{k=\Lambda}[M] = \mathcal{S}[M]$$

• when $k = 0$ all fluctuations are integrated out and the original model is retrieved
$\forall q, \quad R_{k=0}(q) = 0, \quad \Rightarrow \quad \mathcal{Z}_{k=0}[J] = \mathcal{Z}[J]$ and $\Gamma_{k=0} = \Gamma$

define:

- $\mathcal{Z}_k[J] = \int D\phi \; \exp\Big\{ -\mathcal{S}[\phi] - \Delta\mathcal{S}_k[\phi] + \int_x J(x)\phi(x)\Big\}$
- $\mathcal{W}_k[J] = \ln \mathcal{Z}_k[J]$
- $\Gamma_k[M] + \mathcal{W}_k[J] = \int_x J_x M_x - \frac{1}{2}\int_q R_k(q)\phi_q\phi_{-q}$

(effective average action)

with

$$\begin{cases} R_{k=\Lambda}(q) \sim \Lambda^2 \quad (\text{or } \infty) \\ \\ R_{k=0}(q) = 0 \end{cases} \tag{1}$$

then $\Gamma_{k=\Lambda}[M]$ interpolates between the microphysics at $k = \Lambda$ and the macrophysics at $k = 0$:

$$\begin{cases} \Gamma_{k=\Lambda}[M] = S[\phi = M] \\ \\ \Gamma_{k=0}[M] = \Gamma[M] \end{cases} \tag{2}$$

The flow equation for $\Gamma_k[M]$ writes:

$$\partial_k \Gamma_k[M] = \frac{1}{2} \int_q \partial_k R_k(q) G[q; M] \tag{3}$$

where $G[q; M]$ is the full propagator: $G[q; M] = (\Gamma_k^{(2)} + R_k)^{-1}$

Some properties of the Wetterich's equation:
– differential formulation of field theory
– involves only one integral
– the initial condition is the (microscopic) bare theory
– good properties of decoupling of the massive and rapid modes
– starting point of non-perturbative approximation schemes (not linked to an expansion in a small parameter)

– BUT leads to very few exact results.

Approximation schemes require

- to lead to tractable calculations;
- to focus on the “sector” of the model that we want to describe;
- to preserve automatically the good properties of RG;
- to enable a systematic improvement of the results.

Two main (non-perturbative) approximation schemes:
The derivative expansion (DE)
The Blaizot-Mendez-Wschebor scheme (BMW)

The derivative expansion

- $\Gamma_k = \int d^dx \, \left(U_k(M) + \frac{1}{2} Z_k(M)(\nabla M)^2 + O(\nabla^4)\right)$
- flow of $\Gamma_k \Rightarrow$ flow of $U_k(M), Z_k(M), \ldots$

Consists in keeping all $\Gamma_k^{(n)}$ correlation functions and expanding in their momenta (more precisely in $\frac{p_i}{k}$).

Most celebrated: Local Potential Approximation (LPA):

$$\Gamma_k^{\mathrm{LPA}} = \int d^dx \, \left(U_k(M) + \frac{1}{2}(\nabla M)^2\right)$$

- bare momentum dependence of $\Gamma_k^{(2)}(p)$;
- zero momentum approximation for all other correlation functions.

$$\partial_k U_k(M) = \frac{k^{d+1}}{k^2 + U_k''(M)}$$

Some properties of the DE (for the $O(N)$ models)

– it is one-loop exact in $d = 4 - \epsilon$ and in $d = 2 + \epsilon$ (for $N \geq 2$) and exact for $U = U_{k=0}$ at $N = \infty$ (LPA')
– it preserves the convexity of $U = U_{k=0}$ in the broken phase (LPA)
– initial conditions $\Rightarrow$ non-universal physics $\Rightarrow$ computation of T_c
– it reproduces well the physics of the Kosterlitz-Thouless transition ($N = 2, d = 2$)
– it leads to accurate results for the universal quantities: Ising model in $d = 3$:

NPRG:	$\nu = 0.63014$	$\eta = 0.0352$
Monte Carlo:	$\nu = 0.63020(12)$	$\eta = 0.368(2)$
6 loops:	$\nu = 0.6304(13)$	$\eta = 0.335(25)$

But the DE has some drawbacks:

(i) the physical quantities depend on the choice of R_k;

(ii) the momentum dependence of the $\Gamma_k^{(n)}(\{p_i\}, M=0)$ is badly truncated: at T_c (that is $m_R = 0$): $\Gamma_{k=0}^{(2)}(p) \sim p^{2-\eta}$.

(i) $\Rightarrow$ does the DE converge?
(ii) $\Rightarrow$ for which p_i is the DE valid?

Does the DE converge?

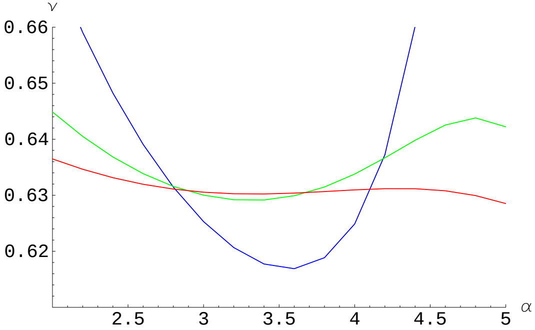
No general answer.

Test:

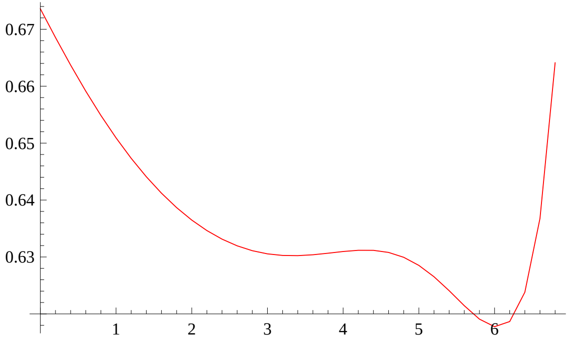
convergence $\Rightarrow$ dependence upon R_k decreases with the order.

But this is an ill-posed problem!

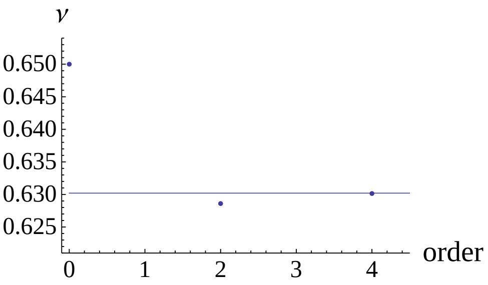
Ising, $d = 3$ resummed 4-, 5- and 6-loop results for ν



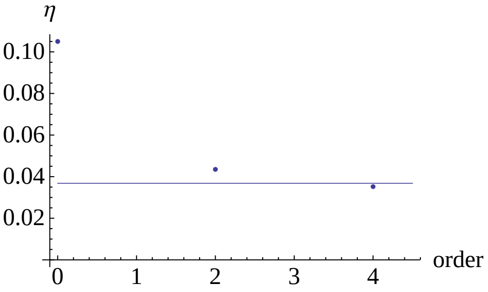
Ising, $d = 3$ resummed 6-loop results for ν



ν_{opt}=0.63023 (MC: $\nu = 0.63020(12)$)



ν
0.650
0.645
0.640
0.635
0.630
0.625
0
1
2
3
4
order



η
0.10
0.08
0.06
0.04
0.02
0
1
2
3
4
order

Range of p_i where the DE is valid?

$$\text{DE} \quad \Rightarrow \quad \text{expansion of the } \Gamma_k^{(n)} \text{ in } p_i/k$$
$$\Downarrow$$
$$\forall p_i \lesssim k$$

Consistent with

$$Z_k(M) \sim \partial_{p^2} \Gamma_k^{(2)}[p, M]|_{p=0, M_{\mathrm{unif.}}}$$

$$\partial_k Z_k(M) \sim \int_q \partial_k R_k(q)\, \partial_{p^2} \Gamma_k^{(3)}(p, p-q, q)|_{p=0} \ldots$$

accuracy of the DE $\quad \Rightarrow \quad \Gamma_k^{(n)}(\{p_i\}) \simeq \Gamma_k^{(n),\mathrm{DE}}(\{p_i\})$ for $p_i \lesssim k$

BUT does not give any information for momenta higher than k !

Remark: better situation in a massive theory: $p_i \lesssim \mathrm{Max}(k, m)$

Blaizot-Mendez-Wschebor (BMW) approximation

- keep all $\Gamma_k^{(n)}$ correlation functions (as in the DE);
- aim at being as accurate as possible for the $\Gamma_k^{(n)}$'s with low n;
  Exact equation on $\Gamma_k^{(2)}(p, M)$ (for uniform field M):

$$\partial_k \Gamma_k^{(2)}(p, M) = \int_q \partial_k R_k(q) G(q; M)^2 \Big(-\frac{1}{2}\Gamma_k^{(4)}(p, -p, q, -q; M) + \Gamma_k^{(3)}(p, q, -p-q; M) G(p+q; M) \Gamma_k^{(3)}(-p, -q, p+q; M) \Big)$$

  Infinite hierarchy of equations on the $\Gamma_k^{(n)}(\{p_i\}, M)$
  $\Rightarrow$ closure requires approx. on $\Gamma_k^{(3)}$ and $\Gamma_k^{(4)}$ in terms of $\Gamma_k^{(2)}$
- truncate the momentum dependence of the $\Gamma_k^{(n)}$'s with "large" n $\Rightarrow$ closed set of equations for $\Gamma_k^{(n)}$ with low n.

First order: LPA, already good for $p \to 0$ in many cases.
Second order: keep $\Gamma_k^{(2)}$, truncate $\Gamma_k^{(3)}$, $\Gamma_k^{(4)}$, ...

The LPA has the first order of approximation of the BMW scheme

0-point function for $M_{|_{\text{unif.}}}$ $\Rightarrow$ $\Gamma_k[M] \propto U_k(M)$

$$\partial_k U_k(M) = \frac{1}{2}\int_q \frac{\partial_k R_k(q)}{\Gamma_k^{(2)}(q,M) + R_k(q)} \tag{4}$$

Truncate the q-dependence of $\Gamma_k^{(2)}(q,M)$ by setting $q = 0$?

No! Keep the bare dependence $\Rightarrow$ $\Gamma_k^{(2)}(q,M) \to q^2 + U_k''(M)$

$\Rightarrow$ This is the LPA equation for U_k.

The second order of approximation of BMW

$$\partial_k \Gamma_k^{(2)}(p;M) = \int_q \partial_k R_k(q) G(q;M)^2 \Big(-\frac{1}{2}\Gamma_k^{(4)}(p,-p,q,-q;M) + \Gamma_k^{(3)}(p,q,-p-q;M) G(p+q;M) \Gamma_k^{(3)}(-p,-q,p+q;M) \Big)$$

Three (crucial) remarks

- $q \lesssim k$ because of $R_k(q)$;
- finite $k \Rightarrow$ no IR divergence $\Rightarrow \Gamma_k^{(n)}(\{p_i\}, M)$ expandable in p_i;
- $\Gamma_k^{(n)}(p_1, \ldots, p_{n-1}, 0; M) = \dfrac{\partial}{\partial M}\Gamma_k^{(n-1)}(p_1, \ldots, p_{n-1}; M)$.

Thus

- for $p \gg k > q \Rightarrow \Gamma_k^{(3)}(p,q,-p-q;M) \simeq \Gamma_k^{(3)}(p,0,-p;M)$
  and $\Gamma_k^{(4)}(p,-p,q,-q;M) \simeq \Gamma_k^{(4)}(p,-p,0,0;M)$
- for $p \ll k$: "LPA regime": again q is neglected and $p \rightarrow 0$

BMW approximation

$$\partial_k \Gamma_k^{(2)}(p; M) = \int_q \partial_k R_k(q) G(q; M)^2 \Big(-\frac{1}{2}\Gamma_k^{(4)}(p, -p, 0, 0; M) + \Gamma_k^{(3)}(p, 0, -p; M) G(p+q; M) \Gamma_k^{(3)}(-p, 0, p; M) \Big)$$

Final result:

$$\boxed{\partial_k \Gamma_k^{(2)}(p; M) = \left(\partial_M \Gamma_k^{(2)}\right)^2 J_3(p; M) - \frac{1}{2}\left(\partial_M^2 \Gamma_k^{(2)}\right) J_2(0; M)}$$

where

$$J_n(p; M) = \int_q \partial_k R_k(q)\ G(p+q; M) G(q; M)^{n-1}$$

That's all folks... (??)

Not fully !

Search for fixed points:

$\Rightarrow$ numerical instabilities

$\Rightarrow$ good variables $\dfrac{\Gamma_k^{(2)}(p,M)-\Gamma_k^{(2)}(0,M)}{p^2}-1$ and $V_k(M)$

$\Rightarrow$ difficulties for $p \to 0$

$\Rightarrow$ necessary to avoid non analytic R_k functions...

Results in $d=3$, $N=1$:

$\eta = 0.039$	$\eta_{\rm MC} = 0.0368(2)$
$\nu = 0.632$	$\nu_{\rm MC} = 0.6302(1)$
$\omega = 0.78$	$\omega_{\rm MC} = 0.821(5)$

Results in $d = 2$, $N = 1$:

$\eta = 0.254 \qquad \eta = 0.25$
$\nu = 1.00 \qquad \nu = 1$

AND... at five loops: $\eta = 0.145(14)$