Special geometries emerging from Yang-Mills type matrix models

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Outline

1. Introduction

2. Curvature from Matrix Models

3. Special Geometries
   - Schwarzschild Geometry
   - Reissner-Nordström Geometry

4. Conclusion and Outlook
Matrix models of Yang-Mills type

\[ S_{YM} = -\text{Tr}[X^a, X^b] [X^c, X^d] \eta_{ac} \eta_{bd} \]

- \( X^a \) Herm. matrices on \( \mathcal{H} \), and \( \eta_{ab} \) is \( D \)-dim. flat metric
- \( X^a = (X^\mu, \Phi^i) \), \( \mu = 1, \ldots, 2n \), \( i = 1, \ldots, D - 2n \), so that \( \Phi^i(X) \sim \phi^i(x) \) define embedding \( \mathcal{M}^{2n} \hookrightarrow \mathbb{R}^D \)
- \( g_{\mu\nu}(x) = \partial_\mu x^a \partial_\nu x^b \eta_{ab} \) (in semi-classical limit)
- \( \mathcal{M}^{2n} \) endowed with a Poisson structure
- \( -i[X^\mu, X^\nu] \sim \{x^\mu, x^\nu\}_{PB} = \theta^{\mu\nu}(x) \)
  \( \Rightarrow \) “effective” metric
  \[ G^{\mu\nu} = e^{-\sigma} \theta^{\mu\rho} \theta^{\nu\sigma} g_{\rho\sigma}, \quad e^{-\sigma} = \frac{\sqrt{\det \theta^{-1}_{\mu\nu}}}{\sqrt{\det G_{\rho\sigma}}} \]
- \( 2n = 4 \): special class of geometries where \( G_{\mu\nu} = g_{\mu\nu} \) corresponds to a self-dual symplectic form \( \theta^{-1}_{\mu\nu} \), i.e.
  \( \Theta = \frac{1}{2} \theta^{-1}_{\mu\nu} dx^\mu \wedge dx^\nu, \quad \star \Theta = \pm i \Theta \)
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may define projectors on the tangential resp. normal bundle of $\mathcal{M} \subset \mathbb{R}^D$ as

$$\mathcal{P}_{T}^{ab} = g^{\mu\nu} \partial_{\mu} x^a \partial_{\nu} x^b, \quad \mathcal{P}_{N}^{ab} = \eta^{ab} - \mathcal{P}_{T}^{ab},$$
Intrinsic Curvature II

- hence, $\nabla^g_\sigma \nabla^g_\nu x^a = \mathcal{P}^{ab}_N \nabla^G_\sigma \nabla^G_\nu x_b$ and

$$R_{\rho \sigma \nu \mu} [g] = \nabla^g_\sigma \nabla^g_\mu x^a \nabla^g_\rho \nabla^g_\nu x_a - \nabla^g_\sigma \nabla^g_\nu x^a \nabla^g_\mu \nabla^g_\rho x_a$$

$$= \mathcal{P}^{ab}_N \nabla^G_\sigma \nabla^G_\mu x_a \nabla^G_\rho \nabla^G_\nu x_b - \mathcal{P}^{ab}_N \nabla^G_\sigma \nabla^G_\nu x_a \nabla^G_\mu \nabla^G_\rho x_b$$

- matrix energy-momentum tensor:

$$T^{ab} = H^{ab} - \frac{1}{4} \eta^{ab} H \sim e^\sigma \left( \frac{(Gg)}{4} \eta^{ab} - G^{\mu \nu} \partial_\mu x^a \partial_\nu x^b \right)$$

$$H^{ab} = \frac{1}{2} \left[ [X^a, X^c], [X^b, X_c] \right]_+ , \quad H = H^{ab} \eta_{ab}$$

- if $G_{\mu \nu} = g_{\mu \nu}$: $T^{ab} \sim e^\sigma \mathcal{P}^{ab}_N$ and $H^{ab} \sim -e^\sigma \mathcal{P}^{ab}_T$
Emerging Geometries
D. Blaschke

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Higher order matrix model terms: self dual case

- It can be shown, that the Einstein-Hilbert action emerges in the effective matrix model action (see Harold's talk)

- order 6 matrix terms:

\[ S_{\mathcal{O}}(X^6) = \text{Tr} \left( \frac{1}{2} [X^c, [X^a, X^b]][X_c, [X_a, X_b]] - \Box X^a \Box X_a \right) \]

\[ \sim \int \sqrt{g} \left( \frac{1}{2} \theta^{\mu\rho} \theta^{\eta\alpha} R_{\mu\rho\eta\alpha} - 2e^\sigma R + 2e^\sigma \partial^\mu \sigma \partial_\mu \sigma \right), \]

where \( \Box Y \equiv [X^a, [X_a, Y]] \)

- order 10 matrix terms:

\[ S_{\mathcal{O}}(X^{10}) = \text{Tr} \left( 2T^{ab} \Box X_a \Box X_b - T^{ab} \Box H_{ab} \right) \sim -2 \int \sqrt{g} e^{2\sigma} R. \]
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Certain combination of order 10 matrix terms semi-classically leads to

\[ S_{\mathcal{O}(X^{10})} \sim \int d^4 x \frac{\sqrt{g}}{(2\pi)^2} e^{2\sigma} (R[g] - 3R^\mu{}^\nu [g] h_{\mu\nu}) + \mathcal{O}(\partial h^2), \]

where \( G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \) is almost self-dual.

degrees of freedom:

\[ \theta_{\mu\nu}^{-1} = \bar{\theta}_{\mu\nu}^{-1} + F_{\mu\nu} = \bar{\theta}_{\mu\nu}^{-1} + \partial_\mu A_\nu - \partial_\nu A_\mu, \]

\[ \delta_\phi g_{\mu\nu} = \delta \phi^i \phi^j \eta_{ij} + \phi^i \delta \phi^j \eta_{ij}, \quad \delta_A F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu, \]

\[ h_{\mu\nu} = -e^{\bar{\sigma}} (\bar{\theta}^{-1} g F)_{\mu\nu} - e^{\bar{\sigma}} (F g \bar{\theta}^{-1})_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (\bar{\theta} F) + \mathcal{O}(F^2). \]

Will now consider two examples of geometries which too a good approximation are solutions to the e.o.m.
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Emerging Geometries
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Higher order matrix model terms II: general case

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\delta A F_{\mu\nu} &= \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu , \\
\delta_\phi & g_{\mu\nu} = -e^{\bar{\sigma}} (\bar{\theta}^{-1} g F)_{\mu\nu} - e^{\bar{\sigma}} (F g \bar{\theta}^{-1})_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (\bar{\theta} F) + \mathcal{O}(F^2) .
\end{align*}
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- Will now consider two examples of geometries which too a good approximation are solutions to the e.o.m.
Embedding of Schwarzschild metric

\[ ds^2 = - \left( 1 - \frac{r_c}{r} \right) dt^2_S + \left( 1 - \frac{r_c}{r} \right)^{-1} dr^2 + r^2 d\Omega^2 \]

Consider Eddington-Finkelstein coordinates and define:

\[ t = t_S + (r^* - r) , \quad r^* = r + r_c \ln \left| \frac{r}{r_c} - 1 \right| , \]

\[ \Rightarrow ds^2 = - \left( 1 - \frac{r_c}{r} \right) dt^2 + \frac{2r_c}{r} dtdr + \left( 1 + \frac{r_c}{r} \right) dr^2 + r^2 d\Omega^2 \]

need at least 3 extra dimensions:

\[ \phi_1 + i\phi_2 = \phi_3 e^{i\omega(t+r)} , \]

\[ \phi_3 = \frac{1}{\omega} \sqrt{\frac{r_c}{r}} , \quad \text{where } \phi_3 \text{ is time-like} \]
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7-dim. embedding given by

\[
    x^a = \begin{pmatrix} t \\ r \cos \varphi \sin \vartheta \\ r \sin \varphi \sin \vartheta \\ r \cos \vartheta \\ \frac{1}{\omega} \sqrt{\frac{r_c}{r}} \cos (\omega (t + r)) \\ \frac{1}{\omega} \sqrt{\frac{r_c}{r}} \sin (\omega (t + r)) \\ \frac{1}{\omega} \sqrt{\frac{r_c}{r}} \end{pmatrix}
\]

with background metric \( \eta_{ab} = \text{diag}(-, +, +, +, +, +, +, -) \).
Embedded Schwarzschild black hole
Symplectic form

Require $\star \Theta = i \Theta$, so that $G^{\mu \nu} = e^{\sigma} \theta^{\mu \rho} \theta^{\nu \sigma} g_{\rho \sigma} = g^{\mu \nu}$ and
\[
\lim_{r \to \infty} e^{-\sigma} = \text{const.} \neq 0.
\]

Solution:
\[
\Theta = i E \wedge dt_s + B \wedge d\varphi,
\]
\[
E = c_1 \left( \cos \vartheta dr - r \gamma \sin \vartheta d\vartheta \right) = d(f(r) \cos \vartheta),
\]
\[
B = c_1 \left( r^2 \sin \vartheta \cos \vartheta d\vartheta + r \sin^2 \vartheta dr \right) = \frac{c_1}{2} d(r^2 \sin^2 \vartheta),
\]
\[
\gamma = \left( 1 - \frac{r_c}{r} \right), \quad f(r) = c_1 r \gamma, \quad f' = c_1 = \text{const.},
\]
from which follows
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\]
Darboux coordinates

\[ x^\mu_D = \{ H_{ts}, t_S, H_\varphi, \varphi \} \] corresponding to Killing vector fields \( V_{ts} = \partial_{t_s}, V_\varphi = \partial_\varphi \) where the symplectic form \( \Theta \) is constant:

\[ \Theta = ic_1 dH_{ts} \wedge dt_S + c_1 dH_\varphi \wedge d\varphi, \]
\[ = c_1 d (iH_{ts}dt_S + H_\varphi d\varphi), \]
\[ H_{ts} = r\gamma \cos \vartheta, \quad H_\varphi = \frac{1}{2} r^2 \sin^2 \vartheta \]

Relations to the Killing vector fields:

\[ E = c_1 dH_{ts} = c_1 E_\mu dx^\mu = i_{V_{ts}} \Theta, \quad E_\mu = V_{ts}^\nu \theta_{\nu \mu}^{-1}, \]
\[ B = c_1 dH_\varphi = c_1 B_\mu dx^\mu = i_{V_\varphi} \Theta, \quad B_\mu = V_\varphi^\nu \theta_{\nu \mu}^{-1}, \]

\[ ds_D^2 = -\gamma dt_S^2 + \frac{e^\sigma}{\gamma} dH_{ts}^2 + r^2 \sin^2 \vartheta d\varphi^2 + \frac{e^\sigma}{r^2 \sin^2 \vartheta} dH_\varphi^2 \]
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Relations to the Killing vector fields:

\[ E = c_1 dH_{ts} = c_1 E_\mu dx^\mu = i_{V_{ts}} \Theta, \quad E_\mu = V_{ts}^\nu \theta_{\nu \mu}^{-1}, \]
\[ B = c_1 dH_\varphi = c_1 B_\mu dx^\mu = i_{V_\varphi} \Theta, \quad B_\mu = V_\varphi^\nu \theta_{\nu \mu}^{-1}, \]

\[ ds_D^2 = -\gamma dt_S^2 + \frac{e^{\bar{\sigma}}}{\gamma} dH_{ts}^2 + r^2 \sin^2 \vartheta d\varphi^2 + \frac{e^{\bar{\sigma}}}{r^2 \sin^2 \vartheta} dH_\varphi^2 \]
Darboux coordinates

\[ x_D^\mu = \{ H_{ts}, t_S, H_\varphi, \varphi \} \] corresponding to Killing vector fields \( V_{ts} = \partial_{t_s}, V_\varphi = \partial_\varphi \) where the symplectic form \( \Theta \) is constant:

\[
\Theta = ic_1 dH_{ts} \wedge dt_S + c_1 dH_\varphi \wedge d\varphi ,
\]

\[
= c_1 d (iH_{ts}dt_S + H_\varphi d\varphi) ,
\]

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Star product

A Moyal type star product can easily be defined as

\[(g \ast h)(x_D) = g(x_D) e^{-i \frac{1}{2} \left( \partial_{\mu} \theta_{D}^{\mu\nu} \partial_{\nu} \right)} h(x_D),\]

with

\[
\theta_{D}^{\mu\nu} = \epsilon \begin{pmatrix}
0 & i & 0 & 0 \\
-i & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix},
\]

where \(\epsilon = 1/c_1 \ll 1\) denotes the expansion parameter.
...or in embedding coordinates:

\[
(g \star h)(x) = g(x) \exp \left\{ \frac{\imath \epsilon}{2} \left( \left( \frac{-i r e^\sigma}{r^2 \gamma} \right) + \frac{\imath e^\sigma}{z} \right) \wedge \frac{\partial}{\partial t} \right\} 
+ \left\{ \left( \frac{1}{r^2} \right) \frac{r e^\sigma}{r} + \left( \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \right) \frac{1}{x^2 + y^2} \right\} \wedge \left( \frac{\partial}{\partial y} y - \frac{\partial}{\partial x} x \right) h(x)
\]

where care must be taken with the sequence of operators and the side they act on.

Higher orders in this star product lead to non-commutative corrections to the embedding geometry, e.g.:

\[
\phi_1 \star \phi_1 + \phi_2 \star \phi_2 \neq \phi_3 \star \phi_3
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...or in embedding coordinates:

\[
(g \star h)(x) = g(x) \exp \left[ \frac{i \epsilon}{2} \left( \left( \frac{\vec{\partial} t}{r^2 \gamma} - i r c \vec{\sigma} e^\bar{\sigma} \right) + \vec{\partial} z i e^\bar{\sigma} \right) \wedge \vec{\partial} t \right]
\]

\[
+ \left( \left( \frac{\vec{\partial} t - \vec{\partial} z}{r} \right) r c e^\bar{\sigma} + \left( \vec{\partial} x x + \vec{\partial} y y \right) \frac{1}{x^2 + y^2} \right) \wedge \left( x \vec{\partial} y - y \vec{\partial} x \right) \right] h(x)
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Star commutators for Schwarzschild geometry

\[ -i \left[ x^a \ast x^b \right] = \epsilon e^{\bar{\sigma}} \]

\[
\begin{pmatrix}
0 & -\frac{r_c y}{r^2} & \frac{r_c x}{r^2} & -i & \frac{iz f_{12}^+(1)}{r} & \frac{iz f_{21}^-(1)}{r} & \frac{iz \phi_3}{2r^2} \\
\frac{r_c y}{r^2} & 0 & e^{-\bar{\sigma}} & -\frac{r_c y z}{r^3} & -y f_{12}^+(\gamma) & -y f_{21}^-(\gamma) & -y \gamma \phi_3 \\
-\frac{r_c x}{r^2} & -e^{-\bar{\sigma}} & 0 & \frac{r_c x z}{r^3} & x f_{12}^+(\gamma) & x f_{21}^-(\gamma) & x \gamma \phi_3 \\
i & \frac{r_c y z}{r^3} & -\frac{r_c x z}{r^3} & 0 & -i \omega \phi_2 & i \omega \phi_1 & 0 \\
-iz f_{12}^+(1) & y f_{12}^+(\gamma) & -x f_{12}^+(\gamma) & i \omega \phi_2 & 0 & -i \omega z \phi_3^{\pm} & -i \omega z \phi_3 \phi_2 \\
-iz f_{21}^-(1) & y f_{21}^-(\gamma) & -x f_{21}^-(\gamma) & -i \omega \phi_1 & i \omega z \phi_3^{\pm} & 0 & -i \omega z \phi_3 \phi_1 \\
-iz \phi_3 & y \gamma \phi_3 & -x \gamma \phi_3 & 0 & i \omega z \phi_3 \phi_2 & -i \omega z \phi_3 \phi_1 & 0 \\
\end{pmatrix} 
+ \mathcal{O}(\epsilon^3),
\]

with

\[
f_{ij}^\pm(Y) = \left( \frac{Y}{2r} \phi_i \pm \omega \phi_j \right).
\]
Embedding of Reissner-Nordström metric

RN metric in spherical coordinates $x^\mu = \{t, r, \vartheta, \varphi\}$:

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) d\tilde{t}^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega$$

which has two concentric horizons at

$$r_h = \left(m \pm \sqrt{m^2 - q^2}\right)$$

Shift the time-coordinate according to

$$t = \tilde{t} + (r^* - r)$$

with $dr^* \equiv \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr$,

and arrive at

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + 2\left(\frac{2m}{r} - \frac{q^2}{r^2}\right) dt dr$$

$$+ \left(1 + \frac{2m}{r} - \frac{q^2}{r^2}\right) dr^2 + r^2 d\Omega.$$
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Embedding of RN metric II

10-dimensional embedding $\mathcal{M}^{1,3} \hookrightarrow \mathbb{R}^{4,6}$ with additional coordinates $\phi_i$ given by

$$\phi_1 + i\phi_2 = \phi_3 e^{i\omega(t+r)},$$
$$\phi_4 + i\phi_5 = \phi_6 e^{i\omega(t+r)},$$

$$\phi_3 = \frac{1}{\omega} \sqrt{\frac{2m}{r}},$$
$$\phi_6 = \frac{q}{\omega r}$$

$\phi_3$, $\phi_4$ and $\phi_5$ are time-like coordinates.

- This is not unique, i.e. could have used a 7-dim. embedding similar to the Schwarzschild case, but which is valid only up to the inner horizon.
- Expect all physically relevant geometries to be embeddable in 10-dim., at least locally (cf. Friedman 1961).
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Symplectic form and Darboux coordinates

\[ \Theta = \frac{1}{\epsilon} \left( i dH_{\tilde{t}} \wedge d\tilde{t} + dH_\varphi \wedge d\varphi \right), \]

\[ H_{\tilde{t}} = \gamma r \cos \vartheta, \quad H_\varphi = \frac{r^2}{2} \left( 1 - \frac{q^2}{r^2} \right) \sin^2 \vartheta, \]

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Star product for RN geometry

A Moyal type star product can again be defined as

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with the same block-diagonal \(\theta^{\mu\nu}\) as before.

\[\ldots\text{and once more, higher orders in the star product lead to non-commutative corrections to the embedding geometry, e.g.}:\]

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- Discussed explicit embeddings of Schwarzschild and RN geometries including self-dual symplectic forms.
- Embeddings should be modified near the horizons to account for nearly constant $e^\sigma$ (work in progress).
- Open questions: deviations from $G = g$, higher order quantum effects, etc. (work in progress).
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References


*Thank you for your attention!*