# Special geometries emerging from Yang-Mills type matrix models

Talk presented by Daniel N. Blaschke

universität wien

Faculty of Physics,
Mathematical Physics Group

*Collaborator:* H. Steinacker

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# Outline

# Matrix models of Yang-Mills type



$$S_{YM} = -\mathrm{Tr}[X^a, X^b][X^c, X^d]\eta_{ac}\eta_{bd}$$

- $X^a$ Herm. matrices on $\mathcal{H}$, and $\eta_{ab}$ is $D$-dim. flat metric
- $X^a = \left(X^\mu, \Phi^i\right)$, $\mu = 1, \ldots, 2n$, $i = 1, \ldots, D-2n$, so that $\Phi^i(X) \sim \phi^i(x)$ define embedding $\mathcal{M}^{2n} \hookrightarrow \mathbb{R}^D$ $g_{\mu\nu}(x) = \partial_\mu x^a \partial_\nu x^b \eta_{ab}$ (in semi-classical limit)
- $\mathcal{M}^{2n}$ endowed with a Poisson structure $-i[X^\mu, X^\nu] \sim \{x^\mu, x^\nu\}_{PB} = \theta^{\mu\nu}(x)$ $\Rightarrow$ "effective" metric

$$G^{\mu\nu} = e^{-\sigma}\theta^{\mu\rho}\theta^{\nu\sigma}g_{\rho\sigma}\,, \qquad e^{-\sigma} \equiv \frac{\sqrt{\det\theta^{-1}_{\mu\nu}}}{\sqrt{\det G_{\rho\sigma}}}$$

- $2n = 4$: special class of geometries where $G_{\mu\nu} = g_{\mu\nu}$ corresponds to a self-dual symplectic form $\theta^{-1}_{\mu\nu}$, i.e. $\Theta = \frac{1}{2}\theta^{-1}_{\mu\nu}dx^\mu \wedge dx^\nu$, $\qquad \star\Theta = \pm i\Theta$

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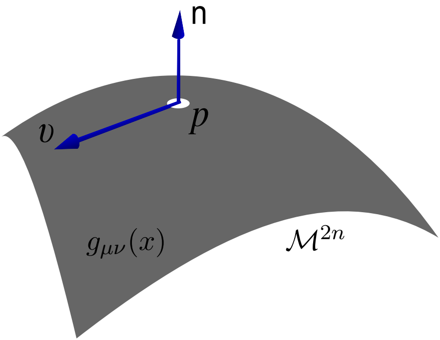
# Intrinsic Curvature






- may define projectors on the tangential resp. normal bundle of $\mathcal{M} \subset \mathbb{R}^D$ as

$$\mathcal{P}_T^{ab} = g^{\mu\nu} \partial_\mu x^a \partial_\nu x^b \,, \qquad \mathcal{P}_N^{ab} = \eta^{ab} - \mathcal{P}_T^{ab} \,,$$

# Intrinsic Curvature II



- hence, $\nabla^g_\sigma \nabla^g_\nu x^a = \mathcal{P}^{ab}_N \nabla^G_\sigma \nabla^G_\nu x_b$ and

$$R_{\rho\sigma\nu\mu}[g] = \nabla^g_\sigma \nabla^g_\mu x^a \nabla^g_\rho \nabla^g_\nu x_a - \nabla^g_\sigma \nabla^g_\nu x^a \nabla^g_\mu \nabla^g_\rho x_a$$
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- matrix energy-momentum tensor:

$$T^{ab} = H^{ab} - \frac{1}{4}\eta^{ab} H \sim e^\sigma \left( \frac{(Gg)}{4} \eta^{ab} - G^{\mu\nu} \partial_\mu x^a \partial_\nu x^b \right)$$

$$H^{ab} = \frac{1}{2}\left[ [X^a, X^c], [X^b, X_c] \right]_+ , \qquad H = H^{ab}\eta_{ab}$$

- if $G_{\mu\nu} = g_{\mu\nu}$: $\qquad T^{ab} \sim e^\sigma \mathcal{P}^{ab}_N$ and $H^{ab} \sim -e^\sigma \mathcal{P}^{ab}_T$

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# Higher order matrix model terms: self dual case



- It can be shown, that the Einstein-Hilbert action emerges in the effective matrix model action (see Harolds talk)
- order 6 matrix terms:

$$S_{\mathcal{O}(X^6)} = \mathrm{Tr}\left(\frac{1}{2}[X^c,[X^a,X^b]][X_c,[X_a,X_b]] - \Box X^a \Box X_a\right)$$
$$\sim \int\sqrt{g}\left(\frac{1}{2}\theta^{\mu\rho}\theta^{\eta\alpha}R_{\mu\rho\eta\alpha} - 2e^\sigma R + 2e^\sigma\partial^\mu\sigma\partial_\mu\sigma\right),$$

where $\Box Y \equiv [X^a,[X_a,Y]]$

- order 10 matrix terms:

$$S_{\mathcal{O}(X^{10})} = \mathrm{Tr}\left(2T^{ab}\Box X_a \Box X_b - T^{ab}\Box H_{ab}\right) \sim -2\int\sqrt{g}\,e^{2\sigma}R\,.$$

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# Higher order matrix model terms II: general case



- Certain combination of order 10 matrix terms semi-classically leads to

$$S_{\mathcal{O}(X^{10})} \sim \int d^4x \frac{\sqrt{g}}{(2\pi)^2} e^{2\sigma} \left( R[g] - 3R^{\mu\nu}[g] h_{\mu\nu} \right) + \mathcal{O}(\partial h^2) \,,$$

$$\text{where} \quad G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \quad \text{is almost self-dual.}$$

- degrees of freedom:

$$\theta^{-1}_{\mu\nu} = \bar{\theta}^{-1}_{\mu\nu} + F_{\mu\nu} = \bar{\theta}^{-1}_{\mu\nu} + \partial_\mu A_\nu - \partial_\nu A_\mu \,,$$

$$\delta_\phi g_{\mu\nu} = \delta\phi^i \phi^j \eta_{ij} + \phi^i \delta\phi^j \eta_{ij} \,, \qquad \delta_A F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu \,,$$

$$h_{\mu\nu} = -e^{\bar{\sigma}} (\bar{\theta}^{-1} g F)_{\mu\nu} - e^{\bar{\sigma}} (F g \bar{\theta}^{-1})_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (\bar{\theta} F) + \mathcal{O}(F^2) \,.$$

- Will now consider two examples of geometries which too a good approximation are solutions to the e.o.m.

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# Embedding of Schwarzschild metric



$$ds^2 = -\left(1 - \frac{r_c}{r}\right) dt_S^2 + \left(1 - \frac{r_c}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Consider Eddington-Finkelstein coordinates and define:

$$t = t_S + (r^* - r)\,, \qquad r^* = r + r_c \ln\left|\frac{r}{r_c} - 1\right|\,,$$

$$\Rightarrow ds^2 = -\left(1 - \frac{r_c}{r}\right) dt^2 + \frac{2r_c}{r} dt dr + \left(1 + \frac{r_c}{r}\right) dr^2 + r^2 d\Omega^2$$

need at least 3 extra dimensions:

$$\phi_1 + i\phi_2 = \phi_3 e^{i\omega(t+r)}\,,$$

$$\phi_3 = \frac{1}{\omega}\sqrt{\frac{r_c}{r}}\,, \qquad \text{where } \phi_3 \text{ is time-like}$$

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# Embedding of Schwarzschild metric II

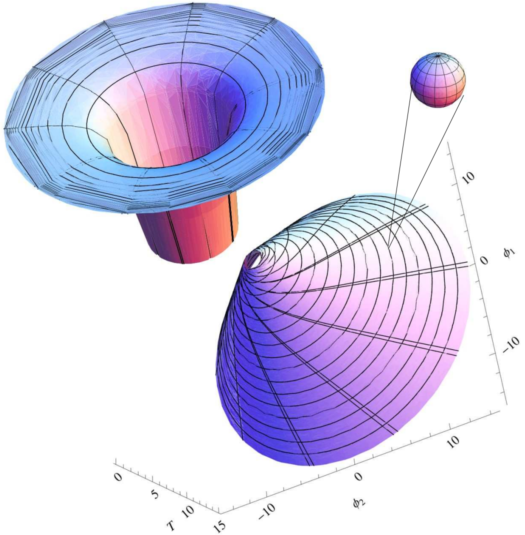


7-dim. embedding given by

$$x^a = \begin{pmatrix} t \\ r\cos\varphi\sin\vartheta \\ r\sin\varphi\sin\vartheta \\ r\cos\vartheta \\ \frac{1}{\omega}\sqrt{\frac{r_c}{r}}\cos\left(\omega(t+r)\right) \\ \frac{1}{\omega}\sqrt{\frac{r_c}{r}}\sin\left(\omega(t+r)\right) \\ \frac{1}{\omega}\sqrt{\frac{r_c}{r}} \end{pmatrix}$$

with background metric $\eta_{ab} = \mathrm{diag}(-,+,+,+,+,+,-)$.

# Embedded Schwarzschild black hole



# Symplectic form



Require $\star\Theta = i\Theta$, so that $G^{\mu\nu} = e^{\sigma}\theta^{\mu\rho}\theta^{\nu\sigma}g_{\rho\sigma} = g^{\mu\nu}$ and $\lim\limits_{r\to\infty} e^{-\sigma} = \text{const.} \neq 0$.

Solution:

$$\Theta = iE \wedge dt_S + B \wedge d\varphi\,,$$
$$E = c_1\left(\cos\vartheta dr - r\gamma\sin\vartheta d\vartheta\right) = d(f(r)\cos\vartheta)\,,$$
$$B = c_1\left(r^2\sin\vartheta\cos\vartheta d\vartheta + r\sin^2\vartheta dr\right) = \frac{c_1}{2}d(r^2\sin^2\vartheta)\,,$$
$$\gamma = \left(1 - \frac{r_c}{r}\right)\,, \qquad f(r) = c_1 r\gamma\,, \qquad f' = c_1 = \text{const.}\,,$$

from which follows

$$e^{-\sigma} = c_1^2\left(1 - \frac{r_c}{r}\sin^2\vartheta\right) \equiv c_1^2 e^{-\bar{\sigma}}\,.$$

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# Darboux coordinates

$x^\mu_D = \{H_{ts}, t_S, H_\varphi, \varphi\}$ corresponding to Killing vector fields $V_{ts} = \partial_{t_s}$ , $V_\varphi = \partial_\varphi$ where the symplectic form $\Theta$ is constant:

$$\Theta = ic_1 dH_{ts} \wedge dt_S + c_1 dH_\varphi \wedge d\varphi \,,$$
$$= c_1 d\left(iH_{ts} dt_S + H_\varphi d\varphi\right) \,,$$

$$H_{ts} = r\gamma \cos\vartheta \,, \qquad H_\varphi = \frac{1}{2} r^2 \sin^2\vartheta$$

Relations to the Killing vector fields:

$$E = c_1 dH_{ts} = c_1 E_\mu dx^\mu = i_{V_{ts}} \Theta \,, \qquad E_\mu = V^\nu_{ts} \theta^{-1}_{\nu\mu} \,,$$
$$B = c_1 dH_\varphi = c_1 B_\mu dx^\mu = i_{V_\varphi} \Theta \,, \qquad B_\mu = V^\nu_\varphi \theta^{-1}_{\nu\mu} \,,$$

$$ds^2_D = -\gamma dt^2_S + \frac{e^{\bar\sigma}}{\gamma} dH^2_{ts} + r^2 \sin^2\vartheta d\varphi^2 + \frac{e^{\bar\sigma}}{r^2 \sin^2\vartheta} dH^2_\varphi$$

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$$H_{ts} = r\gamma\cos\vartheta\,, \qquad H_\varphi = \frac{1}{2} r^2 \sin^2\vartheta$$

Relations to the Killing vector fields:

$$\begin{aligned} E &= c_1 dH_{ts} = c_1 E_\mu dx^\mu = i_{V_{ts}}\Theta\,, &\qquad E_\mu &= V_{ts}^\nu \theta^{-1}_{\nu\mu}\,, \\ B &= c_1 dH_\varphi = c_1 B_\mu dx^\mu = i_{V_\varphi}\Theta\,, &\qquad B_\mu &= V_\varphi^\nu \theta^{-1}_{\nu\mu}\,, \end{aligned}$$

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# Star product



A Moyal type star product can easily be defined as

$$(g \star h)(x_D) = g(x_D) e^{-\frac{i}{2}\left(\overleftarrow{\partial}_\mu \theta_D^{\mu\nu} \overrightarrow{\partial}_\nu\right)} h(x_D) \,,$$

with

$$\theta_D^{\mu\nu} = \epsilon \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} ,$$

where $\epsilon = 1/c_1 \ll 1$ denotes the expansion parameter.

# Star product II



…or in embedding coordinates:

$$(g \star h)(x) = g(x) \exp\left[\frac{i\epsilon}{2}\left(\left(\left(\overleftarrow{\partial}_t \frac{i r_c z e^{\bar{\sigma}}}{r^2\gamma} + \overleftarrow{\partial}_z i e^{\bar{\sigma}}\right) \wedge \overrightarrow{\partial}_t\right.\right.\right.$$
$$\left.\left.\left. + \left(\left(\overleftarrow{\partial}_t - \overleftarrow{\partial}_z \frac{z}{r}\right)\frac{r_c e^{\bar{\sigma}}}{r^2} + \left(\overleftarrow{\partial}_x x + \overleftarrow{\partial}_y y\right)\frac{1}{x^2+y^2}\right) \wedge \left(x \overrightarrow{\partial}_y - y \overrightarrow{\partial}_x\right)\right)\right] h(x)$$

where care must be taken with the sequence of operators and the side they act on.

Higher orders in this star product lead to non-commutative corrections to the embedding geometry, e.g.:

$$\phi_1 \star \phi_1 + \phi_2 \star \phi_2 \neq \phi_3 \star \phi_3$$

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# Star commutators for Schwarzschild geometry

$$-i\left[x^a \stackrel{\star}{,} x^b\right] = \epsilon e^{\bar\sigma}$$

$$\begin{pmatrix}
0 & -\frac{r_c y}{r^2} & \frac{r_c x}{r^2} & -i & \frac{iz f^+_{12}(1)}{r} & \frac{iz f^-_{21}(1)}{r} & \frac{iz\phi_3}{2r^2} \\
\frac{r_c y}{r^2} & 0 & e^{-\bar\sigma} & -\frac{r_c yz}{r^3} & \frac{-y f^+_{12}(\gamma)}{r} & \frac{-y f^-_{21}(\gamma)}{r} & -\frac{y\gamma\phi_3}{2r^2} \\
-\frac{r_c x}{r^2} & -e^{-\bar\sigma} & 0 & \frac{r_c xz}{r^3} & \frac{x f^+_{12}(\gamma)}{r} & \frac{x f^-_{21}(\gamma)}{r} & \frac{x\gamma\phi_3}{2r^2} \\
i & \frac{r_c yz}{r^3} & -\frac{r_c xz}{r^3} & 0 & -i\omega\phi_2 & i\omega\phi_1 & 0 \\
\frac{-iz f^+_{12}(1)}{r} & \frac{y f^+_{12}(\gamma)}{r} & \frac{-x f^+_{12}(\gamma)}{r} & i\omega\phi_2 & 0 & -\frac{i\omega z\phi_3^2}{2r^2} & \frac{-i\omega z\phi_3\phi_2}{2r^2} \\
\frac{-iz f^-_{21}(1)}{r} & \frac{y f^-_{21}(\gamma)}{r} & \frac{-x f^-_{21}(\gamma)}{r} & -i\omega\phi_1 & \frac{i\omega z\phi_3^2}{2r^2} & 0 & \frac{i\omega z\phi_3\phi_1}{2r^2} \\
-\frac{iz\phi_3}{2r^2} & \frac{y\gamma\phi_3}{2r^2} & -\frac{x\gamma\phi_3}{2r^2} & 0 & \frac{i\omega z\phi_3\phi_2}{2r^2} & \frac{-i\omega z\phi_3\phi_1}{2r^2} & 0
\end{pmatrix}$$

$$+\mathcal{O}(\epsilon^3)\,,$$

with

$$f^\pm_{ij}(Y) = \left(\frac{Y}{2r}\phi_i \pm \omega\phi_j\right)\,.$$

# Embedding of Reissner-Nordström metric



RN metric in spherical coordinates $x^\mu = \{t, r, \vartheta, \varphi\}$:

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) d\tilde{t}^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega$$

which has two concentric horizons at

$$r_h = \left(m \pm \sqrt{m^2 - q^2}\right)$$

Shift the time-coordinate according to

$$t = \tilde{t} + (r^* - r)\,, \qquad \text{with } dr^* \equiv \left(1 - \tfrac{2m}{r} + \tfrac{q^2}{r^2}\right)^{-1} dr\,,$$

and arrive at

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + 2\left(\frac{2m}{r} - \frac{q^2}{r^2}\right) dt dr + \left(1 + \frac{2m}{r} - \frac{q^2}{r^2}\right) dr^2 + r^2 d\Omega\,.$$

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# Embedding of RN metric II



10-dimensional embedding $\mathcal{M}^{1,3} \hookrightarrow \mathbb{R}^{4,6}$ with additional coordinates $\phi_i$ given by

$$\phi_1 + i\phi_2 = \phi_3 e^{i\omega(t+r)}\,, \qquad \phi_3 = \frac{1}{\omega}\sqrt{\frac{2m}{r}}\,,$$

$$\phi_4 + i\phi_5 = \phi_6 e^{i\omega(t+r)}\,, \qquad \phi_6 = \frac{q}{\omega r}$$

$\phi_3$, $\phi_4$ and $\phi_5$ are *time-like* coordinates.

- This is not unique, i.e. could have used a 7-dim. embedding similar to the Schwarzschild case, but which is valid only up to the inner horizon.
- Expect all physically relevant geometries to be embeddable in 10-dim., at least locally (cf. Friedman 1961).

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# Symplectic form and Darboux coordinates



$$\Theta = \frac{1}{\epsilon}\left(idH_{\tilde{t}} \wedge d\tilde{t} + dH_{\varphi} \wedge d\varphi\right) ,$$

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# Conclusion and Outlook

- Discussed explicit embeddings of Schwarzschild and RN geometries including self-dual symplectic forms.
- Embeddings should be modified near the horizons to account for nearly constant $e^{\sigma}$ (work in progress).
- Open questions: deviations from $G = g$, higher order quantum effects, etc. (work in progress).

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# References



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*Thank you for your attention!*