Noncommutative field theories with harmonic term

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arXiv:1003.5788

Moyal space in Physics

- Candidate for new Physics
- Noncommutativity of space-time at Planck scale? (Doplicher *et. al.* '94)
- Relation with Quantum Loop Gravity (Freidel Livine '06, Noui '08)
- Effective regime of string theory (Seiberg Witten '99) and matrix theory (Connes *et. al.* '98)
- Models describing quantum Hall effect (Polychronakos '01,...)

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Plan

1. QFT with harmonic term on the Moyal space
2. Langmann-Szabo duality
3. Superalgebraic interpretation

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1. Moyal space: a deformation quantization

- Space of Schwartz functions $f : \mathbb{R}^D \to \mathbb{C}$
- Deformed Moyal product:

$$(f \star g)(x) = \frac{1}{\pi^D \theta^D} \int d^D y d^D z \, f(x+y) g(x+z) e^{-2iy\Theta^{-1}z}$$

$$\Theta = \theta\Sigma, \qquad \Sigma = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & 0 & \\ & & 0 & -1 & \\ & 0 & 1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

- Limit $\theta = 0$: $(f \star g)(x) = f(x) \cdot g(x)$
- Extension to distributions: 1, x_μ, $x_\mu x_\nu$, ...
  $\longrightarrow$ Moyal algebra $\mathcal{M}_\theta$

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- Generic problem on noncommutative spaces: UV/IR mixing
  (Minwalla Van Raamsdonk Seiberg '00)
- Cured by harmonic term: renormalizable theory for $\Omega \neq 0$
  (Grosse Wulkenhaar '04): $(\widetilde{x}_\mu = 2\Theta^{-1}_{\mu\nu} x_\nu)$

$$S[\phi] = \int d^4x \Big(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{\Omega^2}{2}\widetilde{x}_\mu^2 \phi^2 + \frac{m^2}{2}\phi^2 + \lambda\, \phi \star \phi \star \phi \star \phi \Big)$$

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- Effective action associated to the model with harmonic term:

  (A.G. Wallet Wulkenhaar '07, Grosse Wohlgenannt '07)

$$S = \int d^4x \Big(\frac{1}{4} F_{\mu\nu} \star F_{\mu\nu} + \frac{\beta}{4} \{\mathcal{A}_\mu, \mathcal{A}_\nu\}_\star^2 + \frac{\kappa}{2} \mathcal{A}_\mu \star \mathcal{A}_\mu \Big)$$

  where the covariant coordinate: $\mathcal{A}_\mu = A_\mu + \frac{1}{2}\widetilde{x}_\mu$ and the curvature: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star$.

- Good candidate to renormalizability
- Special vacuum: $A^0_\mu = -\frac{1}{2}\widetilde{x}_\mu$ (A.G. '07)

$$S = \int d^4x \Big(\frac{\kappa}{2} \mathcal{A}_\mu \star \mathcal{A}_\mu + \frac{1+\beta}{2} (\mathcal{A}_\mu \star \mathcal{A}_\mu)^2 - \frac{1-\beta}{2} (\mathcal{A}_\mu \star \mathcal{A}_\nu)^2 \Big)$$

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2. Langmann-Szabo duality

- Cyclic symplectic Fourier transformation:

$$\hat{\phi}(k_a) = \frac{1}{(\pi\theta)^{\frac{D}{2}}} \int \mathrm{d}^D x \, \phi(x) e^{-i(-1)^a k_a \wedge x}$$

where $a \in \{1, \dots, 4\}$: cyclic order of the impulsion

- Properties of the Fourier transformation (derivation, Parseval identity,...)
- Quadratic part of the action:

$$\int \mathrm{d}^D x \Big(\frac{1}{2} (\partial^x_\mu \phi)^2 + \frac{\Omega^2}{2} \widetilde{x}^2 \phi^2 + \frac{m^2}{2} \phi^2 \Big) = \int \mathrm{d}^D k \Big(\frac{1}{2} \widetilde{k}^2 \hat{\phi}^2 + \frac{\Omega^2}{2} (\partial^k_\mu \hat{\phi})^2 + \frac{m^2}{2} \hat{\phi}^2 \Big),$$

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2. Metaplectic representation

- Heisenberg algebra: $\mathfrak{h}=\mathbb{R}^{2D}\oplus\mathbb{R}$, $\omega=\begin{pmatrix}0 & \Sigma\\ \Sigma & 0\end{pmatrix}$

$$[(x,p,s),(y,q,t)]=(0,0,x\Sigma q+p\Sigma y)$$

- Symplectic group acts on $\mathfrak{h}$

$$Sp(\mathbb{R}^{2D},\omega)=\{M\in GL(\mathbb{R}^{2D}),\ M^T\omega M=\omega\}$$

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(Bieliavsky Gurau Rivasseau '09, A.G. '10)

- Infinitesimal generator: $Z = \begin{pmatrix} 0 & \frac{4\Omega^2}{\theta^2}\Sigma \\ \Sigma & 0 \end{pmatrix}$
- Corresponding operator: $2id\mu(Z) = -\partial^2 + \Omega^2\widetilde{x}^2$

Proposition (A.G.)

For $M \in Sp(\mathbb{R}^{2D}, \omega)$, the quadratic part of the Grosse-Wulkenhaar action is covariant under $\mu(M) \Leftrightarrow MZM^{-1}$ is of the form of Z, namely $\begin{pmatrix} 0 & \alpha\Sigma \\ \beta\Sigma & 0 \end{pmatrix}$, where $\alpha, \beta \in \mathbb{R}$.

- Position space transformations $O(D) \cap Sp(\mathbb{R}^D, \Sigma)$
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3. Motivation for a grading symmetry

- Scalar theory:

$$S=\int d^4x\Big(\frac{1}{2}(i[\xi_\mu,\phi]_\star)^2+\frac{\Omega^2}{2}(\{\xi_\mu,\phi\}_\star)^2+\dots\Big)$$

Langmann-Szabo duality exchanges the two terms ($\partial_\mu \rightleftarrows \widetilde{x}_\mu$).

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3. The Moyal superalgebra

- $\mathbf{A}_\theta^\bullet = \mathbf{A}_\theta^0 \oplus \mathbf{A}_\theta^1$, where $\mathbf{A}_\theta^0 = \mathcal{M}_\theta$, $\mathbf{A}_\theta^1 = \mathcal{M}_\theta$
- Product: for $f = (f_0, f_1)$, $g = (g_0, g_1) \in \mathbf{A}_\theta^\bullet$,

$$f \cdot g = (f_0 \star g_0 + f_1 \star g_1, f_0 \star g_1 + f_1 \star g_0)$$

- Unit: $\mathbb{1} = (1, 0)$
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3. Differential calculus and scalar action

- Generators: $\xi_\mu = -\frac{1}{2}\widetilde{x}_\mu,\ \eta_{\mu\nu} = \frac{1}{2}\widetilde{x}_\mu\widetilde{x}_\nu$
- Lie superalgebra:
  $\mathfrak{g}^\bullet = \langle \mathbb{1},\ (0,1),\ (i\xi_\mu,0),\ (0,i\xi_\mu),\ (i\eta_{\mu\nu},0)\rangle \subset \mathbf{A}_\theta^\bullet$
  $$\Big[(i\xi_\mu,0),(0,i\xi_\nu)\Big] = i\Theta^{-1}_{\mu\nu}(0,1)$$
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- Graded differential calculus, for the algebra $\mathbf{A}_\theta^\bullet$, restricted to the derivations $(\mathrm{ad}\mathfrak{g}^\bullet)$
  $$df((i\xi_\mu,0)) = (\partial_\mu f_0,\partial_\mu f_1) = \partial_\mu f,$$
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- Scalar action:
  $S = \mathrm{Tr}\Big(\sum_a |d(\phi,\phi)(a)|^2\Big) = \int d^4x\Big(3(\partial_\mu\phi)^2 + (\widetilde{x}_\mu\phi)^2 + \frac{4}{\theta}\phi^2\Big)$

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- Generators: $\xi_\mu = -\frac{1}{2}\widetilde{x}_\mu, \ \eta_{\mu\nu} = \frac{1}{2}\widetilde{x}_\mu\widetilde{x}_\nu$
- Lie superalgebra:
  $\mathfrak{g}^\bullet = \langle \mathbb{1}, \ (0,1), \ (i\xi_\mu, 0), \ (0, i\xi_\mu), \ (i\eta_{\mu\nu}, 0)\rangle \subset \mathbf{A}_\theta^\bullet$
  $$\Big[(i\xi_\mu, 0), (0, i\xi_\nu)\Big] = i\Theta^{-1}_{\mu\nu}(0,1)$$
  $$\Big[(0, i\xi_\mu), (0, i\xi_\nu)\Big] = i(i\eta_{\mu\nu}, 0)$$
- Graded differential calculus, for the algebra $\mathbf{A}_\theta^\bullet$, restricted to the derivations ($\text{ad}\mathfrak{g}^\bullet$)
  $$\mathrm{d}f((i\xi_\mu, 0)) = (\partial_\mu f_0, \partial_\mu f_1) = \partial_\mu f,$$
  $$\mathrm{d}f((0, i\xi_\mu)) = (i\widetilde{x}_\mu f_1, \partial_\mu f_0), \qquad \mathrm{d}f((0,1)) = (-2f_1, 0)$$
- Scalar action:
  $S = \mathrm{Tr}\left(\sum_a |\mathrm{d}(\phi,\phi)(a)|^2 \right) = \int d^4x \Big(3(\partial_\mu\phi)^2 + (\widetilde{x}_\mu\phi)^2 + \frac{4}{\theta}\phi^2 \Big)$

3. Interpretation of the Langmann-Szabo duality

Heisenberg algebra $\mathfrak{h} \to$ Lie superalgebra $\mathfrak{g}^\bullet$ (A.G. '10)

- $\mathfrak{h} = \mathbb{R}^{2D} \oplus \mathbb{R}$ $\quad [(x,p,s),(y,q,t)] = (0,0,x\Sigma q + p\Sigma y)$
- Extension: $\tilde{\mathfrak{h}} = \mathbb{C}^{2D} \oplus \mathbb{C} \oplus \mathbb{C}$

$$[(x,p,s,a),(y,q,t,b)] = (0,0,x\Sigma q + p\Sigma y, i\theta x\Sigma y + i\theta p\Sigma q)$$

  Realization in $\mathbf{A}_\theta^\bullet$ as a Lie algebra:
  $(x,p,s,a) \mapsto x_\mu(i\theta\xi_\mu,0) + p_\mu(0,i\theta\xi_\mu) + s(0,i\theta) + a\mathbb{1}$
- Superization: take the graded bracket in $\mathbf{A}_\theta^\bullet$

$$[(0,1),(0,1)] = -2\mathbb{1}, \quad [(0,i\xi_\mu),(0,1)] = 2(i\xi_\mu,0),$$
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- Closure of $\langle \mathbb{1}, (0,1), (i\xi_\mu,0), (0,i\xi_\mu)\rangle$ in $\mathbf{A}_\theta^\bullet$ $\to \mathfrak{g}^\bullet$
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3. Gauge theory

- Connections: derivation → gauge potential
  $(i\xi_\mu, 0) \to (-iA^0_\mu, 0), \quad (0, i\xi_\mu) \to (0, -iA^1_\mu)$
  $(0, 1) \to (0, -\varphi), \quad (i\eta_{\mu\nu}, 0) \to (-iG_{\mu\nu}, 0)$
- Simplification: $\varphi = 0$, $G_{\mu\nu} = \eta_{\mu\nu}$ (compatible g.t.)
- Gauge transformations $g \in \mathcal{M}_\theta$ with $g^\dagger \star g = g \star g^\dagger = 1$

$$(A^j_\mu)^g = g \star A^j_\mu \star g^\dagger + ig \star \partial_\mu g^\dagger, \quad \text{for } j = 0, 1$$

- Curvature: $(\mathcal{A}^j_\mu = A^j_\mu - \xi_\mu)$

$$\mathcal{F}_{(0,1),(0,1)} = (-2iA^0_\mu, 0), \quad \mathcal{F}_{(i\xi_\mu,0),(i\xi_\nu,0)} = (F_{\mu\nu}, 0),$$

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$$= \int d^4x \Big(3F_{\mu\nu} \star F_{\mu\nu} + \{\mathcal{A}_\mu, \mathcal{A}_\nu\}^2_\star + \frac{168}{\theta}\mathcal{A}_\mu \star \mathcal{A}_\mu\Big)$$

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  (Bieliavsky A.G.)
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