Quantum Field Theory on NC Curved Spacetimes

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Workshop on “Noncommutative Field Theory and Gravity”

Corfu Summer Institute

September 8-12, 2010

Motivation

- QFT on curved spacetimes is important for physics

$\longrightarrow$ cosmology (CMB fluctuations) and black holes (Hawking radiation)

- precise formulation via algebraic approach [Wald, many others]
- But why should we make all of this noncommutative?
  - NC geometry from quantum gravity!?!?

  $\longrightarrow$ include some quantum gravity effects in QFT on CS

  - NC geometry is natural generalization of classical geometry

  $\longrightarrow$ generalize standard methods of QFT on CS as far as possible

  - NC in cosmology and black hole physics is of physical interest

  $\longrightarrow$ provide formal background for phenomenology

  - $\exists$ NC gravity solutions [Schupp, Solodukhin; Ohl, AS; Aschieri, Castellani]

  $\longrightarrow$ test their physical implications by using QFT on CS

Outline of my talk:

1. Deformed QFT on CS: [Ohl, AS] vs. [Dappiaggi, Lechner, Morfa-Morales]

$$\begin{array}{ccc} \text{spacetime} & \xrightarrow{\text{deform}} & \text{deformed spacetime} \\ \text{quantum fields}\big\downarrow & & \big\downarrow\text{quantum fields} \\ \text{QFT} & \xrightarrow{\text{deform}} & \text{deformed QFT} \end{array}$$

2. For formal deformation quantization:

$$\text{QFT on NC CS} \xleftrightarrow{\ast\text{-algebra isomorphism}} \text{QFT on CS}[[\lambda]]$$

3. Example of a convergent deformation:

$$\text{QFT on NC CS} \xrightarrow{\text{injective } \ast\text{-morphism}} \text{QFT on CS}$$

where certain strongly localized observables are excluded!

Scalar field theory on NC curved spacetimes

- *Simple example of a twist:* [Moyal product/twist]

  $\star$-product $\quad h \star k = h\, e^{\frac{i\lambda}{2}\overleftarrow{\partial_\mu}\Theta^{\mu\nu}\overrightarrow{\partial_\nu}}\, k \quad \leftrightsquigarrow \quad$ twist $\quad \mathcal{F}^{-1} = e^{\frac{i\lambda}{2}\Theta^{\mu\nu}\partial_\mu\otimes\partial_\nu}$

- *More general class of twists: (abelian twists)*

$$\mathcal{F}^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha = \exp\left(\frac{i\lambda}{2}\Theta^{ab}X_a \otimes X_b\right) \quad \text{with} \quad [X_a, X_b] = 0$$

NB: most studied NC gravity solutions are of this type

- NC geometry via twist deformation quantization: [Wess group]
  - algebra of functions $(C^\infty(\mathcal{M})[[\lambda]], \star)$, where $h \star k := \bar{f}^\alpha(h) \cdot \bar{f}_\alpha(k)$
  - exterior algebra $(\Omega^\bullet[[\lambda]], \wedge_\star, d)$, where $\omega \wedge_\star \omega' := \bar{f}^\alpha(\omega) \wedge \bar{f}_\alpha(\omega')$
  - pairing $\langle v, \omega\rangle_\star := \langle \bar{f}^\alpha(v), \bar{f}_\alpha(\omega)\rangle$ among vector fields and 1-forms

- deformed action functional:

$$S_\Phi = \int L_\Phi = -\frac{1}{2}\int \big(\langle\langle \mathrm{d}\Phi, g_\star^{-1}\rangle_\star, \mathrm{d}\Phi\rangle_\star + M^2\,\Phi\star\Phi\big)\star \mathrm{vol}_\star$$

- use local basis: $\langle \partial_\mu, \widetilde{\mathrm{d}x}^\nu\rangle_\star = \delta_\mu^\nu$

$$\rightarrow \quad g_\star^{-1} = \partial_\mu^* \otimes_\star g^{\mu\nu}\star\partial_\nu \quad , \qquad \mathrm{d}\Phi = \widetilde{\mathrm{d}x}^\mu \star \partial_{\star\mu}\Phi$$

$$L_\Phi = -\frac{1}{2}\big((\partial_{\star\mu}\Phi)^*\star g^{\mu\nu}\star\partial_{\star\nu}\Phi + M^2\,\Phi\star\Phi\big)\star\mathrm{vol}_\star$$

- deformed equation of motion (top-form valued):

$$0 = \frac{1}{2}\Big(\square_\star[\Phi]\star\mathrm{vol}_\star + \mathrm{vol}_\star\star\big(\square_\star[\Phi^*]\big)^* - M^2\,\Phi\star\mathrm{vol}_\star - M^2\,\mathrm{vol}_\star\star\Phi\Big)$$

$$=: P_\star[\Phi]\star\mathrm{vol}_\star$$

NB: $P_\star$ is formally self adjoint w.r.t. SP $\big(\varphi,\psi\big)_\star = \int \varphi^*\star\psi\star\mathrm{vol}_\star$, i.e.

$$\big(\varphi, P_\star[\psi]\big)_\star = \big(P_\star[\varphi],\psi\big)_\star$$

Reminder: Algebraic approach to commutative QFT

Commutative free scalar QFT in one slide: [Wald; Bär, Ginoux, Pfäffle; ...]

- Let $(\mathcal{M}, g)$ be time-oriented, connected and globally hyperbolic
- start with Klein Gordon operator $P = \Box - M^2$

$\Rightarrow$ unique retarded and advanced Green's operators $\Delta_\pm$

$\Rightarrow$ fundamental solution $\Delta = \Delta_+ - \Delta_-$ (i.e. $\Delta[C_0^\infty(\mathcal{M})] = \mathrm{Sol}_P := \mathrm{Ker}(P)$)

$\Rightarrow$ symplectic vector space (V, ω) with

$$V = C_0^\infty(\mathcal{M}, \mathbb{R})/\mathrm{Ker}(\Delta) \quad \text{and} \quad \omega([\varphi], [\psi]) = \big(\varphi, \Delta(\psi)\big) = \int \varphi\, \Delta[\psi]\, \mathrm{vol}$$

$\Rightarrow$ Weyl algebra is generated by $W(\varphi)$, $\varphi \in V$, such that

(i) $W(0) = 1$

(ii) $W(-\varphi) = W(\varphi)^*$

(iii) $W(\varphi) \cdot W(\psi) = e^{-i\omega(\varphi,\psi)/2}\, W(\varphi + \psi)$

NB: $\omega(\varphi, \psi)$ is called "commutator function" in physics literature, since $[\Phi(\varphi), \Phi(\psi)] = i\,\omega(\varphi, \psi)\, 1$

QFT on NC curved spacetimes

- $(\mathcal{M}, g_\star, \star)\big|_{\lambda=0}$ time-oriented, connected, globally hyperbolic
- let $P_\star = \sum_{n=0}^{\infty} \lambda^n P_{(n)}$ be a deformed Klein-Gordon operator
- technical assumption: $P_{(n)} : C^\infty(\mathcal{M}) \to C^\infty_0(\mathcal{M})$ for all $n > 0$
  - fulfilled for all twists of compact support
  - or $g_\star$ asymptotically (outside compact region) symmetric under $\mathcal{F}$
- based on strong results for the commutative case we find:

there exist unique Green's operators $\Delta_{\star\pm} := \sum_{n=0}^{\infty} \lambda^n \Delta_{(n)\pm}$ satisfying

(i) $P_\star \circ \Delta_{\star\pm} = \mathrm{id}_{C^\infty_0(\mathcal{M})[[\lambda]]}$,

(ii) $\Delta_{\star\pm} \circ P_\star\big|_{C^\infty_0(\mathcal{M})[[\lambda]]} = \mathrm{id}_{C^\infty_0(\mathcal{M})[[\lambda]]}$,

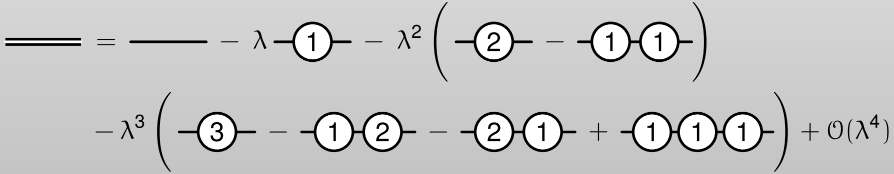
(iii) $\mathrm{supp}\big(\Delta_{(n)\pm}(\varphi)\big) \subseteq J^\pm\big(\mathrm{supp}(\varphi)\big)$, for all $n \in \mathbb{N}^0$ and $\varphi \in C^\infty_0(\mathcal{M})$,

where $J^\pm$ is the causal future/past with respect to the metric $g_\star|_{\lambda=0}$.

Explicit formula for $\Delta_{\star\pm}$ in terms of $\Delta_{\pm} := \Delta_{(0)\pm}$:

$$
\begin{aligned}
\Delta_{\star\pm} = \Delta_{\pm} & \\
& - \lambda\, \Delta_{\pm} \circ P_{(1)} \circ \Delta_{\pm} \\
& - \lambda^2 \left(\Delta_{\pm} \circ P_{(2)} \circ \Delta_{\pm} - \Delta_{\pm} \circ P_{(1)} \circ \Delta_{\pm} \circ P_{(1)} \circ \Delta_{\pm} \right) \\
& + \mathcal{O}(\lambda^3) \quad \text{[higher orders follow the same structure]}
\end{aligned}
$$

Graphically:



$\rightarrow$ perturbative approach to deformed Green's operators

- define fundamental solution $\Delta_\star := \Delta_{\star+} - \Delta_{\star-}$
- we obtain $\Delta_\star[C_0^\infty(\mathcal{M})[[\lambda]]] = \mathrm{Sol}_{P_\star}^{\mathbb{C}} := \mathrm{Ker}(P_\star)$
- space of “physical sources”:

$$H := \left\{\varphi \in C_0^\infty(\mathcal{M})[[\lambda]] : \left(\Delta_{\star\pm}(\varphi)\right)^* = \Delta_{\star\pm}(\varphi)\right\}$$

NB: Let ψ be a real solution of the deformed wave equation, then there is a $\varphi \in H$, such that $\psi = \Delta_\star(\varphi)$.

Proposition (T. Ohl, AS)

$(V_\star, \omega_\star)$ *with* $V_\star := H/\mathit{Ker}(\Delta_\star)$ *and*

$$\omega_\star([\varphi],[\psi]) := \left(\varphi, \Delta_\star(\psi)\right)_\star = \int \varphi^* \star \Delta_\star(\psi) \star \mathrm{vol}_\star$$

is a symplectic vector space.

Proposition

There is a symplectic isomorphism $S : (V_\star, \omega_\star) \to (V[[\lambda]], \omega)$,
i.e. $\omega(S[\varphi], S[\psi]) = \omega_\star([\varphi],[\psi])$, *between the NC and com. field theory*$[[\lambda]]$.

- Consider unital $*$-algebras over $\mathbb{C}[[\lambda]]$ [math/0408217 (Waldmann), …]

Def: Let (V, ρ) be a symplectic vector space over $\mathbb{R}[[\lambda]]$. Then $\mathcal{A}_{(V,\rho)}$ is $*$-algebra of field polynomials if it is generated by $\Phi(\varphi)$, $\varphi \in V$, such that

$$\Phi(\alpha\,\varphi + \beta\,\psi) = \alpha\,\Phi(\varphi) + \beta\,\Phi(\psi)\ ,$$
$$\Phi(\varphi)^* = \Phi(\varphi)\ ,$$
$$\big[\Phi(\varphi), \Phi(\psi)\big] = i\,\rho(\varphi,\psi)\,1\ .$$

- $\mathcal{A}_{(V_\star,\omega_\star)} \hat{=}$ NC QFT , $\mathcal{A}_{(V[[\lambda]],\omega)} \hat{=}$ com. QFT$[[\lambda]]$

Corollary

The symplectic isomorphism $S : (V_\star, \omega_\star) \to (V[[\lambda]], \omega)$ induces via $\mathfrak{S}\big(\Phi_\star(\varphi)\big) := \Phi(S\varphi)$ a $*$-algebra isomorphism $\mathfrak{S} : \mathcal{A}_{(V_\star,\omega_\star)} \to \mathcal{A}_{(V[[\lambda]],\omega)}$ between NC and commutative QFT$[[\lambda]]$.

- Consequences:
  - Induction of (algebraic) states $\Omega_\star := \Omega \circ \mathfrak{S} : \mathcal{A}_{(V_\star,\omega_\star)} \to \mathbb{C}[[\lambda]]$
  - NC correlation functions from commutative correlation functions

$$\langle 0|\Phi_\star(\varphi_1)\cdots\Phi_\star(\varphi_n)|0\rangle = \langle 0|\Phi(S\varphi_1)\cdots\Phi(S\varphi_n)|0\rangle$$

An example of a convergent deformation

- spacetime: $\mathcal{M} = (0,\infty)\times S_1$ with $g = -dt\otimes dt + t^2 d\phi\otimes d\phi$
- deformation: $\mathcal{F}^{-1} = e^{i\lambda(t\partial_t\otimes\partial_\phi - \partial_\phi\otimes t\partial_t)} \quad\rightarrow\quad t\star e^{i\phi} = e^{-2\lambda}\, e^{i\phi}\star t$
- in Fourier space ($\phi\rightarrow k\in\mathbb{Z}$) we have

$$\widetilde{P}_\star = \cosh(3\lambda k)\circ\widetilde{\square}\,,\quad \widetilde{\Delta}_{\pm\star} = \widetilde{\Delta}_\pm\circ\cosh(3\lambda k)^{-1}$$

- symplectic VS $(V_\star,\omega_\star) = (C_0^\infty(\mathcal{M},\mathbb{R})/\mathrm{Ker}(\Delta_\star),\omega_\star)$ with

$$\omega_\star(\varphi,\psi) = -\int_0^\infty dt\, t\int_0^\infty d\tau\,\tau\,\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\widetilde{\varphi}(t,-k)\,\frac{\Delta(t,\tau,k)}{\cosh(3\lambda k)}\widetilde{\psi}(\tau,k)$$

- symplectic map $S:(V_\star,\omega_\star)\rightarrow(V,\omega)$ in Fourier space

$$(\widetilde{S}\widetilde{\varphi})(t,k) = \frac{\widetilde{\varphi}(t,k)}{\sqrt{\cosh(3\lambda k)}}\quad\Rightarrow\quad \omega(S\varphi,S\psi) = \omega_\star(\varphi,\psi)$$

- S is injective *but not* surjective (all $\widetilde{S}\,\widetilde{\varphi}$ fall of faster than $e^{-3\lambda|k|/2}$ for $|k|\gg 1$)
- Interpretation:

  QFT on NC CS is $*$-isomorphic to a **reduced** QFT on CS, where strongly localized observables are excluded.

- QFT on NC curved spacetimes [arXiv:0912.2252]:
  - formally self adjoint EOM operators $P_\star$
  - existence, uniqueness and construction of the deformed Green's operators
  - symplectic structure on the space of real solutions of $P_\star$
  - quantization via $\ast$-algebras of field polynomials
  - QFT on NC CS / QFT on CS$[[\lambda]]$-correspondence via $\ast$-algebra isomorphism

- Convergent deformation of a FRW toy-model [arXiv:1009.1090]:
  - construction of the deformed symplectic vector space
  - quantization in terms of Weyl algebras
  - $\exists$ $\ast$-isomorphism between QFT on NC CS and a reduced QFT on CS, where strongly localized observables are excluded
  - relation between NC and commutative field operators:

$$\widehat{\Phi}_\star(t,k) = \sqrt{\cosh(3\lambda k)^{-1}}\,\widehat{\Phi}(t,k) \overset{|k|\gg 1}{\approx} \sqrt{2}\,e^{-3\lambda|k|/2}\,\widehat{\Phi}(t,k)$$

$\Rightarrow$ the NC theory is strongly improved in the UV (i.e. $|k| \gg 1$)

Some details . . .

- **Remember:** $H := \left\{\varphi \in C_0^\infty(\mathcal{M})[[\lambda]] : \left(\Delta_{\star\pm}(\varphi)\right)^* = \Delta_{\star\pm}(\varphi)\right\}$
- **Hodge operators:**
  $\alpha, \alpha_\star : C^\infty(\mathcal{M})[[\lambda]] \to \Omega^d(\mathcal{M})[[\lambda]]$, $\alpha(\varphi) = \varphi\,\mathrm{vol}$; $\alpha_\star(\varphi) = \varphi \star \mathrm{vol}_\star$

  We have the following lemmas:

I: $\kappa := \alpha_\star^{-1} \circ \alpha : C_0^\infty(\mathcal{M}, \mathbb{R})[[\lambda]] \to H$ is a vector space isomorphism

II: $j_\pm : C_0^\infty(\mathcal{M}, \mathbb{R})[[\lambda]] \to C_0^\infty(\mathcal{M}, \mathbb{R})[[\lambda]]$ defined by $j_\pm := \kappa^{-1} \circ P_\star \circ \Delta_\pm$ is a VS isomorphism with $j_\pm\left(P[C_0^\infty(\mathcal{M}, \mathbb{R})[[\lambda]]]\right) = (\kappa^{-1} \circ P_\star)[C_0^\infty(\mathcal{M}, \mathbb{R})[[\lambda]]]$

III: $\hat{\omega}_\star := \omega_\star \circ \left(\kappa \circ j_\pm \otimes \kappa \circ j_\pm\right)$ is a symplectic structure on $V := C_0^\infty(\mathcal{M}, \mathbb{R})[[\lambda]]/P[C_0^\infty(\mathcal{M}, \mathbb{R})[[\lambda]]]$

IV: $\hat{\omega}_\star([\varphi], [\psi]) = \left(\varphi, \hat{\Delta}_\star(\psi)\right)$, where $\hat{\Delta}_\star = \left(\mathrm{id} - \Delta_\mp \circ P\right) \circ \Delta_\star \circ P_\star \circ \Delta_\pm$ satisfies $P \circ \hat{\Delta}_\star = \hat{\Delta}_\star \circ P = 0$ on $C_0^\infty(\mathcal{M}, \mathbb{R})[[\lambda]]$

V: $S = \sum \lambda^n S_{(n)} : V \to V$ given by $S_{(0)} = \mathrm{id}$ and
$S_{(n)} = \frac{1}{2}\left(\Delta^{-1} \circ \hat{\Delta}_{(n)} - \sum\limits_{m=1}^{n-1} S_{(m)} \circ S_{(n-m)}\right)$ satisfies
$\omega(S[\varphi], S[\psi]) = \hat{\omega}_\star([\varphi], [\psi])$ for all $[\varphi], [\psi] \in V$