Beyond linear theory: an analytical approach to cosmological perturbations

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We need to go beyond linear theory

- As we saw in the M. Pietroni talk, the crucial role played by the gravitational instability makes the **dark matter density field non-linear** on the relevant range of scales.
- The study of the LSS requires to go **beyond the linear theory**.
- **N-body simulations work** also in this mildly non linear regime but they need very large volumes and high resolutions, with the consequence that due to time limitations, **only few cosmologies have been investigated** so far.
- Perturbation theory improves on the linear one for $z > 2$, but fails at smaller redshifts.

$$\Downarrow$$

- New promising semi-analytical approaches: **Resummation methods**

Dark matter hydrodynamics

The dark matter hydrodynamics is described by the Euler-Poisson system.

$$\frac{\partial \delta}{\partial \tau} + \nabla \cdot [(1+\delta)\mathbf{v}] = 0\,, \qquad \frac{\partial \mathbf{v}}{\partial \tau} + \mathcal{H}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\phi\,,$$

$$\nabla^2\phi = \frac{3}{2}\,\Omega_m\,\mathcal{H}^2$$

where

$\delta = \delta\rho/\bar{\rho}$: density contrast
$\tau = a(\tau)/d\tau$: conformal time
$\mathcal{H} = d\log a/d\tau$: conformal expansion rate

$\mathbf{v}$: peculiar velocity
ϕ: peculiar gravitational potential

In Fourier-space

Defining the velocity divergence $\theta(\mathbf{x}, \tau) = \nabla \cdot \mathbf{v}(\mathbf{x}, \tau)$ one gets, in Fourier space

$$\frac{\partial \delta(\mathbf{k},\tau)}{\partial \tau} + \theta(\mathbf{k},\tau) + \int d^3\mathbf{q}\, d^3\mathbf{p}\, \delta_D(\mathbf{k}-\mathbf{q}-\mathbf{p})\alpha(\mathbf{q},\mathbf{p})\theta(\mathbf{q},\tau)\delta(\mathbf{p},\tau) = 0\,,$$

$$\frac{\partial \theta(\mathbf{k},\tau)}{\partial \tau} + \mathcal{H}\,\theta(\mathbf{k},\tau) + \frac{3}{2}\Omega_m \mathcal{H}^2 \delta(\mathbf{k},\tau) + \int d^3\mathbf{q}\, d^3\mathbf{p}\, \delta_D(\mathbf{k}-\mathbf{q}-\mathbf{p})\beta(\mathbf{q},\mathbf{p})\theta(\mathbf{q},\tau)\theta(\mathbf{p},\tau) = 0$$

The mode-mode coupling is given by the following functions:

$$\alpha(\mathbf{p},\mathbf{q}) = \frac{(\mathbf{p}+\mathbf{q})\cdot\mathbf{p}}{p^2}\,, \qquad \beta(\mathbf{p},\mathbf{q}) = \frac{(\mathbf{p}+\mathbf{q})^2\,\mathbf{p}\cdot\mathbf{q}}{2\,p^2 q^2}\,,$$

Linear Theory

- It consist in neglecting mode-mode coupling in fluid equations: $\alpha = \beta = 0$
- This approximation **holds at early times and/or on large scale**.
- We take $\Omega_m = 1 \rightarrow$ Einstein de Sitter Cosmology ($\mathcal{H} \sim a^{-1/2}$)

$$\frac{\partial\,\delta(\mathbf{k},\tau)}{\partial\,\tau} + \theta(\mathbf{k},\tau) = 0$$

$$\frac{\partial\,\theta(\mathbf{k},\tau)}{\partial\,\tau} + \mathcal{H}\,\theta(\mathbf{k},\tau) + \frac{3}{2}\Omega_m\mathcal{H}^2\delta(\mathbf{k},\tau) = 0$$

The solutions are:

$$\delta(\mathbf{k},\,\tau) = \delta_0(\mathbf{k},\,\tau_i)\left(\frac{a(\tau)}{a(\tau_i)}\right)^m$$

$$-\frac{\theta(\mathbf{k},\,\tau)}{\mathcal{H}} = m\,\delta(\mathbf{k},\,\tau)$$

Two possible values for m.
$m = 1$: growing mode
$m = -3/2$: decaying mode

- In our Quantum Field Theory language we will call this solutions the tree-level ones.

Compact form of equations of motion

Crocce & Scoccimarro, 2006

- Assuming EdS cosmology, **the hydrodynamical equations** for density and velocity perturbations **can be written in a compact form**
(repeated indices/momenta are summed/integrated over)

$$(\delta_{ab}\partial_\eta + \Omega_{ab})\,\varphi_b(\mathbf{k},\eta) = e^\eta \gamma_{abc}(\mathbf{k},\,-\mathbf{p},\,-\mathbf{q})\varphi_b(\mathbf{p},\eta)\,\varphi_c(\mathbf{q},\eta)$$

where

$$\begin{pmatrix} \varphi_1(\mathbf{k},\eta) \\ \varphi_2(\mathbf{k},\eta) \end{pmatrix} \equiv e^{-\eta} \begin{pmatrix} \delta(\mathbf{k},\eta) \\ -\theta(\mathbf{k},\eta)/\mathcal{H} \end{pmatrix} \qquad \eta = \log \frac{a}{a_{in}} \qquad \mathbf{\Omega} = \begin{pmatrix} 1 & -1 \\ -3/2 & 3/2 \end{pmatrix}$$

a is the scale factor and the only non zero components of the vertex are

$$\gamma_{121}(\mathbf{k},\,\mathbf{p},\,\mathbf{q}) = \gamma_{112}(\mathbf{k},\,\mathbf{q},\,\mathbf{p}) = \frac{1}{2}\,\delta_D(\mathbf{k}+\mathbf{p}+\mathbf{q})\,\alpha(\mathbf{p},\mathbf{q})\,,$$
$$\gamma_{222}(\mathbf{k},\,\mathbf{p},\,\mathbf{q}) = \delta_D(\mathbf{k}+\mathbf{p}+\mathbf{q})\,\beta(\mathbf{p},\mathbf{q})\,,$$

Without changing the equations structure there is a simple way to **extend this approach to non EdS cosmology**.

The linear propagator

- In order to solve the linear hydrodynamical equations (compact form) we have

$$(\delta_{ab}\partial_{\eta_a} + \Omega_{ab})\, g_{bc}(\eta_a, \eta_b) = \delta_{ac}\, \delta_D(\eta_a - \eta_b)\,,$$

so that $\varphi_a^0(\mathbf{k}, \eta_a) = g_{ab}(\eta_a, \eta_b)\varphi_b^0(\mathbf{k}, \eta_b)$ is the solution of the **linear** equation

- The linear retarded propagator (θ step function):

$$g_{ab}(\eta_a, \eta_b) = \left[\mathbf{B} + \mathbf{A}\, e^{-5/2(\eta_a - \eta_b)}\right]_{ab} \theta(\eta_a - \eta_b)$$

$$\mathbf{B} = \frac{1}{5}\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} \qquad \mathbf{A} = \frac{1}{5}\begin{pmatrix} 2 & -2 \\ -3 & 3 \end{pmatrix}$$

- The formal solutions of the **non linear equations of motion** are:

$$\varphi_a(\mathbf{k}, \eta_a, \eta_b) =$$
$$g_{ab}(\eta_a, \eta_b)\varphi_b^0(\mathbf{k}, \eta_b) + \int_{\eta_b}^{\eta_a} e^{\eta'} g_{ab}(\eta_a, \eta')\gamma_{bcd}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\varphi_c(\mathbf{p}, \eta')\, \varphi_d(\mathbf{q}, \eta')$$

that can be solved iteratively recovering the traditional perturbation theory.

QFT-like formulation and Feynman diagrams

- It can be shown that **hydrodynamical equations can be derived by varying a suitable action** with respect to the fields.

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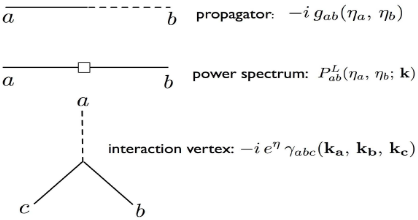
- Like in QFT it is possible to **define the generating functional, generator of n-point connected functions and so on...**

- Following the standard (Quantum Field Theory) procedure one can read out the Feynman Rules



Three fundamental building blocks

- Loop corrections take into account the effect of non-linearities, averaged over the statistics of the initial conditions (power spectrum only, if gaussian).
- All the known results in cosmological perturbation theory are expressible in terms of these Feynman diagrams.

The full propagator: exact time evolution equation

S.A., M. Pietroni and S. Matarrese, to appear

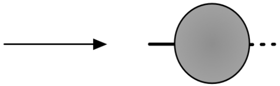
- **The propagator is the cross correlation** between the perturbation at a given time and length scale and the initial condition at the same scale.

- It is **not an observable quantity** but it is a fundamental ingredient for the computation of the power spectrum.

- **The evolution equations in times** (or scale) are useful tools to study this quantity. The full propagator obeys the **exact evolution equation**:

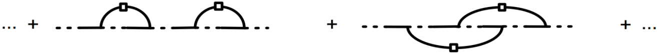
$$\partial_{\eta_a} G_{ab}(k\,,\eta_a,\eta_b) = -\Omega_{ac} G_{cb}(k\,,\eta_a,\eta_b) + \int_{\eta_b}^{\eta_a} ds\, \Sigma_{ac}(k\,,\eta_a\,,s)\, G_{cb}(k\,,s,\eta_b)$$

where Σ is the full two-point 1PI function:



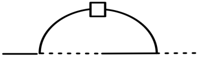
Propagator in the large-k limit - **linear PS** (1)

- We solve, in an approximate way, the propagator evolution equation.
- Take $z_{in} = 100$: $\Longrightarrow$ consider only the leading time dependence (growing mode).
- In the large-k limit **we resum all the infinite series of the dominant diagrams**



- Each of the $2n$ vertixes contributes a factor: $u_c\,\gamma_{acb}(\mathbf{k}, -\mathbf{q}, \mathbf{q}-\mathbf{k}) \xrightarrow{\textit{large } k} \frac{1}{2}\frac{\mathbf{k}\cdot\mathbf{q}}{q^2}\delta_{ab}$
  $\Rightarrow$ *Each loop integral decouples* and the *Diagrams* $\sim k^{2n}$.
- In the large-k limit we compute the RHS of the exact evol. eq. at n-loop order
  $\Longrightarrow$ we get the **FACTORIZATION PROPERTY**

$$\sum_{j=0}^{n-1}\int_0^\eta ds\Sigma_{ac}^{(n-j)}(k,\eta,s)G_{cb}^{(j)}(k,s,0) \xrightarrow{\textit{large } k} G_{ab}^{(n-1)}(k,\eta,0)\int_0^\eta ds\,\Sigma_{ac}^{(1)}(k,\eta,s)u_c$$

$\Sigma_{ab}^{(1)}$  ; $(u_c = \{1,1\})$

Propagator in the large-k limit - **linear PS** (2)

- Hence, summing over all the loop order, in the large-k limit we get

$$\partial_\eta \, G_{ab}(k\,,\eta,0) = -\Omega_{ac} G_{cb}(k\,,\eta,0) + G_{ab}(k\,,\eta,0) \int_0^\eta ds \, \Sigma_{ac}^{1-loop}(k\,,\eta\,,s) u_c$$

where $\int_0^\eta ds \, \Sigma_{ac}^{(1)}(k\,,\eta\,,s) u_c \xrightarrow{large\,k} -k^2 \sigma_v^2 e^{2\eta}$ and

$\sigma_v^2 \equiv \frac{1}{3} \int d^3q \, \frac{P^0(\mathbf{q})}{q^2}$ is the velocity dispersion.

- This result can be integrated to get

$$G_{ab}(k,\eta,0) \xrightarrow{large\,k} g_{ab}(\eta,0) e^{\frac{-k^2 \sigma_v^2 e^{2\eta}}{2}} \, .$$

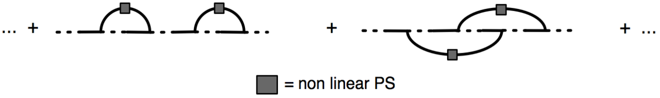
- This matches with the diagrammatic results found by (Crocce & Scoccimarro, 2006) obtained resumming the dominant diagrams about the propagator in the large-k limit
- With our approach, in a simple way, **we can go beyond this diagrammatic resummation**.

Propagator in the large-k limit - **non linear PS**

- In the previous computation we partially renormalized the vertex and the propagator but we did not renomalize the PS.

  $\Longrightarrow$ Now we take into account also the PS renormalization (in this way we avoid the double counting problem).

  $\Longrightarrow$ We consider the **dominant diagrams with the non linear PS** instead of the linear one.



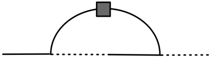
- Cause of this behavior: $A_c\,\gamma_{acb}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k}) \xrightarrow{large\ k} A_2\,\frac{1}{2}\,\frac{\mathbf{k}\cdot\mathbf{q}}{q^2}\,\delta_{ab}$
  $\Rightarrow$ *Each loop integral decouples*; the *Diagrams* $\sim k^{2n}$; only P_{22}^{NL} is selected.

- We have shown that the **FACTORIZATION PROPERTY STILL HOLDS**

Propagator in the large-k limit - **non linear PS**

$$\sum_{j=0}^{n-1}\int_0^\eta ds\tilde{\Sigma}_{ac}^{(n-j)}(k,\eta,s)G_{cb}^{(j)}(k,s,0)\stackrel{large\ k}{\longrightarrow} G_{ab}^{(n-1)}(k,\eta,0)\int_0^\eta ds\,\tilde{\Sigma}_{ac}^{(1)}(k,\eta,s)u_c$$

where now the up indexes count the number of the loops computed with the non linear PS (They don't count the loops due to the renormalization of PS^{NL}).

$\tilde{\Sigma}_{ab}^{(1)} \quad \longrightarrow$ 

- We get this **new evolution equation**

$$\partial_\eta\, G_{ab}(k\,,\eta,0) = -\Omega_{ac}\, G_{cb}(k\,,\eta,0) + G_{ab}(k\,,\eta,0)\int_0^\eta ds\,\tilde{\Sigma}_{ac}^{(1)}(k\,,\eta\,,s)u_c$$

where $\int_0^\eta ds\,\tilde{\Sigma}_{ac}^{(1)}(k\,,\eta\,,s)u_c \stackrel{large\ k}{\longrightarrow} -k^2\tilde{\sigma}_v^2(\eta)e^\eta$ and

$\tilde{\sigma}_v^2(\eta) \equiv \frac{1}{3}\int_0^\eta ds\int d^3q\,\frac{P_{22}^{NL}(\mathbf{q},\eta,s)}{q^2}$

Propagator in the small-k limit - **linear PS**

- In the small-k limit (large scale) high order contributions to the propagator are suppressed, and **linear perturbation theory is recovered.**

  $\Rightarrow$ We can compute the evolution equation at the 1-loop level.

- Modulo terms at least of two-loop order we get

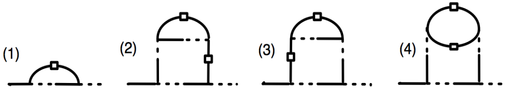
$$\partial_\eta \, G_{ab}(k\,,\eta,0) = -\Omega_{ac}\, G_{cb}(k\,,\eta,0) + G_{ab}(k\,,\eta,0) \int_0^\eta ds\, \Sigma^{(1)}_{ac}(k\,,\eta\,,s) u_c$$

$\Longrightarrow$ **In the large and small k limit we get the same evolution equation**

Propagator in the small-k limit - **non linear PS** (1)

- **In general, with the non linear PS, the small-k limit of $\tilde{\Sigma}_{ab}$ is not zero.**
  $\Longrightarrow$ It doesn't give the linear perturbation theory!!!
- FIRST EXAMPLE: If we consider $\mathrm{P}^{NL}(k, \eta, s)$=$\mathrm{P}^{1loop}(k, \eta, s)$ we get

$\tilde{\Sigma}_{ab}$



$$(1) + (2) + (3) \xrightarrow{small\ k} 0 \quad \text{but} \quad (4) \overset{small\ k}{\neq} 0$$

If we take into account also this 2-loop diagram (easiest way)

$\Sigma^{(V)}_{ab}$

we get $(1) + (2) + (3) + (4) + (V) \xrightarrow{small\ k} 0$.

We have also the right limit for large k

$$\tilde{\Sigma}^{P^{1loop}}_{ab} = \tilde{\Sigma}_{ab} + \Sigma^{(V)}_{ab}$$

where the $\Sigma^{(V)}_{ab}$ contribution to $\tilde{\Sigma}^{P^{1loop}}_{ab}$ is subdominant.

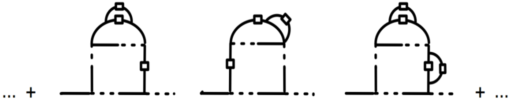
Propagator in the small-k limit - **non linear PS** (2)

- SECOND EXAMPLE

Resummed PS: $P^{NL}(k, \eta, s) = g(\eta, s)P^{TRG}(k, s, s)$ (M. Pietroni, 2008).

Now we are resumming diagrams like these.

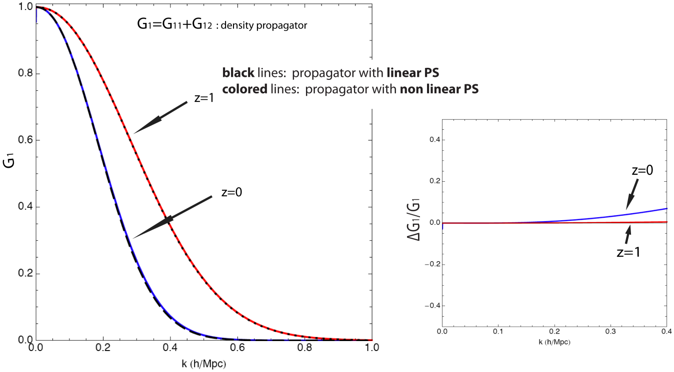
$\tilde{\Sigma}_{ab}$



With this non linear PS we get automatically the right small-k limit.

Numerical Results: density propagator

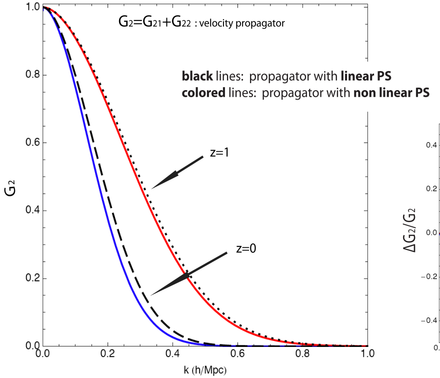
To solve (numerically) the equations we take the P^{TRG} non linear power spectrum computed in (M. Pietroni, 2008).

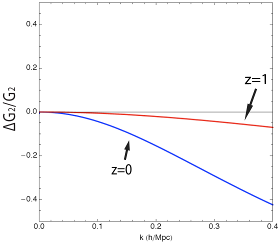


⇒ The density propagator **slightly feel** the effects of the non linear power spectrum.

Numerical Results: velocity propagator

- The effects of the non linear power spectrum **are much important** when we compute the velocity propagator.





Short term goal: the power spectrum

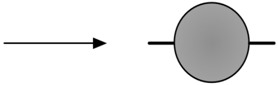
The full power spectrum equation have this structure

$$P_{ab} = P^{I}_{ab} + P^{II}_{ab}$$

where

$$P^{I}_{ab}(k;\eta_a,\eta_b) = G_{ac}(k;\eta_a,0)G_{bd}(k;\eta_b,0)P^{0}_{cd}(k)$$
$$P^{II}_{ab}(k;\eta_a,\eta_b) = \int_0^{\eta_a} ds_1 \int_0^{\eta_b} ds_2\; G_{ac}(k;\eta_a,s_1)G_{bd}(k;\eta_b,s_2)\Phi_{cd}(k;s_1,s_2)$$

where Φ is the full two-point 1PI function:



- In the power spectrum computation the propagator is a fundamental key ingredient.
- The idea is to **couple the time evolution equation** of the power spectrum with the time evolution equation of the propagator.

Conclusions

- Very important **quantifying departures from linear theory** to compare cosmological models with future galaxy surveys.
  The $0 < z < 1$ range is crucial for DE studies. (Dark Energy dominates)

- **Quantum Field Theory techniques** apply to cosmology are very **useful tools to study the (mild) non linear regime** of the hydrodynamical equations.

- The **time evolution equations** can be derived for the **correlation functions**.

- A simple approximation scheme allows us to **compute the propagator taking into account also the non linear power spectrum**.

- **Numerical results about the propagator**: we have shown that the effects of the further class diagrams resummed are more relevant for the velocity propagator and less relevant for the density propagator.

- The immediate line of development is: **computation of the non linear power spectrum considering the results for the non linear propagator**.