# Polyakov action from the fRG

A. Codello

Institute of Physics
University of Mainz

September 14, 2010

# Outline

# Motivations

- Study quantum gravity in $d = 2$ using the fRG
- Learn how to treat non-local terms in the fRG framework
- A simple situation where we can carry over the fRG "recipe"

# Minimally coupled scalar field

- Scalar field $\phi$ on an arbitrary $2d$ Riemannian manifold $(\mathcal{M}, g)$
- The scalar interacts with the background metric $g_{\mu\nu}$:

$$S[\phi, g] = \frac{1}{2}\int d^2x\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \frac{1}{2}\int d^2x\sqrt{g}\phi\Delta\phi$$

- Laplace-Beltrami operator:

$$\Delta\phi = -\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right)$$

# Effective average action

- Introduce a cutoff action

$$\Delta S_k[\phi, g] = \frac{1}{2} \int d^d x \sqrt{g} \phi\, R_k(\Delta)\, \phi$$

and define:

$$e^{W_k[J,g]} = \int D_g \phi \, \exp\left( -S[\phi, g] - \Delta S_k[\phi, g] + \int d^d x \sqrt{g} J \phi \right)$$

- The cutoff kernel $R_k$ is constructed using $\Delta$: the flow will be covariant

# Effective average action

- Define the effective average action:

$$\Gamma_k[\varphi,g]+\Delta S_k[\varphi,g]=\int d^dx\sqrt{g}J(\varphi)\varphi-W_k[J(\varphi),g]$$

- fRG quantization:

$$\lim_{k\to 0}\Gamma_k[\varphi,g]=\Gamma[\varphi,g] \qquad \lim_{k\to\Lambda}\Gamma_k[\varphi,g]=S[\varphi,g]$$

# Flow equation

- One-loop flow is exact in the simple case we are considering:

$$\partial_t \Gamma_k[\varphi, g] = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2 S[\varphi, g]}{\delta\varphi\delta\varphi} + R_k[g]\right)^{-1} \partial_t R_k[g]$$

- Since $S^{(2,0)}[0,g] = \Delta$ we have:

$$\partial_t \Gamma_k[0, g] = \frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k(\Delta)}{\Delta + R_k(\Delta)}$$

- Functional trace of a function of the Laplace-Beltrami operator: evaluate using the non-local heat kernel expansion

# Truncation

- Truncation ansatz for the effective average action:

$$\Gamma_k[0,g] = \int d^2x \sqrt{g}\,(a_k + b_k R + R\, c_k(\Delta) R) + O(R^3)$$

- The scalar interaction are not generated along the flow (in this simple case only!)
- Exact expansion to order $R^2$ involving two couplings $a_k$, $b_k$ and a running structure function $c_k(\Delta)$

# Evaluation of the trace

- Curvature expansion of the functional trace:

$$\begin{aligned}\partial_t\Gamma_k[0,g] \;=\;& \frac{1}{8\pi}Q_1[h_k]\int d^2x\sqrt{g}+\frac{1}{48\pi}Q_0[h_k]\int d^2x\sqrt{g}R \\ &+\frac{1}{8\pi}\int d^2x\sqrt{g}R\left[\int_0^\infty ds\,\tilde{h}_k(s)\,s\,f_{R2d}(s\Delta)\right]R \\ &+O(R^3)\end{aligned}$$

- Non-local heat kernel structure function:

$$f_{R2d}(x)=\frac{1}{32}f(x)+\frac{1}{8x}f(x)-\frac{1}{16x}+\frac{3}{8x^2}f(x)-\frac{3}{8x^2}$$

$$f(x)=\int_0^1 d\xi\,e^{-x\xi(1-\xi)}$$

- $\tilde{h}(s)$ is the inverse Laplace transform of $h_k(z)=\frac{\partial_t R_k(z)}{z+R_k(z)}$

# Beta functions of $a_k$ and $b_k$

- Beta function for the couplings:

$$\partial_t a_k = \frac{1}{8\pi} Q_1[h_k] \qquad \partial_t b_k = \frac{1}{48\pi} Q_0[h_k]$$

- Using the optimized cutoff:

$$\partial_t a_k = \frac{k^2}{4\pi} \qquad \partial_t b_k = \frac{1}{24\pi}$$

# Flow of $c_k(x)$

- Flow equation for the running structure function:

$$
\begin{aligned}
8\pi\,\partial_t c_k(x) \;=\;& \frac{1}{32}\int_0^1 d\xi\, Q_{-1}\left[h_k(z+x\xi(1-\xi))\right] \\
&+\frac{1}{8x}\int_0^1 d\xi\, Q_0\left[h_k(z+x\xi(1-\xi))\right]-\frac{1}{16x}Q_0[h_k] \\
&+\frac{3}{8x^2}\int_0^1 d\xi\, Q_1\left[h_k(z+x\xi(1-\xi))\right]-\frac{3}{8x^2}Q_1[h_k]
\end{aligned}
$$

- $x$ stands for $\Delta$

# Flow of $c_k(x)$

- The flow can be written as:

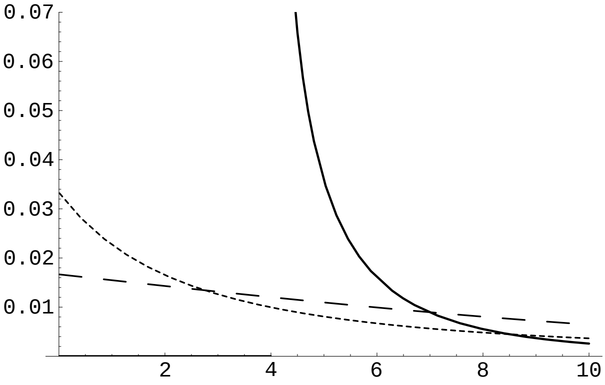
$$\partial_t c_k(x) = \frac{1}{8\pi k^2} f\left(\frac{x}{k^2}\right)$$

- The function $f(u)$, $u = x/k^2$, depends explicitly on the cutoff shape function used:

$$\begin{aligned} f_{opt}(u) &= \frac{1}{8u}\left[\sqrt{\frac{u}{u-4}} - \frac{u+4}{u}\sqrt{\frac{u-4}{u}}\right]\theta(u-4) \\ f_{mass}(u) &= \frac{\sqrt{u(u+4)}(u+6) + 8(u+3)\,\mathrm{artanh}\sqrt{\frac{u}{u+4}}}{(u+4)^{3/2}u^{5/2}} \end{aligned}$$

$f_{exp}(u)$ is found numerically

# Flow of $c_k(x)$




$f(u)$ evaluated using the exponential cutoff (long dashed), the mass cutoff (short dashed) and the optimized cutoff (thick)

# Flow of $c_k(x)$

- All three functions are analytic around the origin and $f_{opt}(u)$ is zero in the entire interval $[0,4)$
- If we expand $f(u)$ as a power series in $u$ about $u=0$, it follows that we have a non-zero running of local terms of the form $c_k^{(n)} \int \sqrt{g} R \Delta^n R$ only for the exponential and the mass cutoffs
- For example, we can expand for small $u$

$$f_{mass}(u) = \frac{1}{30} - \frac{u}{70} + \frac{u^2}{210} + O(u^3)\,,$$

and read off the resulting beta functions for the couplings $c_k^{(n)}$ in the mass cutoff case

# Flow of $c_k(x)$

- For the optimized cutoff none of the couplings $c_k^{(n)}$ has a non-zero beta function
- The running of the couplings $c_k^{(-n)}$, $n > 0$ (which multiply non-local terms involving inverse powers of $\Delta$) is zero for all three cutoff choices. In particular, the beta function of the coupling $c_k^{(-1)}$ pertaining to the operator $\int \sqrt{g} R \frac{1}{\Delta} R$ is zero, even if this is the form the EAA is expected to reach at $k = 0$!
- To capture the non-local features of the EAA we need to consider the running of the whole structure function $c_k(x)$

## Integrating the flow

- Integrate the flow equations from the UV scale $\Lambda$ to the IR scale $k \to 0$
- Impose initial conditions (renormalize) on the flow of the couplings:

$$
\begin{aligned}
a_k &= a_\Lambda - \frac{1}{4\pi}(\Lambda^2 - k^2) \\
b_k &= b_\Lambda - \frac{1}{24\pi}\log\frac{\Lambda}{k}
\end{aligned}
$$

- Set $a_\Lambda = \frac{\Lambda^2}{4\pi}$ so that the renormalized $a_0$ vanishes and conformal invariance is preserved
- Introduce the arbitrary scale $k_0$ and set $b_\Lambda = \frac{1}{24\pi}\log\frac{\Lambda}{k_0}$

# Integrating the flow

- Integrate the flow equation for the structure function:

$$c_k(x) = c_\Lambda(x) - \frac{1}{16\pi x}\int_{x/\Lambda^2}^{x/k^2} du\, f(u)$$

- We can take the limit $\Lambda \to \infty$
- Set the initial condition $c_\infty(x) = 0$

# Effective average action

- Complete effective average action to order $R^2$:

$$\Gamma_k[0,g] = \frac{k^2}{4\pi}\int d^2x\sqrt{g} + \frac{\chi}{12}\log\frac{k}{k_0}$$

$$-\frac{1}{96\pi}\int d^2x\sqrt{g}R\left[\frac{\sqrt{\Delta/k^2-4}(\Delta/k^2+2)}{\Delta\,(\Delta/k^2)^{3/2}}\theta(\Delta/k^2-4)\right]R+O(R^3)$$

- $\chi = \frac{1}{2\pi}\int d^2x\sqrt{g}R$ is the Euler characteristic of the manifold $\mathscr{M}$
- Safe $k\to 0$ limit on the torus $\chi=0$. In the spherical case, $\chi=2$, or in all higher genus topologies, the limit $k\to 0$ can be taken only if we also send $k_0\to 0$ in such a way that $\frac{k}{k_0}$ remains constant

# Effective average action

- For $k \to 0$ we find (for all cutoff shapes):

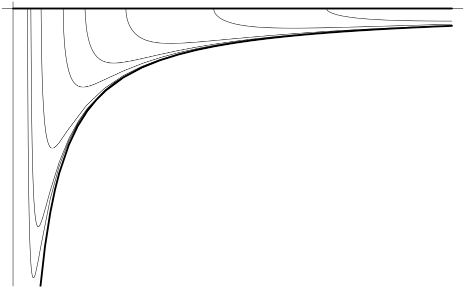
$$c_0(x) = -\frac{1}{96\pi x}$$

- We recover Polyakov's effective action:

$$\Gamma_0[0,g] = -\frac{1}{96\pi}\int d^2x\sqrt{g}\,R\frac{1}{\Delta}R$$

- In principle we still have to show that all higher terms in the truncation vanish at $k=0$

# Effective average action



Flow of the structure function $c_k(x)$ from $c_\infty(x) = 0$ to $c_0(x) = -\frac{1}{96\pi x}$ for different values of the IR cutoff in the range $\infty \geq k \geq 0$

- Convergence of the effective average action to the effective action is non-uniform: $c_k(x) \sim c_0(x)$ for $x < 4k^2$

# Summary

- We explained how the Polyakov effective action for a minimally coupled scalar field on a curved two dimensional manifold emerges within the functional RG approach
- We calculated the RG flow of the structure function $c_k(\Delta)$ using the non-local heat kernel expansion

## Summary

- We learned that in order to be able to recover, at the IR scale, special non-local terms in the EAA, $\int \sqrt{g} R \frac{1}{\Delta} R$ in our example, it is necessary to include the running of the complete structure function which allows for an arbitrary dependence on $\Delta$
- We also saw that, quite remarkably, individual non-local terms in a Laurent series expansion, $\int \sqrt{g} R \Delta^{-n} R$, $n > 0$, have no RG running, even though the $k \to 0$ limit of the EAA is precisely of this type
- We learned that convergence of the effective average action to the effective action is non-uniform in general

## Summary

- First step in view of applications of this framework to quantized gravity in $d = 2$ and in $d = 4$
- Along the same line the low energy effective action for quantum gravity in $d = 4$ can be recovered
- To know more: A.C. arXiv:1004.2171 and A. Satz, A.C., F.D. Mazzitelli arXiv:1006.3808