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- σ-models in low dimensions appear to be
  - low-dimensional analogues of Yang-Mills theories.
  - Arise as approximate models in the contexts of both particle physics and solid state physics.
  - Used in the construction of models in high-T_c superconductivity and of the quantum Hall effect.
  - Simple examples of harmonic maps studied in differential geometry.
  - Integrable in $2+0$-dimensions, only very special cases are integrable in $2+1$-dimensions.
- Point of particle physics and relativity, the most interesting models are those which are Lorentz invariant; but all Lorentz invariant σ-models in $2+1$-dimensions are non-integrable, ie. it is natural to consider numerical simulations of time evolution.
- Baby Skyrme Model is a modified σ-model that includes a Skyrme term , which is quartic in derivatives, in order to yield stable soliton (localized, finite-energy solution of a non-linear field theory) solutions.
- Baby Skyrme Model a dimensional reduction of the Skyrme model: which is a non-linear field theory for pions in $3+1$ dimensions which soliton solutions called Skyrmions.
- Quantised Skyrmions are models for physical baryons.

Baby Skyrme Model

- Lagrangian density of the model is of the form

$$\mathcal{L} = \frac{1}{2}\,\partial_\alpha \boldsymbol{\phi}\,\partial^\alpha \boldsymbol{\phi} - \frac{\kappa^2}{4}\,(\partial_\alpha \boldsymbol{\phi} \times \partial_\beta \boldsymbol{\phi})(\partial^\alpha \boldsymbol{\phi} \times \partial^\beta \boldsymbol{\phi}) - V(\boldsymbol{\phi}).$$

- The field $\boldsymbol{\phi}$ is a map from $M^3 \to S^2$, where M^3 is the 3-dimensional Minkowski space with metric $\eta^{\alpha\beta} = \mathrm{diag}(1, -1, -1)$ and the target space S^2 is the 2-sphere of unit radius embedded in Euclidean 3-space.
- The baby Skyrme field $\boldsymbol{\phi}$ is a scalar 3-vector with norm one, ie. $|\boldsymbol{\phi}|^2 = 1$.
- The constant κ is a free parameter which have the dimension of length.
- First term in the Lagrangian is the familiar $O(3)$ sigma model, possesses metastable states which when perturbed may shrink or spread out due to conformal (scale) invariance of the model. Effective description of quantum Hall effect the kinetic part corresponds to spin stiffness term.
- Second term is the 2-dimensional analogue of the Skyrme term which breaks the scale invariance. Analogous to Coulomb term.
- The resulting energy functional has no minima.
- Last term is the potential (or mass term) is needed to stabilize the size of the corresponding solitons (so-called baby Skyrmions). One choice is $V = 1 - (\boldsymbol{n} \cdot \boldsymbol{\phi})^2$. Analogous to Zeeman interaction.

Topological Properties

- Finiteness of the energy requires the potential term to vanish at infinity, implying that

$$\lim_{r\to\infty} \boldsymbol{\phi}(t,x,y) = \boldsymbol{n}, \qquad r = \sqrt{x^2+y^2}$$

For simplicity we choose $\boldsymbol{n}$ to be the vacuum state, that is $\boldsymbol{n} = (0,0,1)$.

- Thus a topological number exists since the field $\boldsymbol{\phi}$, due to the boundary conditions, can be considered as a map from $S^2 \longrightarrow S^2$.

- Topological charge is the homotopy invariant of the field

$$\deg[\boldsymbol{\phi}] = \frac{1}{4\pi}\int \boldsymbol{\phi}\cdot(\partial_x\boldsymbol{\phi}\times\partial_y\boldsymbol{\phi})\,dx\,dy.$$

and thus, conserved.

Noncommutative Deformation

- Both the Skyrme and Baby Skyrme model are perturbatively non-renormalizable field theories.
- No easy to access the quantum fluctuations. Semiclassical treatments are needed.
- Full quantization requires a cutoff which can be attained by its lattice version.
- Noncommutative deformation introduces a regulating parameter into the quantum theory.
  - Noncommutative version may improve its renormalizability properties at short distances.
  - It may even render it finite.
- Moyal-deformed fields have a much richer soliton spectrum than their commutative counterparts.
- The noncommutative deformation replaces the potential term (or Zeeman interactions), since it introduces new new length scale into the theory, which also stabilizes solitons against collapse or spreading.
- We study the noncommutative baby skyrme model without the potential term, obtain explicitly class of exact analytic solitonic solutions with no analogues in the commutative theory.
- We compute static energy, discuss their stability and evaluate the two-Skyrmion interaction potential at large distances.

Grassmannian-valued Baby Skyrme Model without Potential Term

- Equation of motion

$$\partial^\mu \left(g^\dagger \partial_\mu g \ + \kappa^2 \left[g^\dagger \partial^\nu g \, , \, [g^\dagger \partial_\mu g \, , \, g^\dagger \partial_\nu g] \right] \right) = 0$$

where $g = g(t, x, y)$ a map from $\mathbb{R}^3 \to U(n)$ or $Gr(2,1)$.

- Energy for static $(g_t = 0)$ configurations

$$E \ = \ \int \mathrm{d}^2 x \, \left\{ \partial_i g^\dagger \, \partial_i g \ - \ \frac{\kappa^2}{4} [g^\dagger \partial_i g \, , \, g^\dagger \partial_j g] \, [g^\dagger \partial_i g \, , \, g^\dagger \partial_j g] \right\}$$

where the Granssmannian group $Gr(n,k) = \frac{U(C^n)}{U(\mathrm{imP}) \times \mathrm{U}(\ker \mathrm{P})}$ for a projector P of rank k.

- Look for extrema of E that lie inside some Grassmannian, hence

$$g^2 = \mathbb{1}_n \ \Leftrightarrow \ g^\dagger = g \ \Leftrightarrow \ g = \mathbb{1}_n - 2P$$

where $P^\dagger = P = P^2$.

- Then putting $\delta E = 0$ the equation of motion simplifies to

$$[\, P_{ii} \, , \, P \,] \ + \ 4\kappa^2 \, F[P] \ = \ 0$$

where

$$F[P] \ = \ 2P_{ij}[P_i, P]P_j \ + \ P_j[P_{ii}, P]P_j \ - \ \partial_i(P_j P_j)[P_i, P] \ - \ P_j P_j [P_{ii}, P] \ - \ \mathrm{h.c.}$$

Harmonic Maps

- Complex coordinates

$$z = x + \mathrm{i}y, \quad \bar{z} = x - \mathrm{i}y \Longrightarrow \partial_x = \partial_z + \partial_{\bar{z}}, \quad -\mathrm{i}\partial_y = \partial_z - \partial_{\bar{z}}$$

- At $\kappa = 0$ we connect with the chiral sigma model. Its energy is extremized by BPS projectors

$$0 = P_{\bar{z}}\, P \Longleftrightarrow 0 = P\, P_z$$

  Bogomolny equation (first order). Harmonic Map Ansatz: Monopoles, Skyrmions, Instanton, Models obtained as dimensional or algebraic reduction of above.

- This implies various identities, such as $[P_z, P_{\bar{z}}] = P_{z\bar{z}}$ and satisfy the non-Bogomolny equation (second order)

$$0 = [P_{z\bar{z}}, P].$$

- Turn on κ and compute the failure of the BPS projectors to extremize the baby Skyrme energy:

$$\frac{1}{8}F = P_{\bar{z}}\, P_{zz}\, P_{\bar{z}} - P_z\, P_{\bar{z}\bar{z}}\, P_z$$

One Example: 1-Baby Skyrmion

- Evaluate this expression in the CP^1 model for the (rank-one) BPS projectors, which are based on holomorphic functions f

$$P \;=\; \frac{1}{1+|f|^2}\begin{pmatrix} 1 & \bar f \\ f & f\bar f \end{pmatrix} \quad\Longleftrightarrow\quad 0 \;=\; [P_{z\bar z}, P].$$

- Then the baby Skyrme term is equal to

$$\frac{1}{8}F = \frac{1}{(1+|f|^2)^4}\begin{pmatrix} \bar f f'^2 \bar f'' - f \bar f'^2 f'' & \bar f^2 f'^2 \bar f'' + \bar f'^2 f'' - 2\bar f f'^2 \bar f'^2 \\ -f^2 \bar f'^2 f'' - f'^2 \bar f'' + 2 f f'^2 \bar f'^2 & f \bar f'^2 f'' - \bar f f'^2 \bar f'' \end{pmatrix}.$$

  This vanishes only for constant f.

- In the simplest case, $f = z$, one finds

$$\tfrac{1}{8}F = \frac{2}{(1+|z|^2)^4}\begin{pmatrix} 0 & -\bar z \\ z & 0 \end{pmatrix}.$$

- Thus, the sigma-model BPS solitons never obey the baby Skyrme equation of motion.

Moyal Deformation

- A Moyal deformation of Euclidean $\mathbb{R}^2$ space with coordinates (x, y) is achieved by replacing the ordinary pointwise product of smooth functions on it with the noncommutative but associative Moyal star product.

- Introduce a constant positive real parameter θ which appears in the star commutation relation between the coordinates,

$$x \star y - y \star x \equiv [x\,,\,y]_\star = \mathrm{i}\,\theta \qquad \Longrightarrow \qquad [z\,,\,\bar{z}]_\star = 2\theta\ .$$

- It is convenient to work with the dimensionless coordinates

$$a = \frac{z}{\sqrt{2\theta}} \quad \text{and} \quad a^\dagger = \frac{\bar{z}}{\sqrt{2\theta}} \qquad \Longrightarrow \qquad [a\,,\,a^\dagger]_\star = 1\ .$$

- A different realization of this Heisenberg algebra promotes the coordinates (and thus all their functions) to noncommuting operators acting on an auxiliary Fock space $\mathcal{H}$ but keeps the ordinary operator product.

Fock Space

- The Fock space is a Hilbert space with orthonormal basis states

$$|m\rangle \;=\; \frac{1}{\sqrt{m!}}\,(a^\dagger)^m\,|0\rangle \qquad \text{for} \quad m\in\mathbb{N}_0 \quad \text{and} \quad a|0\rangle = 0$$

$$a\,|m\rangle \;=\; \sqrt{m}\,|m{-}1\rangle\ , \qquad a^\dagger\,|m\rangle \;=\; \sqrt{m{+}1}\,|m{+}1\rangle\ ,$$

$$N\,|m\rangle = a^\dagger a\,|m\rangle \;=\; m\,|m\rangle\ ,$$

  therewith characterizing a and $a^\dagger$ as standard annihilation and creation operators.

- The star-product and operator formulations are tightly connected through the Moyal-Weyl map: Coordinate derivatives correspond to commutators with coordinate operators,

$$\sqrt{2\theta}\,\partial_z \;\leftrightarrow\; -\mathrm{ad}(a^\dagger) = [\,\cdot\,, a^\dagger] \quad , \qquad \sqrt{2\theta}\,\partial_{\bar z} \;\leftrightarrow\; \mathrm{ad}(a) = [a,\,\cdot\,] \quad ,$$

  and the integral over the noncommutative plane reads

$$\int \mathrm{d}^2x\ f_\star(x) \;=\; 2\pi\,\theta\ \mathrm{Tr}_{\mathcal{H}}\, f_{\mathrm{op}}\ ,$$

  where the function $f_\star$ corresponds to the operator f_{op} via the Moyal-Weyl map and the trace is over the Fock space $\mathcal{H}$.

- The time coordinate t of the full baby Skyrme model remains commutative.

Abelian Model

- The noncommutative target space is much bigger than the original one, new possibilities for BPS projectors arise.
- Two types of classical solutions to the deformed theory
  - Non-abelian solutions are continuously (in θ) connected to their commutative counterparts (tensored with $\mathbb{1}_{\mathcal{H}}$) and represent smooth deformations of it.
  - Abelian solutions become singular at $\theta \to 0$ and are genuinely noncommutative.
- Since novel features can be expected only in the abelian case, we focus on it from now on and choose $n=1$, i.e. the Moyal-deformed U(1) baby Skyrme model.
- The Moyal deformation introduces the dimensional parameter θ into the theory, which invalidates Derrick's argument.
- Scaling of spatial coordinates now relates theories with different strengths of noncommutativity. Therefore, classical solutions at a fixed value of θ are safe against shrinking or spreading.

Exact Noncommutative Baby Skyrmions

- The ordinary equations carry over to the deformed abelian baby Skyrme model (with replacing $\mathbb{1}_n$ by $\mathbb{1}_{\mathcal{H}}$), since on a formal level its noncommutativity resembles the non-abelianness in the standard U(n) model.
- The failure of a standard noncommutative U(1) sigma-model BPS solution, $g = \mathbb{1} - 2P$, is again measured by $\frac{1}{8}F$.
- Moyal-deformed theory, this expression may vanish, and surprisingly does so if the projector is a function of the number operator $N = a^\dagger a$ only!
- In the star-product picture, this corresponds to functions only of the radial variable $r = \sqrt{z\bar{z}}$, and so they are called radial projectors.
- $F[P]$ vanishes for $P = P(r)$, but in the commutative theory only trivial projectors can be radial. In the Fock-space basis, radial projectors are simply diagonal.
- The BPS projector

$$P^{(k)} = \sum_{n=0}^{k-1} |n\rangle\langle n| \qquad \text{obeys} \qquad P^{(k)}_{\bar{z}} P^{(k)}_{zz} P^{(k)}_{\bar{z}} = 0 = P^{(k)}_{z} P^{(k)}_{\bar{z}\bar{z}} P^{(k)}_{z}$$

  as well as $[P^{(k)}_{z\bar{z}}, P^{(k)}] = 0$.
- $F[P^{(k)}] = 0$, and the noncommutative baby Skyrme equation of motion is satisfied.

Energy and Stability

- The Grassmannian energy functional reads

$$E[P^{(k)}] \;=\; 8\pi \left(k + \tfrac{4\kappa^2}{\theta}\, k^2\right) \;.$$

  The energy depends only on the dimensionless parameter κ^2/θ.

- The Bogomolny bound of $8\pi k$, and the k^2 dependence of the Skyrme term signals an instability of the higher-charge baby Skyrmions against decay into those of charge one.

- Interpreting $P^{(k)}$ as describing k charge-one baby Skyrmions sitting on top of each other, they can lower their energy by passing to a configuration of near-infinite mutual separation, which is again a (near-exact) baby Skyrme solution.

- Are our noncommutative baby Skyrmions stable?

- The energy formula shows that the solution $P^{(k)}$ will decay into k well-separated copies of $P^{(1)}$, so only the single baby Skyrmion may be (and probably is) stable.

- It can lower its energy by changing its shape away from being round and becoming non-BPS. One can settle this issue by computing $\delta^2 E$ restricted to Gr$(P^{(1)})$, future work.

Interacting Forces

- The long-range forces between two noncommutative baby Skyrmions, is the energy of a two-center BPS soliton, because for large separation this configuration approaches a superposition of two rank-one BPS solitons.

- Two-center configuration the projector has the form

$$P^{(\alpha,\beta)} \;=\; \tfrac{1}{1-|\sigma|^2}\left\{|\alpha\rangle\langle\alpha| \;+\; |\beta\rangle\langle\beta| \;-\; \sigma|\alpha\rangle\langle\beta| \;-\; \bar{\sigma}|\beta\rangle\langle\alpha|\right\} \qquad \text{with} \quad \sigma = \langle\alpha|\beta\rangle$$

  the lumps are centered at positions α and β in the complex Moyal plane, and the coherent states $|\alpha\rangle$ and $|\beta\rangle$ are normalized to one. $F[P^{(\alpha,\beta)}] \neq 0$ unless $\alpha-\beta \to 0$ or ∞.

- Energy between two baby Skyrmions is

$$E[P^{(\alpha,\beta)}] \;=\; 8\pi\left\{2 \;+\; 8\,\frac{\kappa^2}{\theta}\Big(1+\tfrac{1}{4}\,r^4\,\sinh^{-2}\frac{r^2}{2}\Big)\right\}, \qquad r = |\alpha-\beta|.$$

- This expression interpolates smoothly between

$$E[P^{(r=0)}] \;=\; 8\pi\,(2+\tfrac{4\kappa^2}{\theta}\cdot 4) \;=\; E[P^{(2)}]$$

$$E[P^{(r\to\infty)}] \;=\; 2\cdot 8\pi\,(1+\tfrac{4\kappa^2}{\theta}) \;=\; 2\cdot E[P^{(1)}]$$

  which again underscores the decay channel $P^{(2)} \to P^{(1)} + P^{(1)}$.

- The interaction potential is exponentially repulsive,

$$V(r) \;\sim\; 64\pi\,\tfrac{\kappa^2}{\theta}\,r^4\,\mathrm{e}^{-r^2/2}, \qquad r \to \infty.$$

Conclusions

- Find other exact abelian noncommutative baby Skyrmions or rule out this possibility.
- Determine whether $P^{(1)}$ has minimal energy in the rank-one Grassmannian (i.e. is stable)
- Work out the scattering of two such lumps
- To deform the *full Skyrme model (on $\mathbb{R}^{1,3}$)* and to construct noncommutative Skyrmions $\longrightarrow$ noncommutative instantons.

Work in collaboration with Olaf Lechtenfeld.