BRANES, QUANTIZATION AND FUZZY SPHERES

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Generalized Nahm equations

- In Type IIB string theory, monopoles $\equiv$ D1-branes ending on D3-branes; from perspective of N D1-branes described by **Nahm equations:** (Diaconescu '97)

$$\frac{\mathrm{d}X^i}{\mathrm{d}s} + \varepsilon^{ijk}\,[X^j, X^k] \;=\; 0$$

Solution: $X^i(s) \;=\; \frac{1}{s}\,\tau^i \;\;,\;\; \tau^i \;=\; \varepsilon^{ijk}\,[\tau^j, \tau^k]$ fuzzy S^2 (Myers '99)

- M2-branes ending on M5-branes described by **Basu–Harvey equations:** (Basu & Harvey '05)

$$\frac{\mathrm{d}X^i}{\mathrm{d}s} + \varepsilon^{ijkl}\,[X^j, X^k, X^l] \;=\; 0$$

Solution: $X^i(s) \;=\; \frac{1}{\sqrt{2s}}\,\tau^i \;\;,\;\; \tau^i \;=\; \varepsilon^{ijkl}\,[\tau^j, \tau^k, \tau^l]$ "fuzzy S^3"?

- Generalized Nahm equations are built on *n*-Lie algebras

n-Lie algebras

- Vector space A with totally antisymmetric n-ary map $[-,\dots,-] : A^{\wedge n} \longrightarrow A$ satisfying **fundamental identity** ("n-Jacobi identity") (Filippov '85)

- Fundamental identity ensures that inner derivations $\delta_{a^1\wedge\cdots\wedge a^{n-1}}(b) := [a^1,\dots,a^{n-1},b]$ form Lie algebra $\mathfrak{g}_A$

- $(A\,,\,\mathfrak{g}_A) \longleftrightarrow \mathcal{L}$
  (fund. id. , Jacobi id.) $\longleftrightarrow$ homotopy Jacobi identities
  — hence $\mathcal{L}$ is an L_∞-algebra ("strong homotopy Lie algebra")
  Generalized Nahm equations $\longleftrightarrow$ homotopy Maurer–Cartan equation for L_∞-algebra $\mathcal{L}\otimes\Omega^\bullet(\mathbb{R})$ (Lazaroiu *et al.* '09)

- Example: $(A_{n+1};\tau^1,\dots,\tau^{n+1})$, $[\tau^{i_1},\dots,\tau^{i_n}] = \varepsilon^{i_1\cdots i_n i_{n+1}}\,\tau^{i_{n+1}}$
  $\mathfrak{g}_{A_{n+1}} = \mathfrak{so}(n+1)$ (unique simple n-Lie algebra over $\mathbb{C}$)

Geometrical meaning of n-Lie algebras

- Nambu–Poisson structures: Smooth manifold M with n-Lie algebra structure $\{-,\dots,-\} : C^\infty(M)^{\wedge n} \longrightarrow C^\infty(M)$ satisfying:

  1. Fundamental identity:

  $$\begin{aligned}\{f_1,\dots,f_{n-1},\{g_1,\dots,g_n\}\} &= \{\{f_1,\dots,f_{n-1},g_1\},\dots,g_n\} \\ &\quad + \cdots + \{g_1,\dots,\{f_1,\dots,f_{n-1},g_n\}\}\end{aligned}$$

  2. Generalized Leibniz rule:

  $$\{f_1\, f_2, f_3,\dots,f_{n+1}\} = f_1\,\{f_2,\dots,f_{n+1}\} + \{f_1,\dots,f_{n+1}\}\, f_2$$

- Example:
  $S^n \subset (\mathbb{R}^{n+1}; x^1,\dots,x^{n+1})$, $\{x^{i_1},\dots,x^{i_n}\} = \varepsilon^{i_1\dots i_n i_{n+1}}\, x^{i_{n+1}}$
  Extend by linearity and generalized Leibniz rule
  $\mathfrak{g}_{A_{n+1}} =$ isometries of S^n

Quantum geometry of branes

- D1–D3 geometry $\equiv$ fuzzy S^2; definition of fuzzy S^3 for M2–M5?
- Nahm equations $\equiv$ boundary condition for open strings (Chu & Smith '09); const. B-field induces shift in Nahm equations, accounted for by Heisenberg algebra:

$$[X^i, X^j] = \mathrm{i}\,\theta^{ij}\,, \qquad \theta \sim B$$

Quantization of Poisson bracket on $\mathbb{R}^2$ (Seiberg & Witten '99, . . .)

- Basu–Harvey equations $\equiv$ boundary condition of open membranes; modification from const. C-field on M5-brane accounted for by **Nambu–Heisenberg algebra:** (Nambu '73)

$$[X^i, X^j, X^k] = \mathrm{i}\,\Theta^{ijk}\,, \qquad \Theta \sim C$$

Quantization of Nambu–Poisson 3-bracket on $\mathbb{R}^3$

Quantization

$$C^\infty(M) \longrightarrow \mathrm{End}(\mathcal{H}) \;, \qquad f \longmapsto \hat{f}$$

M — Poisson manifold, $\mathcal{H}$ — complex Hilbert space

1. $f \longmapsto \hat{f}$ $\mathbb{C}$-linear, $f = f^* \implies \hat{f} = \hat{f}^\dagger$

2. $f = 1 \longmapsto \mathrm{id}_{\mathcal{H}}$

3. Correspondence principle: $[\hat{f}, \hat{g}] = -\mathrm{i}\,\hbar\,\widehat{\{f, g\}}$

4. $\hat{x}^i$ act irreducibly on $\mathcal{H}$ (rep. of isometries G if $M = G/H$)

- Berezin quantization (fuzzy geometry):
  - Drop irreducibility (prequantization)
  - Quantize only $\Sigma \subsetneq C^\infty(M)$ (polarization/geometric quantization)
  - Correspondence principle only to $O(\hbar)$ (deformation quantization)

Berezin–Toeplitz quantization of $\mathbb{CP}^n$

- Hilbert space: $\mathcal{H}_L = H^0(M,L) =$ global holomorphic sections of "quantum line bundle" L over Kähler manifold (M,ω), with $c_1(L) = [\omega]$

- For $M = \mathbb{CP}^n$, $L := \mathcal{O}(k)$, $\omega \longleftrightarrow$ Fubini–Study metric:

  $$\mathcal{H}_k = \mathrm{span}_{\mathbb{C}}(z_{\alpha_1}\cdots z_{\alpha_k})_{\alpha_i=0}^{n} = \mathrm{span}_{\mathbb{C}}\big(\hat{a}^\dagger_{\alpha_1}\cdots\hat{a}^\dagger_{\alpha_k}|0\rangle\big)$$

  k-particle Hilbert space of "lowest Landau level states"

  (Karabali & Nair '02)

- Rawnsley coherent states: $|z\rangle \in \mathcal{H}_L \qquad \forall z \in M$

- Here $|z\rangle = \frac{1}{k!}\left(\bar{z}_\alpha\hat{a}^\dagger_\alpha\right)^k|0\rangle$ (Perelomov coherent states)

Berezin–Toeplitz quantization of $\mathbb{CP}^n$

- Quantization:

$$f(z) = \sigma(\hat{f}) = \frac{\langle z|\hat{f}\,|z\rangle}{\langle z|z\rangle}, \qquad \hat{f} = Q(f) = \int \frac{\omega^n}{n!}\,\frac{|z\rangle\langle z|}{\langle z|z\rangle}\,f$$

obeys $\lim_{k\to\infty} \Big\| \mathrm{i}\,k\,\big[Q(f),\,Q(g)\big] - Q\big(\{f,g\}\big)\Big\|_{\mathrm{HS}} = 0$
(Bordemann, Meinrenken & Schlichenmaier '94)

- Bergman metric:

$$g = \frac{1}{k}\,\partial\,\overline{\partial}\,\log\langle z|z\rangle$$

Expansion for $k \longrightarrow \infty$ in powers of curvature approximates Einstein metrics for Kähler manifolds in $\mathbb{CP}^n$ (Zelditch '98)

Quantization of S^2 = The fuzzy sphere

- For $\mathbb{CP}^1 \cong S^2$, quantizable functions Σ = spherical harmonics $Y_{l,m}$ with $l \leq k$ (Berezin '75; Hoppe '82; Madore '92)

- Poisson bracket $\{x^i, x^j\} = \varepsilon^{ijk}\, x^k$
  $\Longrightarrow$ Lie algebra $[\hat{x}^i, \hat{x}^j] = -\,\mathrm{i}\,\hbar\,\varepsilon^{ijk}\,\hat{x}^k$

- $x^i = \dfrac{\bar{z}_\alpha\, \sigma^i_{\alpha\beta}\, z_\beta}{|z|^2} \in S^2 \subset \mathbb{R}^3$ (Jordan–Schwinger transformation)
  $\longmapsto \hat{x}^i = \frac{1}{k!}\, \sigma^i_{\alpha\beta}\, \hat{a}^\dagger_\alpha\, \hat{a}^\dagger_{\rho_1} \cdots \hat{a}^\dagger_{\rho_{k-1}} |0\rangle\langle 0| \hat{a}_\beta\, \hat{a}_{\rho_1} \cdots \hat{a}_{\rho_{k-1}}$

- Gives $\hbar = \dfrac{2}{k}$, classical limit is $k \longrightarrow \infty$

- Generalizes to any $\mathbb{CP}^n$ with $\sigma^i_{\alpha\beta} \longrightarrow \lambda^i_{\alpha\beta}$ = Gell-Mann matrices of $SU(n+1)$ (coherent state star-product $f \star g = \sigma(\hat{f}\,\hat{g})$ (Balachandran *et al.* '02))

Generalized quantization

(DeBellis, Sämann & RS '10)

- Extend Berezin quantization to Nambu–Poisson manifolds, keeping data $\mathcal{H}$ and $\mathrm{End}(\mathcal{H})$

- Modify correspondence principle:

$$\lim_{\hbar\to 0} \Big\| \frac{\mathrm{i}}{\hbar}\,\sigma\big([\hat{f}_1,\dots,\hat{f}_n]\big) - \{f_1,\dots,f_n\}\Big\|_{L^2} \ = \ 0$$

- Satisfied by **truncating** Nambu–Poisson algebra on $\mathbb{C}[x^i]$ to n-Lie algebra (Ho, Hou & Matsuo '08), and defining:

$$[\hat{f}_1,\dots,\hat{f}_n] \ := \ \sigma^{-1}\big(-\mathrm{i}\,\hbar\,\{\sigma(\hat{f}_1),\dots,\sigma(\hat{f}_n)\}_K\big)$$

with $K \longrightarrow \infty$ as $\hbar \longrightarrow 0$; deformation of totally antisymmetric operator product $\varepsilon^{i_1\cdots i_n}\,\hat{f}_{i_1}\cdots\hat{f}_{i_n}$

Quantization of S^4

- Use Clifford algebra $Cl(\mathbb{R}^5)$ to isometrically embed $S^4 \subset \mathbb{CP}^3$:

$$x^i = \frac{1}{|z|^2}\,\gamma^i_{\alpha\beta}\,\bar{z}^\alpha\,z^\beta\ , \qquad x^i\,x^i = 1$$

(cf. $\mathbb{CP}^3 \xrightarrow{S^2} S^4$ (Medina & O'Connor '03; Dolan & O'Connor '03; Abe '04))

- Restricted coherent state projection of Berezin symbol $\sigma(\hat{f})$, $\hat{f} \in \mathcal{H}_k$ to $\Sigma \cap C^\infty(S^4)$ gives quantization map:

$$\hat{x}^i := \tfrac{1}{k!}\,\gamma^i_{\alpha\beta}\,\hat{a}^\dagger_\alpha\,\hat{a}^\dagger_{\rho_1}\cdots\hat{a}^\dagger_{\rho_{k-1}}|0\rangle\langle 0|\hat{a}_\beta\,\hat{a}_{\rho_1}\cdots\hat{a}_{\rho_{k-1}}$$

End$(\mathcal{H}_k)$ $=$ noncommutative polynomials of degree $3k$

- Satisfies $\hat{x}^i\,\hat{x}^i = \big(1+\frac{4}{k}\big)\,\mathrm{id}_{\mathcal{H}_k}$, linearly identical to totally antisymmetric operator product, correspondence principle with truncated Nambu–Poisson 4-bracket $\{-,\dots,-\}_k$

Quantization of S^3

- $S^4 \longrightarrow S^3$ by $x^5 = 0 \Longrightarrow S^3 \subset \mathbb{CP}^1 \times \mathbb{CP}^1 \subset \mathbb{CP}^3$

- $x^5 = 0$ not holomorphic (so can't factorize $\mathcal{H}_k$ by holomorphic ideal) — project onto maximal set of irreps on which $\hat{x}^i \, \hat{x}^i \propto \mathrm{id}_{\mathcal{H}_k}$ (nonassociative!) (Guralnik & Ramgoolam '01)

- 3-Lie algebra: (Basu & Harvey '05)

$$[\hat{x}^i, \hat{x}^j, \hat{x}^k] := -[\hat{x}^i, \hat{x}^j, \hat{x}^k, \hat{x}^5] = \mathrm{i}\,\hbar(k)\,\varepsilon^{ijkl}\,\hat{x}^l$$

- Radial fuzziness: Radial modes projected out after multiplication (Grosse, Klimcik & Presnajder '96; Guralnik & Ramgoolam '01) or dynamically suppressed (Medina & O'Connor '03)

- Radial fuzziness of quantum S^3 allows consistent solutions to Basu–Harvey equations (Natase, Papageorgakis & Ramgoolam '09)

Fuzzy scalar field theory on S^4

- $\mathcal{H}_k = H^0(\mathbb{CP}^3, \mathcal{O}(k))$ carries irrep of $SU(4)$ isometries of $\mathbb{CP}^3$ and spinor rep of $SO(5)$ isometries of S^4

- Laplace operator Δ on $C^\infty(S^4) =$ quadratic Casimir operator of $SO(5)$ in spinor rep; carries over to linear operator on $\mathrm{End}(\mathcal{H}_k)$ by "Berezin push": (Lazaroiu, McNamee & Sämann '08)

$$\Delta^B := Q \circ \Delta \circ \sigma$$

- Action: $\displaystyle\int_{S^4} \mathrm{d}\mu_{S^d}\, f = \frac{1}{\mathrm{vol}} \int_{\mathbb{CP}^3} \frac{\omega^3}{3!}\, \rho(f) = \frac{\mathrm{vol}(S^4)}{k}\, \mathrm{tr}\,(\hat{f})$

  $\rho(f) =$ image of $f \in C^\infty(S^4)$ in $C^\infty(\mathbb{CP}^3)$

- Path integral: Integration domain $= \Sigma \cap C^\infty(S^4)$

Quantization of $\mathbb{R}^3$

- Geometry of quantized Nambu–Heisenberg algebra $[\hat{x}^1, \hat{x}^2, \hat{x}^3] = -\,\mathrm{i}\,\hbar$: 3-algebra structure only linearly

- Use fuzzy S^2 with Hilbert space $\mathcal{H}_k = H^0(\mathbb{CP}^1, \mathcal{O}(k))$;

$$[\hat{x}^1, \hat{x}^2, \hat{x}^3] = \varepsilon^{ijk}\,\hat{x}^i\,\hat{x}^j\,\hat{x}^k = -\frac{6\,\mathrm{i}}{k}\,\mathrm{id}_{\mathcal{H}_k}$$

- $\mathcal{H} = \bigoplus_k \mathcal{H}_k \ , \qquad \mathcal{A} = \bigoplus_k \mathrm{End}(\mathcal{H}_k)$

- $\mathbb{R}^3_\hbar$ = "discrete foliation" of $\mathbb{R}^3$ by fuzzy spheres with radii $R_{F,k} = \sqrt{1+\frac{2}{k}}\,\sqrt[3]{\frac{\hbar\,k}{6}}$
(Hammou, Lagraa & Sheikh-Jabbari '02; Batista & Majid '03)

Quantization of hyperboloids

- $H^{p,q} \;=\; SO(p,q)/SO(p-1,q)$, $\;x^\mu\, x^\nu\, \eta_{\mu\nu} \;=\; 1\;$ signature $\;(p,q)$

- Allow for indefinite metric in Clifford algebra
  $\Longrightarrow$ non-hermitian operators, non-unitary reps!!

- Example: fuzzy AdS, M5-geometry $\;=\; \mathbb{R}^{1,2}_\hbar \times \mathbb{R}^3_\hbar$

- n-Lie algebra $\;A_{p,q}$, $\;n \;=\; p+q-1\;$ (unique simple over $\mathbb{R}$)

- Extension to superspheres (in projective superspaces),
  n-Lie superalgebras, ...

Quantization of loop spaces

(Brylinski '93; Sergeev '08)

- Action principle for Nambu mechanics on $\mathbb{R}^3$ as dynamics of loops (Takhtajan '94)
- Quantization of open membranes ending on M5-branes in const. C-fields $\Longrightarrow$ noncommutative loop space (complicated, nonassociative) (Bergshoeff *et al.* '00; Kawamoto & Sasakura '00)
- ADHMN construction for self-dual strings based on loop spaces of $\mathbb{R}^4$ and S^3 (Gustavsson '08; Sämann '10)

Gerbes and quantization

(Sämann & RS '1?)

- Symplectic 2-form $\omega \in H^2(S^2)$ encodes line bundle with connection (L, ∇), $F_\nabla = \omega$

  Multisymplectic 3-form $\varpi \in H^3(S^3)$ encodes bundle gerbe with connection $(\mathcal{G}, A, B)$, $\mathrm{d}B = \varpi$

- Transgress $\mathcal{G}$ to line bundle over loop space of S^3

- $L \longleftrightarrow$ Atiyah Lie algebroids (Hawkins '08)

  $\mathcal{G} \longleftrightarrow$ Courant algebroid, with associated L_∞-algebra (Rogers '10)