Supergravity

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Corfu, 9-10 September 2010

Mostly based on forthcoming book with D. Freedman;

1973

Wess, Zumino

Supersymmetry and supergravity

- supersymmetry

  Bosons and fermions

  in one multiplet

$$\delta A(x) = \bar{\varepsilon}\psi(x)$$

$$\delta\psi(x) = \gamma^{\mu}\varepsilon\frac{\partial}{\partial x^{\mu}}A(x)$$

- commutator gives general coordinate transformations

$$[\delta(\varepsilon_1),\delta(\varepsilon_2)] = \bar{\varepsilon}_2\gamma^{\mu}\varepsilon_1\frac{\partial}{\partial x^{\mu}} \quad \text{or} \quad \{Q, Q\} = \gamma^{\mu}P_{\mu}$$

$\Rightarrow$ gauge theory contains gravity: Supergravity

1976

Freedman, van Nieuwenhuizen, Ferrara

To a main tool in superstring theory

- Philosophy of the 70's: a symmetry should be gauged
- Now supergravity is not seen as a fundamental theory, but as a basic tool in applications of superstring theory.

Present day supergravity

- The AdS/CFT developped to models for quark-gluon plasma and non-perturbative QCD in general based on duality with supergravity solutions
- Phenomenological models based on compactifications on Calabi-Yau
- Since 2000: many cosmological models using CMB:
  - models use supergravity limit of string theory

→ often string theory-inspired supergravity.

(but not all sugra theories are presently related to a string theory)

- Supergravity has also e.g. black holes, cosmic string solutions, domain walls, Randall-Sundrum scenarios, …

1. Scalar fields and symmetries: 1.1 Poincaré group

$$x^\mu = \Lambda^\mu{}_\nu x'^\nu + a^\mu$$

Expand

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \lambda^\mu{}_\nu + \mathcal{O}(\lambda^2)\,.$$

Note metric $(-+\ldots+)$. Algebra

$$[m_{[\mu\nu]}, m_{[\rho\sigma]}] = \eta_{\nu\rho} m_{[\mu\sigma]} - \eta_{\mu\rho} m_{[\nu\sigma]} - \eta_{\nu\sigma} m_{[\mu\rho]} + \eta_{\mu\sigma} m_{[\nu\rho]}\,.$$

Differential operators

$$L_{[\rho\sigma]} \equiv x_\rho \partial_\sigma - x_\sigma \partial_\rho\,.$$

Act on scalars with

$$U(\Lambda) \equiv \mathrm{e}^{-\frac{1}{2}\lambda^{\rho\sigma} L_{[\rho\sigma]}}\,.$$

Using rule $\phi(x) = \phi'(x')$ gives

$$\phi^i(x) \to \phi'^i(x) = U(\Lambda)\phi^i(x) = \phi^i(\Lambda x).$$

More general

$$M_{[\rho\sigma]} = L_{[\rho\sigma]} \mathbb{1} + m_{[\rho\sigma]}\,,$$

$$\psi(x) \to \psi'(x) = U(\Lambda)\psi(x) = \mathrm{e}^{-\frac{1}{2}\lambda^{\rho\sigma} m_{[\rho\sigma]}} \psi(\Lambda x + a)\,.$$

1.2 Other symmetries and currents

Generic infinitesimal

$$\delta\phi^i(x) \equiv \epsilon^A \Delta_A \phi^i(x) \,,$$

(constant parameters).

Transformation of Lagrangian:

$$\delta\mathcal{L} \equiv \epsilon^A \left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i} \partial_\mu \Delta_A \phi^i + \frac{\delta\mathcal{L}}{\delta\phi^i} \Delta_A \phi^i \right] = \epsilon^A \partial_\mu K^\mu_A \,.$$

Leads to conserved currents

$$J^\mu{}_A = -\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i} \Delta_A \phi^i + K^\mu_A \,, \qquad \partial_\mu J^\mu{}_A \approx 0 \,.$$

2. The Dirac field

$$\partial\!\!\!/\Psi(x) \equiv \gamma^\mu \partial_\mu \Psi(x) = m\Psi(x)\,.$$

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,\eta^{\mu\nu}\,\mathbb{1}$$

Lorentz transformations generated by

$$\Sigma^{\mu\nu} \equiv \tfrac{1}{4}\,[\gamma^\mu, \gamma^\nu]\,,$$

which satisfies Lorentz algebra.

For actions we need

$$\bar\Psi = \Psi^\dagger\beta = \Psi^\dagger \mathrm{i}\gamma^0\,,$$

such that spinor bilinears can be formed that are Lorentz invariants:

$$\delta\Psi = -\tfrac{1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu}\Psi\,, \qquad \delta\bar\Psi = \tfrac{1}{2}\lambda^{\mu\nu}\bar\Psi\Sigma_{\mu\nu}$$

3. Clifford algebras and spinors

- Determines the properties of
  - the spinors in the theory
  - the supersymmetry algebra
- We should know
  - how large are the smallest spinors in each dimension
  - what are the reality conditions
  - which bispinors are (anti)symmetric
    (can occur in superalgebra)

3.1 The Clifford algebra in general dimension

Some extra important choices:

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0\,.$$

i.e. Hermitian for spacelike.

For even dimension, define

$$\gamma_* \equiv (-\mathrm{i})^{m+1}\gamma_0\gamma_1\cdots\gamma_{D-1}\,,$$

which satisfies $\gamma_*^2 = \mathbb{1}$.
E.g. $D=4$: $\gamma_* = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$.

Projections

$$P_L = \tfrac{1}{2}(\mathbb{1}+\gamma_*)\,, \qquad P_R = \tfrac{1}{2}(\mathbb{1}-\gamma_*)\,.$$

$$\gamma^{\mu_1\cdots\mu_r} = \gamma^{[\mu_1}\cdots\gamma^{\mu_r]}\,, \qquad \text{e.g.} \qquad \gamma^{\mu\nu} = \tfrac{1}{2}\gamma^\mu\gamma^\nu - \tfrac{1}{2}\gamma^\nu\gamma^\mu\,,$$

$$\varepsilon_{012(D-1)} = 1\,, \qquad \varepsilon^{012(D-1)} = -1$$

3.2 Supersymmetry and symmetry of bi-spinors (intro)

- E.g. a supersymmetry on a scalar is a symmetry transformation depending on a spinor ε:

$$\delta(\epsilon)\phi(x) = \bar\epsilon\,\psi(x)$$

- For the algebra we should obtain a GCT

$$[\delta(\epsilon_2),\delta(\epsilon_1)]\,\phi(x) = \underbrace{\bar\epsilon_1\gamma^\mu\epsilon_2}_{\xi^\mu}\partial_\mu\phi(x)$$

- Then the GCT parameter should be antisymmetric in the spinor parameters

$$\xi^\mu = \bar\epsilon_1\gamma^\mu\epsilon_2 = -\bar\epsilon_2\gamma^\mu\epsilon_1$$

Thus, to see what is possible, we have to know the symmetry properties of bi-spinors

3.2 Spinors in general dimension

Majorana conjugate

$\bar{\lambda} = \lambda^T C$

C is a matrix such that $C\gamma_{\mu_1 \ldots \mu_r}$ are all symmetric or antisymmetric, depending only on D and r.

- with anticommuting spinors

$\bar{\lambda}\gamma_{\mu_1 \ldots \mu_r}\chi = t_r \bar{\chi}\gamma_{\mu_1 \ldots \mu_r}\lambda$

D (mod 8)	$t_r = -1$	$t_r = +1$
0	0,3	2,1
	0,1	**2,3**
1	0,1	2,3
	0,1	2,3
	1,2	**0,3**
	1,2	0,3
	1,2	**0,3**
	2,3	0,1
5	**2,3**	**0,1**
6	2,3	0,1
	0,3	**1,2**
7	0,3	1,2

Since symmetries of spinor bilinears are important for supersymmetry, we use the Majorana conjugate to define $\bar{\lambda}$.

$$\bar{\lambda}\gamma_{\mu_1\ldots\mu_r}\chi = t_r\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\lambda$$

D (mod 8)	$t_r = -1$	$t_r = +1$
0	0,3	2,1
	0,1	**2,3**
1	0,1	2,3
2	0,1	2,3
10	**1,2**	**0,3**
3 =11	1,2	0,3
4	**1,2**	**0,3**
	2,3	0,1
5	**2,3**	**0,1**
6	2,3	0,1
	0,3	**1,2**
7	0,3	1,2

$$\chi = \Gamma^{(r_1)}\Gamma^{(r_2)}\ldots\Gamma^{(r_p)}\lambda \implies \bar{\chi} = t_0^p t_{r_1}t_{r_2}\cdots t_{r_p}\bar{\lambda}\Gamma^{(r_p)}\ldots\Gamma^{(r_2)}\Gamma^{(r_1)}$$

Spinor indices

$$\lambda^\alpha = \mathcal{C}^{\alpha\beta}\lambda_\beta\,, \qquad \lambda_\alpha = \lambda^\beta \mathcal{C}_{\beta\alpha}\,.$$

NW-SE convention

Note that $\mathcal{C}_{\alpha\beta}$ are components of C^{-1} and $\mathcal{C}^{\alpha\beta}$ of C^T.

Translations:

$$\bar{\chi}\gamma_\mu\lambda = \chi^\alpha(\gamma_\mu)_\alpha{}^\beta\lambda_\beta\,,$$

and also

$$(\gamma_\mu)_{\alpha\beta} = (\gamma_\mu)_\alpha{}^\gamma\mathcal{C}_{\gamma\beta}$$

Have symmetry $-t_1$.

Reality and charge conjugation

Complex conjugation can be replaced by charge conjugation, an operation that acts as complex conjugation on scalars, and has a simple action on fermion bilinears. For example, it preserves the order of spinor factors.

In fact complex conjugation uses

$$B = \mathrm{i}t_0 C\gamma^0$$

We use

$$\lambda^C \equiv B^{-1}\lambda^*\,, \qquad (\gamma_\mu)^C \equiv B^{-1}\gamma_\mu^* B = (-t_0 t_1)\gamma_\mu\,.$$

It works like this:

$$(\bar\chi M\lambda)^* \equiv (\bar\chi M\lambda)^C = (-t_0 t_1)\overline{\chi^C} M^C \lambda^C$$

see e.g.: AVP: 'Tools for supersymmetry', hep-th/ 9910030

3.3 Majorana spinors

- A priori a spinor ψ has $2^{\mathrm{Int}[D/2]}$ (complex) components
- Using e.g. 'left' projection $P_L = (1+\gamma_*)/2$

  'Weyl spinors' $P_L\,\psi = \psi$ **if *D* is even** (otherwise trivial)
- In some dimensions (and signature) there are reality conditions

  $\psi = \psi^C = B^{-1}\,\psi^*$

  consistent with Lorentz algebra: 'Majorana spinors'
- consistency requires $t_1 = -1$.

Other types of spinors

- If t_1=1: Majorana condition not consistent
- Define other reality condition with even number of spinors:

$$\chi^i = \varepsilon^{ij} B^{-1} (\chi^j)^*$$

- 'Symplectic Majorana spinors'
- In some dimensions Weyl and Majorana can be combined, e.g. reality condition for Weyl spinors: 'Majorana-Weyl spinors'

$D = 2$ mod 8:

Majorana: $\psi^C = \psi\,,$ Weyl: $P_{L,R}\psi = \psi$

$D = 4$ mod 4

$$(P_L\psi)^C = P_R\psi\,, \qquad (P_R\psi)^C = P_L\psi$$

Possibilities for susy depend on the properties of irreducible spinors in each dimension

- Dependent on signature. Here: Minkowski
- M: Majorana
  MW: Majorana-Weyl
  S: Symplectic
  SW: Symplectic-Weyl

Dim	Spinor	min.# comp
2	MW	1
3	M	2
4	M	4
5	S	8
6	SW	8
7	S	16
8	M	16
9	M	16
10	MW	16
11	M	32

Majorana OR Weyl fields in D=4

- Any field theory of a Majorana spinor field ª can be rewritten in terms of a Weyl field P_Lª and its complex conjugate.
- Conversely, any theory involving the chiral field Â=P_LÂ and its conjugate Â^C=P_RÂ^C can be rephrased as a Majorana equation if one defines the Majorana field ª =P_LÂ +P_RÂ ^{C}.
- Supersymmetry theories in D=4 are formulated in both descriptions in the physics literature.

4. The Maxwell and Yang-Mills Gauge Fields

4.1 Abelian

Couple to

$$\Psi(x) \to \Psi'(x) \equiv \mathrm{e}^{\mathrm{i}q\theta(x)}\Psi(x)\,.$$

Due to

$$A_\mu(x) \to A'_\mu(x) \equiv A_\mu(x) + \partial_\mu\theta(x)\,.$$

with covariant derivatives

$$D_\mu\Psi(x) \equiv (\partial_\mu - \mathrm{i}qA_\mu(x))\Psi(x)\,,$$

Field strengths couple to currents

$$\partial^\mu F_{\mu\nu} = -J_\nu\,, \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

Typical action

$$S[A_\mu, \bar{\Psi}, \Psi] = \int \mathrm{d}^D x \left[-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \bar{\Psi}(\gamma^\mu D_\mu - m)\Psi\right]$$

4.2. Electromagnetic duality

Vector field strengths are in *2m* – symplectic vectors

$$e^{-1}\mathcal{L}_1 = -\tfrac{1}{4}(\mathrm{Re}\, f_{AB})F_{\mu\nu}^{A}F^{\mu\nu\,B} + \tfrac{1}{8}(\mathrm{Im}\, f_{AB})\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^{A}F_{\rho\sigma}^{B}$$

coupling constants or functions of scalars

$$\begin{array}{llll} \partial^\mu \,\mathrm{Im}\, F_{\mu\nu}^{A-} & = & 0 & \text{Bianchi identities} \\ \partial_\mu \,\mathrm{Im}\, G_A^{\mu\nu-} & = & 0 & \text{Equations of motion.} \end{array}$$

$$F_{\mu\nu}^{\pm A} \equiv \tfrac{1}{2}\left(F_{\mu\nu}^{A} \pm \tfrac{1}{2}\mathrm{i}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma\,A}\right)$$

$$G_A^{\mu\nu\,-} \equiv -2\mathrm{i}\frac{\delta S(F^+,F^-)}{\delta F_{\mu\nu}^{-A}} = \mathrm{i} f_{AB}F^{\mu\nu\,-\,B}$$

Invariance under Gl(*2m*,R)

$$\begin{pmatrix} F'^- \\ G'^- \end{pmatrix} = \mathcal{S}\begin{pmatrix} F^- \\ G^- \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} F^- \\ G^- \end{pmatrix}$$

For consistency:

$$\begin{aligned} G'^- &= (C+\mathrm{i}Df)F^- = (C+\mathrm{i}Df)(A+\mathrm{i}Bf)^{-1}F'^- \\ \to\ & \boxed{\mathrm{i}f' = (C+\mathrm{i}Df)(A+\mathrm{i}Bf)^{-1}} \end{aligned}$$

should be symmetric

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2m,\mathbb{R})$$

4.3 Non-abelian gauge symmetry

- Simplest: act by matrices and $[t_A, t_B] = f_{AB}{}^C t_C$
- Gauge fields for any generator

$$\delta(\epsilon) A_\mu{}^A = \partial_\mu \epsilon^A + \epsilon^C A_\mu{}^B f_{BC}{}^A$$

- Curvatures $[D_\mu, D_\nu] = -\delta_A(R_{\mu\nu}{}^A)$

$$R_{\mu\nu}{}^A = 2\partial_{[\mu} A_{\nu]}{}^A + A_\nu{}^C A_\mu{}^B f_{BC}{}^A$$

Are the field strengths F

- Gauge transformations commutators

$$[\delta_A(\epsilon_1^A), \delta_B(\epsilon_2^B)] = \delta_C\left(\epsilon_2^B \epsilon_1^A f_{AB}{}^C\right)$$

5. The free Rarita-Schwinger field

$$\Psi_\mu(x) \to \Psi_\mu(x) + \partial_\mu \epsilon(x)\,.$$

$$S = -\int \mathrm{d}^D x\, \bar{\Psi}_\mu \left[\gamma^{\mu\nu\rho}\partial_\nu - m\gamma^{\mu\rho}\right] \Psi_\rho$$

Degrees of freedom

- on-shell degrees of freedom : number of helicity states
- off-shell degrees of freedom :

  number of field components − gauge transformations.

spin	off-shell		on-shell	
		$D=4$		$D=4$
0	1	1	1	1
1/2	$2^{[D/2]}$	4	$\frac{1}{2}2^{[D/2]}$	2
1	$D-1$	3	$D-2$	2
3/2	$(D-1)2^{[D/2]}$	12	$\frac{1}{2}(D-3)2^{[D/2]}$	2
2	$\frac{1}{2}D(D-1)$	6	$\frac{1}{2}D(D-3)$	2

SO(D-1) SO(D-2)

Rarita-Schwinger, more details

$$\Psi_\mu(x) \to \Psi_\mu(x) + \partial_\mu \epsilon(x)$$

$$S = -\int \mathrm{d}^D x \, \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho$$

Field equation

$$\begin{aligned}
\delta\mathcal{L} = & \; -\delta\bar{\Psi}_\mu \gamma^{\mu\nu\rho}\partial_\nu \Psi_\rho & -\bar{\Psi}_\mu \gamma^{\mu\nu\rho}\partial_\nu \delta\Psi_\rho \\
= & \qquad \ldots & +\partial_\nu \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \delta\Psi_\rho \\
= & \qquad \ldots & +\delta\bar{\Psi}_\rho \gamma^{\mu\nu\rho}\partial_\nu \Psi_\mu
\end{aligned}$$

Thus

$$\gamma^{\mu\nu\rho}\partial_\nu \Psi_\rho = 0$$

Equivalent to

$$\gamma^\mu (\partial_\mu \Psi_\nu - \partial_\nu \Psi_\mu) = 0$$

On-shell Degrees of freedom by initial conditions

- On-shell= nr. of helicity states
- count number of initial conditions, divide by 2.
- E.g. scalar: field equation $\partial_1\partial^1$Á =0.
  Initial conditions Á($t=0,x^i$) and ∂_0Á ($t=0,x^i$)
- Dirac: first order: determines time derivatives: 4 initial conditions: 2 dof on shell
- PS: $\partial^i\partial^i$Á =0 gives Á=0, since we consider $\partial^i\partial^i=\vec{k}^2$.

On-shell Degrees of freedom massless Rarita-Schwinger

$$\gamma^\mu(\partial_\mu \Psi_\nu - \partial_\nu \Psi_\mu) = 0$$

Gauge fix

$$\gamma^i \Psi_i = 0$$

Evaluate field equations $\nu = 0$ and $\nu = i$

$$\begin{aligned} \Psi_0(\vec{x}, 0) &= 0\,, \\ \partial\!\!\!/ \Psi_i &= 0\,, \\ \gamma^i \Psi_i(\vec{x}, 0) &= 0\,, \\ \partial^i \Psi_i(\vec{x}, 0) &= 0\,. \end{aligned}$$

Hence $2^{[\frac{D}{2}]}(D-3)$ initial components.

6. N=1 Global supersymmetry in D=4

- Classical algebra $\{Q_\alpha, Q_\beta\} = -\frac{1}{2}\gamma^\mu_{\alpha\beta}P_\mu$

$$[P, Q] = 0$$

$$[M_{\mu\nu}, Q] = -\frac{1}{2}\gamma_{\mu\nu}Q$$

6.2. The chiral multiplet

- Transformation under SUSY

$$\delta Z = \frac{1}{\sqrt{2}}\bar{\epsilon}P_L\chi\,, \qquad \delta\bar{Z} = \frac{1}{\sqrt{2}}\bar{\epsilon}P_R\chi\,,$$

$$\delta P_L\chi = \frac{1}{\sqrt{2}}P_L(\partial\!\!\!/ Z + F)\epsilon\,, \qquad \delta P_R\chi = \frac{1}{\sqrt{2}}P_R(\partial\!\!\!/ \bar{Z} + \bar{F})\epsilon\,,$$

$$\delta F = \frac{1}{\sqrt{2}}\bar{\epsilon}\,\partial\!\!\!/ P_L\chi \qquad \delta\bar{F} = \frac{1}{\sqrt{2}}\bar{\epsilon}\,\partial\!\!\!/ P_R\chi$$

- Algebra

$$[\delta_1, \delta_2]\begin{pmatrix} Z \\ P_L\chi \\ F \end{pmatrix} = -\frac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\partial_\mu\begin{pmatrix} Z \\ P_L\chi \\ F \end{pmatrix}$$

- Simplest action $S_{\mathrm{kin}} = \int \mathrm{d}^4x[-\partial^\mu\bar{Z}\partial_\mu Z - \bar{\chi}\partial\!\!\!/ P_L\chi + \bar{F}F]$
- Potential term $S_F = \int \mathrm{d}^4x[FW'(Z) - \frac{1}{2}\bar{\chi}P_L W''(Z)\chi]$

$$S = S_{\mathrm{kin}} + S_F + S_{\bar{F}}$$

The algebra

$$\delta(\epsilon)\phi(x) = \bar{\epsilon}\,\psi(x) = \epsilon^\alpha Q_\alpha \phi(x)$$
$$\rightarrow Q_\alpha \phi(x) = \psi_\alpha(x)$$

- A transformation is a parameter times a generator

$$\delta(\epsilon) = \epsilon^\alpha Q_\alpha$$

- Calculating a commutator

$$\begin{aligned}\delta(\epsilon_1)\,\delta(\epsilon_2) &= \overbrace{\epsilon_1^\alpha Q_\alpha}^{\text{bosonic}} \epsilon_2^\beta Q_\beta \\ &= \epsilon_2^\beta \epsilon_1^\alpha Q_\alpha Q_\beta\end{aligned}$$

$$[\delta(\epsilon_1),\,\delta(\epsilon_2)] = \epsilon_2^\beta \epsilon_1^\alpha Q_\alpha Q_\beta - \epsilon_1^\alpha \epsilon_2^\beta Q_\beta Q_\alpha$$

$$\begin{aligned} &= \epsilon_2^\beta \epsilon_1^\alpha \left(Q_\alpha Q_\beta + Q_\beta Q_\alpha\right) \\ &= -\tfrac{1}{2}\bar{\epsilon}_1 \gamma^\mu \epsilon_2\, \partial_\mu\end{aligned}$$

6.3. Susy gauge thoeries

- Gauge multiplet (in WZ gauge)

$$
\begin{aligned}
S_{\text{gauge}} &= \int \mathrm{d}^4x \left[-\tfrac{1}{4}F^{\mu\nu A}F_{\mu\nu}^{A} - \tfrac{1}{2}\bar{\lambda}^A\gamma^\mu D_\mu\lambda^A + \tfrac{1}{2}D^A D^A\right], \\
\delta A_\mu^A &= \tfrac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^A, \\
\delta\lambda^A &= \left[-\tfrac{1}{4}\gamma^{\rho\sigma}F_{\rho\sigma}^A + \tfrac{1}{2}\mathrm{i}\gamma_* D^A\right]\epsilon, \\
\delta D^A &= \tfrac{1}{2}\bar{\epsilon}\mathrm{i}\gamma_*\gamma^\mu D_\mu\lambda^A, \qquad D_\mu\lambda^A \equiv \partial_\mu\lambda^A + \lambda^C A_\mu{}^B f_{BC}{}^A
\end{aligned}
$$

$$
\begin{aligned}
\delta(\theta)A_\mu^A &= \partial_\mu\theta^A + \theta^C A_\mu{}^B f_{BC}{}^A, \\
\delta(\theta)\lambda^A &= \theta^C\lambda^B f_{BC}{}^A, \\
\delta(\theta)D^A &= \theta^C D^B f_{BC}{}^A
\end{aligned}
$$

$$
\begin{aligned}
[\delta_1,\delta_2]\,A_\mu^A &= -\tfrac{1}{2}\bar{\epsilon}_1\gamma^\nu\epsilon_2 F_{\nu\mu}^A, \\
[\delta_1,\delta_2]\,\lambda^A &= -\tfrac{1}{2}\bar{\epsilon}_1\gamma^\nu\epsilon_2 D_\nu\lambda^A, \\
[\delta_1,\delta_2]\,D^A &= -\tfrac{1}{2}\bar{\epsilon}_1\gamma^\nu\epsilon_2 D_\nu D^A.
\end{aligned}
$$

Susy gauge theories

Full theory

$$S = S_{\mathsf{gauge}} + S_{\mathsf{matter}} + S_{\mathsf{coupling}} + S_W + S_{\overline{W}}\,.$$

$$\begin{aligned}
S_{\mathsf{matter}} &= \int \mathrm{d}^4x \left[-D^\mu \bar{Z} D_\mu Z - \bar{\chi}\gamma^\mu P_L D_\mu \chi + \bar{F}F\right] , \\
S_{\mathsf{coupling}} &= \int \mathrm{d}^4x \left[-\sqrt{2}(\bar{\lambda}^A \bar{Z} t_A P_L \chi - \bar{\chi} P_R t_A Z \lambda^A) + \mathrm{i}\, D^A \bar{Z} t_A Z\right] , \\
S_F &= \int \mathrm{d}^4x \left[F^\alpha W_\alpha + \tfrac{1}{2}\bar{\chi}^\alpha P_L W_{\alpha\beta}\chi^\beta\right] , \\
S_{\bar{F}} &= \int \mathrm{d}^4x \left[\bar{F}_\alpha \overline{W}^\alpha + \tfrac{1}{2}\bar{\chi}_\alpha P_R \overline{W}^{\alpha\beta}\chi_\beta\right] .
\end{aligned}$$

Modified chiral multiplet

$$\begin{aligned}
\delta Z &= \frac{1}{\sqrt{2}}\bar{\epsilon} P_L \chi\,, \\
\delta P_L \chi &= \frac{1}{\sqrt{2}} P_L(\gamma^\mu D_\mu Z + F)\epsilon\,, \\
\delta F &= \frac{1}{\sqrt{2}}\bar{\epsilon} P_R \gamma^\mu D_\mu \chi - \bar{\epsilon} P_R \lambda^A t_A Z\,,
\end{aligned}$$

6.4 Extended supersymmetry

Notation left-right

$$Q_i = P_L Q_i\,, \qquad Q^i = P_R Q^i\,.$$

Algebra:

$$\begin{aligned}
\left\{Q_{i\alpha},\bar{Q}^{j\beta}\right\} &= -\tfrac{1}{2}\delta_i^j (P_L\gamma_\mu)_\alpha{}^\beta P^\mu\,, &\qquad \left\{Q^i_\alpha,\bar{Q}_j^\beta\right\} &= -\tfrac{1}{2}\delta_j^i (P_R\gamma_\mu)_\alpha{}^\beta P^\mu\,,\\
\left\{Q_{i\alpha},\bar{Q}_j^\beta\right\} &= 0\,, & \left\{Q^i_\alpha,\bar{Q}^{j\beta}\right\} &= 0\,,\\
\left[M_{[\mu\nu]},Q_{i\alpha}\right] &= -\tfrac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_{i\beta}\,, & \left[M_{[\mu\nu]},Q^i_\alpha\right] &= -\tfrac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q^i_\beta\,,\\
\left[P_\mu,Q_{i\alpha}\right] &= 0 & \left[P_\mu,Q^i_\alpha\right] &= 0\,.
\end{aligned}$$

Spin content of representations of supersymmetry with maximal spin s_{max} · 2.

		s=2	s = 3/2	s = 1	s = 1/2	s = 0
$\mathcal{N} = 1$	$s_{max} = 2$	1	1			
	$s_{max} = 3/2$		1	1		
	$s_{max} = 1$			1	1	
	$s_{max} = 1/2$				1	1+1
$\mathcal{N} = 2$	$s_{max} = 2$	1	2	1		
	$s_{max} = 3/2$		1	2	1	
	$s_{max} = 1$			1	2	1 + 1
	$s_{max} = 1/2$				2	2 + 2
$\mathcal{N} = 3$	$s_{max} = 2$	1	3	3	1	
	$s_{max} = 3/2$		1	3	3	1+1
	$s_{max} = 1$			1	3 +1	3 + 3
$\mathcal{N} = 4$	$s_{max} = 2$	1	4	6	4	1+1
	$s_{max} = 3/2$		1	4	6 +1	4 + 4
	$s_{max} = 1$			1	4	6
$\mathcal{N} = 5$	$s_{max} = 2$	1	5	10	10 +1	5 + 5
	$s_{max} = 3/2$		1	5 +1	10 + 5	10 +10
$\mathcal{N} = 6$	$s_{max} = 2$	1	6	15 + 1	20 + 6	15 + 15
	$s_{max} = 3/2$		1	6	15	20
$\mathcal{N} = 7$	$s_{max} = 2$	1	7 + 1	21 + 7	35 + 21	35 + 35
$\mathcal{N} = 8$	$s_{max} = 2$	1	8	28	56	70

7. Differential geometry
7.1 Scalars, vectors, tensors

General coordinate transformations

$$\begin{aligned}
\delta\phi(x) &\equiv \phi'(x) - \phi(x) = \mathcal{L}_\xi \phi = \xi^\mu \partial_\mu \phi\,, \\
\delta U^\mu(x) &\equiv U'^\mu(x) - U^\mu(x) = \mathcal{L}_\xi U^\mu = \xi^\rho \partial_\rho U^\mu - (\partial_\rho \xi^\mu) U^\rho\,, \\
\delta\omega_\mu(x) &\equiv \omega'_\mu(x) - \omega_\mu(x) = \mathcal{L}_\xi \omega_\mu = \xi^\rho \partial_\rho \omega_\mu + (\partial_\mu \xi^\rho)\omega_\rho\,, \\
\delta T_\nu^\mu(x) &\equiv T_\nu'^\mu(x) - T_\nu^\mu(x) = \mathcal{L}_\xi T_\nu^\mu = \xi^\rho \partial_\rho T_\nu^\mu - (\partial_\rho \xi^\mu) T_\nu^\rho + (\partial_\nu \xi^\rho) T_\rho^\mu
\end{aligned}$$

7.2 The algebra and calculus of differential forms

$$\omega^{(p)} = \frac{1}{p!}\omega_{\mu_1\mu_2\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\mu_2}\wedge\ldots\mathrm{d}x^{\mu_p}$$

$$\mathrm{d}\omega^{(p)} = \frac{1}{p!}\partial_\mu\omega_{\mu_1\mu_2\cdots\mu_p}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\mu_2}\wedge\ldots\mathrm{d}x^{\mu_p}$$

7.3 The metric and frame field on a manifold

$$g_{\mu\nu}(x) = e_\mu^a(x)\eta_{ab}e_\nu^b(x)$$

$$\begin{aligned}
\varepsilon_{\mu_1\mu_2\cdots\mu_D} &\equiv e^{-1}\varepsilon_{a_1a_2\cdots a_D}e_{\mu_1}^{a_1}e_{\mu_2}^{a_2}\cdots e_{\mu_D}^{a_D}, \\
\varepsilon^{\mu_1\mu_2\cdots\mu_D} &\equiv e\varepsilon^{a_1a_2\cdots a_D}e_{a_1}^{\mu_1}e_{a_2}^{\mu_2}\cdots e_{a_D}^{\mu_D}
\end{aligned}$$

The local Lorentz basis of 1-forms is

$$e^a \equiv e_\mu^a(x)\mathrm{d}x^\mu$$

7.4 Hodge duality of forms

$$^*e^{a_1} \wedge \ldots e^{a_p} = \frac{1}{q!} e^{b_1} \wedge \ldots e^{b_q} \varepsilon_{b_1 \cdots b_q}{}^{a_1 \cdots a_p}$$

$$\int {}^*\omega^{(p)} \wedge \omega^{(p)} = \frac{1}{p!} \int \mathrm{d}^D x \sqrt{-g}\, \omega^{\mu_1 \cdots \mu_p} \omega_{\mu_1 \cdots \mu_p}$$

7.5 p-form gauge fields

$$S_p = -\frac{1}{2}\int {}^*F^{(p+1)} \wedge F^{(p+1)}\,, \qquad F^{(p+1)} \equiv \mathrm{d}A^{(p)}$$

- they are all gauge fields

$$\delta A_\mu(x) = \partial_\mu \Lambda(x) \qquad \delta A_{\mu\nu}(x) = \partial_{[\mu}\Lambda_{\nu]}(x)$$

Reducible symmetry

$$\Lambda_\nu = \partial_\nu \Lambda(x)$$

- Degrees of freedom:

off-shell: as antisymmetric tensor in SO(*D-1)*

(massive representation)

on-shell: as antisymmetric tensor in *SO(D-2)*

(massless representation)

e.g. 2-index antisymm. tensor *D=4:*

• off-shell: SO(3): (3×2)/2=3 components: as vector

• on shell: SO(2): 1 component: as scalar

Dualities

- *n*- tensor leads to *(n+1)*-field strength

$$A_{\mu_1 \dots \mu_n} \to F_{\mu_1 \dots \mu_{n+1}} = (n+1)\partial_{[\mu_1} A_{\mu_2 \dots \mu_{n+1}]}$$

Satisfies Bianchi identity $\partial_{[\mu_1} F_{\mu_2 \dots \mu_{n+2}]} = 0$

And usually a field equation $\partial^{\mu_1} F_{\mu_1 \dots \mu_{n+1}} = 0$

- dual- field strength is *(D-n-1)*

$$\tilde{F}^{\mu_1 \dots \mu_{D-n-1}} \equiv \frac{1}{(n+1)!} \varepsilon^{\mu_1 \dots \mu_D} F_{\mu_{D-n-2} \dots \mu_D}$$

Field equation becomes Bianchi

$$\partial^{[\mu_1} \tilde{F}^{\mu_2 \dots \mu_{D-n}]} = 0$$

- Is field strength of *D-n-2* tensor

in 4 dim: (*n=2*) Antisymmetric tensor $\$$ scalar (*n=0*)
(*n=1*) vector (electric) $\$$ vector (magnetic)

Dualities

- *n*-tensor dual to (*d-n-2*) -tensor

in 4 dim: (*n=2*) Antisymmetric tensor $\$$ scalar (*n=0*)
(*n=1*) vector (electric) $\$$ vector (magnetic)

- Electric – magnetic dualities in 4 dimensions play important role.

7.6 Connections and covariant derivatives

$$\begin{aligned}\nabla_\mu V^\rho &= \partial_\mu V^\rho + \Gamma^\rho_{\mu\nu} V^\nu\,,\\ \nabla_\mu V_\nu &= \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu} V_\rho\,,\end{aligned}$$

Lorentz covariant derivatives on objects with frame indices

$$\begin{aligned}D_\mu V^a &= \partial_\mu V^a + \omega_\mu{}^a{}_b V^b\,,\\ D_\mu U_a &= \partial_\mu U_a - U_b \omega_\mu{}^b{}_a = \partial_\mu U_a + \omega_{\mu a}{}^b U_b\,,\\ D_\mu T_{ab} &= \partial_\mu T_{ab} - T_{cb}\omega_\mu{}^c{}_a - T_{ac}\omega_\mu{}^c{}_b\,.\end{aligned}$$

On fermions and on mixed quantities

$$\begin{aligned}D_\mu \Psi_\nu &\equiv \left(\partial_\mu + \tfrac{1}{4}\omega_{\mu ab}\gamma^{ab}\right)\Psi_\nu\,,\\ \nabla_\mu \Psi_\nu &= D_\mu \Psi_\nu - \Gamma^\rho_{\mu\nu}\Psi_\rho\,.\end{aligned}$$

Torsion

Vielbein postulate

$$\nabla_\mu e_\nu^a = \partial_\mu e_\nu^a + \omega_\mu{}^a{}_b e_\nu^b - \Gamma^\sigma_{\mu\nu} e_\sigma^a = 0 .$$

Torsion is 2× antisymmetric part of Γ.

In form language

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b \equiv T^a , \qquad \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} = T_{\mu\nu}{}^\rho .$$

Relations:

$$\begin{aligned}
\omega_\mu{}^{ab} &= \omega_\mu{}^{ab}(e) + K_\mu{}^{ab} , \\
\omega_\mu{}^{ab}(e) &= 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\nu e_\sigma{}^c , \\
K_{\mu[\nu\rho]} &= -\tfrac{1}{2}(T_{[\mu\nu]\rho} - T_{[\nu\rho]\mu} + T_{[\rho\mu]\nu}) , \\
\Gamma^\rho_{\mu\nu} &= \Gamma^\rho_{\mu\nu}(g) - K_{\mu\nu}{}^\rho , \\
\Gamma^\rho_{\mu\nu}(g) &= \tfrac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) .
\end{aligned}$$

7.7 Curvature tensor

$$R_{\mu\nu ab} \equiv \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu ac}\omega_\nu{}^c{}_b - \omega_{\nu ac}\omega_\mu{}^c{}_b$$

$$R_{\mu\nu}{}^\rho{}_\sigma \equiv \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\tau}\Gamma^\tau_{\nu\sigma} - \Gamma^\rho_{\nu\tau}\Gamma^\tau_{\mu\sigma}$$

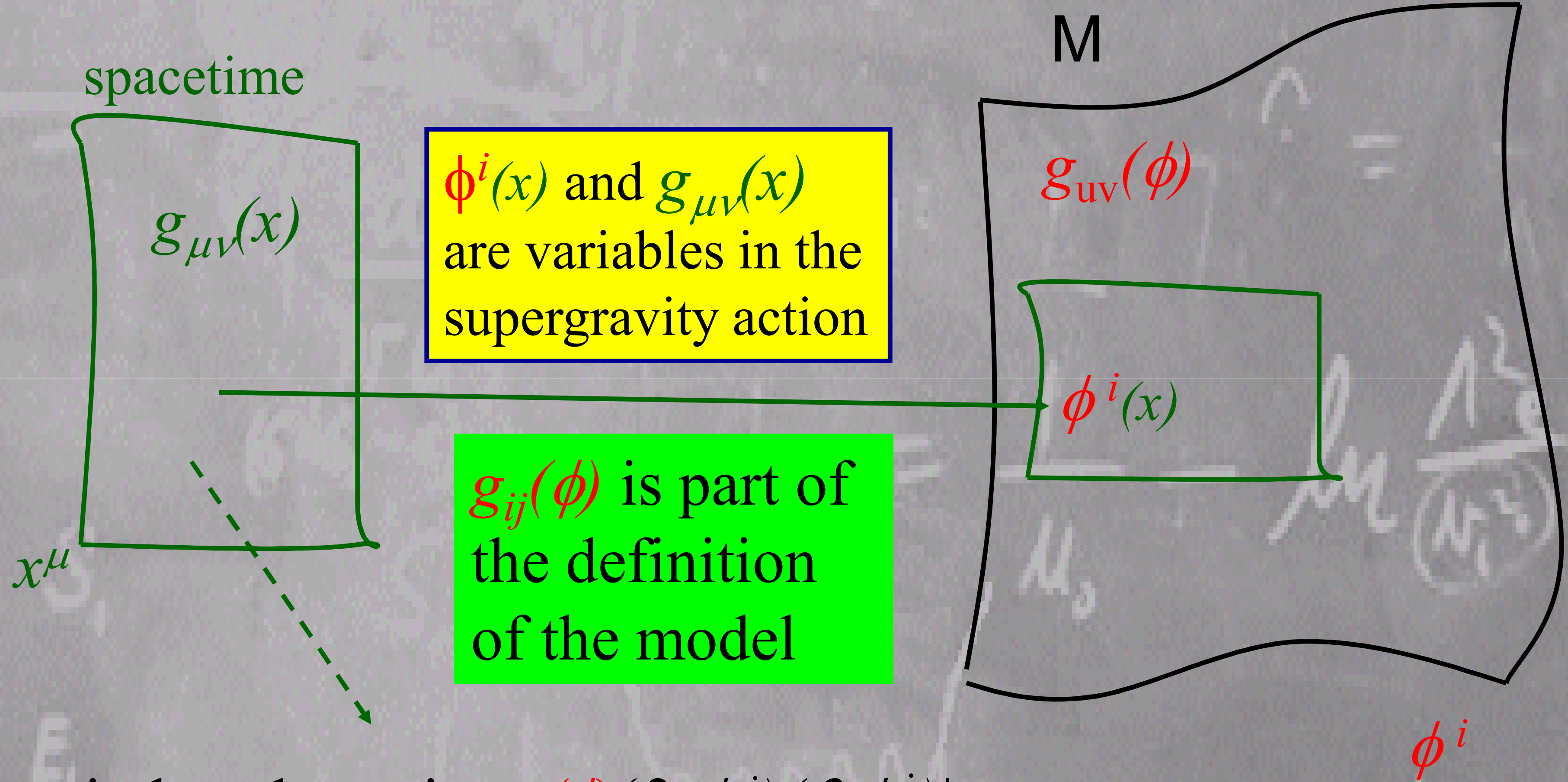
Acting with ∇_ν on

$$\nabla_\mu e^a_\rho = \partial_\mu e^a_\rho + \omega_\mu{}^a{}_b e^b_\rho - \Gamma^\sigma_{\mu\rho} e^a_\sigma = 0 .$$

and antisymmetrizing, gives

$$R_{\mu\nu}{}^\rho{}_\sigma = R_{\mu\nu ab} e^{a\rho} e^b_\sigma$$

7.8 The nonlinear ¾-model



induced metric $g_{ij}(\phi)(\partial_\mu\phi^i)(\partial_\nu\phi^j)|_{\phi=\phi(x)}$

appears in action

Symmetries of target space

Symmetries are defined from

$$\delta(\theta)\phi^i = \theta^A k_A{}^i(\phi)$$

Each $k_A{}^i$ is a vector field that satisfies
(for invariance of the action)

$$\nabla_i k_{jA} + \nabla_j k_{iA} = 0\,, \qquad k_{iA} = g_{ij} k^j_A\,, \qquad \nabla_i k_{jA} = \partial_i k_{jA} - \Gamma^k_{ij}(g) k_{kA}$$

(We assume torsionless connection here).

8. The first and second order formulations of general relativity

8.1 Second order formalism for gravity and bosonic matter

$$
\begin{aligned}
S &= \int \mathrm{d}^D x \sqrt{-\det g}\left(\frac{1}{2\kappa^2} g^{\mu\nu} R_{\mu\nu}(g) + L\right), \\
L &= -\tfrac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}
\end{aligned}
$$

$R(g)$ is first derivative in $\Gamma(g)$ and thus second order in $g_{\mu\nu}$.

Einstein field equations

$$
\begin{aligned}
G_{\mu\nu} &\equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu}, \\
T_{\mu\nu} &\equiv -2 \frac{1}{\sqrt{-\det g}} \frac{\delta(\sqrt{-\det g}\, L)}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi + F_\mu{}^\rho F_{\nu\rho} + g_{\mu\nu} L
\end{aligned}
$$

8.2 Second order formalism for gravity and fermions

$$\begin{aligned} S &= S_2 + S_{1/2} \\ &= \int \mathrm{d}^D x\, e \left[\frac{1}{2\kappa^2} e_a^\mu e_b^\nu R_{\mu\nu}{}^{ab}(e) - \frac{1}{2}\bar{\Psi}\gamma^\mu \nabla_\mu \Psi + \frac{1}{2}\bar{\Psi}\overleftarrow{\nabla}_\mu \gamma^\mu \Psi\right]. \end{aligned}$$

$R(\omega(e))$ second order.

Note $\gamma_\mu = e_{a\mu}\gamma^a$ or $\gamma^\mu = e_a^\mu \gamma^a$.

Calculations simplify due to

$$\left.\frac{\delta S_2}{\delta \omega_{\mu ab}}\right|_{\omega=\omega(e)} = 0\,.$$

We obtain

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu} \equiv \kappa^2 \tfrac{1}{4}\bar{\Psi}\left[\gamma_\mu \overleftrightarrow{\nabla}_\nu + \gamma_\nu \overleftrightarrow{\nabla}_\mu\right]\Psi\,.$$

8.3 The first order formalism for gravity and fermions

e_μ^a and $\omega_\mu{}^{ab}$ independent.
Without fermionic part: field equation of ω would give $\omega = \omega(e)$.
Now we get

$$T_{ab}{}^\nu = \tfrac{1}{2}\kappa^2\bar{\Psi}\,\gamma_{ab}{}^\nu\,\Psi = -2K^\nu{}_{ab}\,.$$

This implies that first order leads to torsion.
Rewrite Lagrangian with torsion to torsionless second order form:

$$S = \frac{1}{2}\int \mathrm{d}^D x\, e\,\left[\frac{1}{\kappa^2}R(g) - \bar{\Psi}\gamma^\mu\overleftrightarrow{\nabla}_\mu\Psi + \frac{1}{16}\kappa^2(\bar{\Psi}\gamma_{\mu\nu\rho}\Psi)(\bar{\Psi}\gamma^{\mu\nu\rho}\Psi)\right]\,.$$

In principle a physical effect!

9. N=1 pure supergravity in 4 dimensions

- Local ²
- N=1 supergravity in D=4 only frame field and gravitino.
- We need in general a specific type of spinor.
  Here Majorana.
- First part of calculation is universal.
- The approach below is most easily extended to
  N>2, D=4 and to higher dimension,
  while the superconformal approach is better suited to the derivation and understanding of matter couplings.

9.1 The universal part of supergravity

$$
\begin{aligned}
S &= S_2 + S_{3/2}\,, \\
S_2 &= \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\, e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega) \\
S_{3/2} &= -\frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\, \bar\psi_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho\,,
\end{aligned}
$$

postulate the rules

$$
\begin{aligned}
\delta e_\mu^a &= \tfrac{1}{2}\bar\epsilon\gamma^a\psi_\mu\,, \\
\delta\psi_\mu &= D_\mu\epsilon(x) \equiv \partial_\mu\epsilon + \tfrac{1}{4}\omega_{\mu ab}\gamma^{ab}\epsilon\,.
\end{aligned}
$$

Later group theoretical argument.

First consider without torsion: $\omega = \omega(e)$.

The universal part of supergravity: susy invariance

Linear terms in ψ_μ are universal:
frame field variation of S_2
and the gravitino variation of $S_{3/2}$

$$
\begin{aligned}
\delta S_2 &= \int \mathrm{d}^D x \frac{1}{\kappa^2}\left(e^{b\nu} R_{\mu\nu ab} - \frac{1}{2} e_{a\mu} R\right) \delta e^{a\mu} \\
&= \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)\left(-\bar{\epsilon}\gamma^\mu\psi^\nu\right) .
\end{aligned}
$$

$$
\begin{aligned}
\delta S_{3/2} &= -\frac{1}{\kappa^2}\int \mathrm{d}^D x\, e\, \bar{\epsilon}\overleftarrow{D}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho \\
&= \frac{1}{\kappa^2}\int \mathrm{d}^D x\, e\, \bar{\epsilon}\gamma^{\mu\nu\rho} D_\mu D_\nu \psi_\rho \\
&= = \frac{1}{8\kappa^2}\int \mathrm{d}^D x\, e\, \bar{\epsilon}\gamma^{\mu\nu\rho} R_{\mu\nu ab}\gamma^{ab}\psi_\rho .
\end{aligned}
$$

Use Bianchi and symmetry of Ricci tensor, and everything cancels.
Notice that the calculation involves an intimate mix of the key properties of Riemannian geometry and Dirac algebra!

9.2 Supergravity in the first order formalism

- Beyond linear order it is complicated. Look for simplifications.
- First order formalism. ! independent.
- We will obtain a physical equivalent second order action, since we find torsion!
  Thus different!

$$\delta S_{3/2} = -\frac{1}{8\kappa^2}\int \mathrm{d}^D x\, e\, (\bar\psi_\mu \gamma^{\mu\nu\rho}\gamma_{ab}\psi_\rho)\delta\omega_\nu{}^{ab}\,.$$

Leads to

$$T_{ab}{}^{\nu} = \tfrac{1}{2}\bar\psi_a\gamma^\nu\psi_b + \tfrac{1}{4}\bar\psi_\mu\gamma^{\mu\nu\rho}{}_{ab}\psi_\rho\,.$$

For $D = 4$:

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4 x\, e\left[R(e) - \bar\psi_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho + \mathcal{L}_{\text{SG,torsion}}\right],$$

$$\mathcal{L}_{\text{SG,torsion}} = -\frac{1}{16}\left[(\bar\psi^\rho\gamma^\mu\psi^\nu)(\bar\psi_\rho\gamma_\mu\psi_\nu + 2\bar\psi_\rho\gamma_\nu\psi_\mu) - 4(\bar\psi_\mu\gamma\cdot\psi)(\bar\psi^\mu\gamma\cdot\psi)\right],$$

with torsion-free connection

$$D_\nu\psi_\rho \equiv \partial_\nu\psi_\rho + \tfrac{1}{4}\omega_{\nu ab}(e)\gamma^{ab}\psi_\rho\,.$$

9.3 The 1.5 order formalism

Statement: complete and invariant under

$$
\begin{aligned}
\delta\psi_\mu &= D_\mu\epsilon \equiv \partial_\mu\epsilon + \tfrac{1}{4}\omega_{\mu ab}\gamma^{ab}\epsilon\,,\\
\omega_{\mu ab} &= \omega_{\mu ab}(e) + K_{\mu ab}\,,\\
K_{\mu\nu\rho} &= -\tfrac{1}{4}(\bar\psi_\mu\gamma_\rho\psi_\nu - \bar\psi_\nu\gamma_\mu\psi_\rho + \bar\psi_\rho\gamma_\nu\psi_\mu)\,,
\end{aligned}
$$

Structure of the proof ?

First, third and fifth order in gravitino.

Looks too difficult.

Computer?

First order formalism: no fifth order

but one needs a rule for $\delta\omega$.

Intermediate: really 2nd order, but use trick

The 1.5 order formalism

$$\delta S[e,\,\omega(e){+}K,\,\psi] = \int \mathsf{d}^D x \left[\frac{\delta S}{\delta e}\delta e + \frac{\delta S}{\delta \omega}\delta(\omega(e)+K) + \frac{\delta S}{\delta \psi}\delta\psi \right] .$$

Second term vanishes !

± ! can be neglected in the variation of the action
if ! takes the value ! (e,Ã) determined by its field equation.

Procedure:

1. Use the **first order form of the action** S[e,!,Ã] and the transformation rules ± e, ±Ã with connection ! unspecified.
2. Ignore the connection variation and **calculate**

$$\delta S = \int \mathsf{d}^D x \left[\frac{\delta S}{\delta e}\delta e + \frac{\delta S}{\delta \psi}\delta\psi \right] .$$

3. **Substitute** ! (e,Ã) in the result, which must vanish for a consistent supergravity theory.

9.4 Local supersymmetry of N=1, D=4 supergravity

Rewrite using

$$\gamma^{abc} = -\mathrm{i}\varepsilon^{abcd}\gamma_*\gamma_d\,, \qquad \gamma^{\mu\nu\rho} = -\mathrm{i}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\gamma_*\gamma_\sigma$$

$$S_{3/2} = \frac{\mathrm{i}}{2\kappa^2}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_*\gamma_\sigma D_\nu\psi_\rho\,.$$

Frame field occurs less:
(only in γ_σ.)

Still a lot of calculations,
Fierzing and geometry with torsion.
3 pages in book.

9.5 The algebra of local supersymmetry

$$[\delta_1, \delta_2]\, e_\mu^a = D_\mu \xi^a \,, \qquad \xi^a = \tfrac{1}{2}\bar{\epsilon}_2 \gamma^a \epsilon_1 = -\tfrac{1}{2}\bar{\epsilon}_1 \gamma^a \epsilon_2$$

On the other hand

$$\delta_\xi e_\mu^a = \xi^\rho \partial_\rho e_\mu^a + \partial_\mu \xi^\rho e_\rho^a \,.$$

Using $\nabla_\rho e_\mu^a = 0$ we get

$$\delta_\xi e_\mu^a \;=\; \nabla_\mu \xi^\rho e_\rho^a - \xi^\rho \omega_\rho{}^a{}_b e_\mu^b + \xi^\rho T_{\rho\mu}{}^a \,, \qquad \xi^\rho T_{\rho\mu}{}^a = \tfrac{1}{2}(\xi^\rho \bar{\psi}_\rho)\gamma^a \psi_\mu \,.$$

We obtain

$$[\delta_1, \delta_2]\, e_\mu^a \;=\; \left(\delta_\xi - \delta_{\hat{\lambda}} - \delta_{\hat{\epsilon}}\right) e_\mu^a \,, \qquad \hat{\lambda}_{ab} = \xi^\rho \omega_{\rho ab} \,, \qquad \hat{\epsilon} = \xi^\rho \psi_\rho$$

This is a **covariant general coordinate transformation**.

The algebra of local supersymmetry, II

- On Ã$_1$: more complicated. Fierzing.
  Only modulo equations of motion.
  ‘*closes only on-shell*’. ‘*on-shell multiplet*’: see 2+2 dof, while off-shell 6+12 dof.
- ‘*off-shell multiplet*’: adding auxiliary fields.
  Works for N = 1, D=4. Not general.
- Important for the coupling of chiral and gauge multiplets to supergravity.
  We will obtain a set of auxiliary fields for N = 1, D=4 supergravity later.

10. D=11 supergravity

- Extensions after N = 1, D=4:
  - couple to chiral and gauge multiplets
  - extended susy
  - higher dimensions
- E.g. D=10: low-energy limits of string theory
- In general complicated, but D=11 (largest one) is simple
- Reduces to N = 8, D=4

10.1 Dimensional reduction from D=11

- In fact: *full* dimensional reduction e.g. (D+1)-dim. fields on (Minkowski$_D$ £ S^1) gives *Fourier modes*, *massive*.
- Now only parts *independent of extra dimensions*. massless→massless. Is a *consistent truncation* on circles. (field equations of omitted fields are satisfied by truncation).
- D=11 is the *maximal dimension*: to show this we consider M_{11}! M_4 £ T^7, and show that we get to N=8, D=4.
- Any higher dimension has larger spinors: cannot fit in N=8, D=4. Hence would need spin 5/2. No consistent interactions known in Minkowski space.

D=11 theory, first ideas

- Putative D=11 theory: we expect that it contains *graviton and gravitino*.
  Start with ¡^{M}, M=0,…,10: 32£ 32 matrices

$$\Gamma^{\mu} = \gamma^{\mu} \times \mathbb{1}\,, \qquad \mu = 0,1,2,3\,,$$
$$\Gamma^{i} = \gamma_{*} \times \hat{\gamma}^{i}\,, \qquad i = 4,5,6,7,8,9,10\,.$$

Gravitino is ª$_{M ® a}$ in which

®=1,2,3,4 is a 4-dim. spinor index

a=1…,8 is the index on which the 7-dim. gammas act:

8 gravitinos (¹ ® a) + 56 spin ½ (i ® a).

All fermions of N=8 !

10.2 The field content of D=11supergravity

- We already know a lot.
  Graviton: ½D(D-3)=44.
  Gravitino: 128.
  Missing 84 bosonic
- We saw also *p*-forms with $\binom{D-2}{p}$ components.
- We need 3-form A_{MNP} . (Cremmer-Julia)
- Further check by reducing:

$D=11$	spin 2	spin 1	spin 0
g_{MN} A_{MNP}	$g_{\mu\nu}$ 1	$g_{\mu i}$ 7 $A_{\mu ij}$ 21	g_{ij} 28 $A_{\mu\nu i}$ 7 A_{ijk} 35
	1	28	70

10.3 Construction of the action and transformation rules

Ansatz $$S=\frac{1}{2\kappa^2}\int \mathrm{d}^{11}x\, e\left[e^{a\mu}e^{b\nu}R_{\mu\nu ab}-\bar\psi_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho-\frac{1}{24}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}+\ldots\right].$$

$$\begin{aligned}
\delta e_\mu^a &= \tfrac{1}{2}\bar\epsilon\gamma^a\psi_\mu\,,\\
\delta\psi_\mu &= D_\mu\epsilon+\left(a\,\gamma^{\alpha\beta\gamma\delta}{}_\mu+b\,\gamma^{\beta\gamma\delta}\delta_\mu^\alpha\right)F_{\alpha\beta\gamma\delta}\epsilon\,,\\
\delta A_{\mu\nu\rho} &= -c\,\bar\epsilon\gamma_{[\mu\nu}\psi_{\rho]}=-\tfrac{1}{3}c\,\bar\epsilon(\gamma_{\mu\nu}\psi_\rho+\gamma_{\nu\rho}\psi_\mu+\gamma_{\rho\mu}\psi_\nu)\,.
\end{aligned}$$

Initially torsion free.
Graviton-gravitino system as our `universal‘ calculation.
Then check ± S/ Ã F∂ $. After integrations by parts, Bianchi identities and °-matrix algebra: a and b determined in terms of c:

Check algebra

$$[\delta_1,\delta_2]A_{\mu\nu\rho}=-\tfrac{4}{9}c^2\bar\epsilon_1\gamma^\sigma\epsilon_2F_{\sigma\mu\nu\rho}\,.$$

First term $\partial_\sigma A_{\mu\nu\rho}$ is the spacetime translation.
Remainder: $\theta_{\mu\nu}=-\bar\epsilon_1\gamma^\sigma\epsilon_2A_{\sigma\mu\nu}$, as before:
$\rightarrow\ c^2=9/8$.

Constructing the D=11 action

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^{11}x\, e \left[e^{a\mu} e^{b\nu} R_{\mu\nu ab} - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{1}{24} F^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} + \ldots \right].$$

effective supercurrent.

$$\mathcal{J}^\nu = \frac{\sqrt{2}}{96} \left(\gamma^{\alpha\beta\gamma\delta\nu\rho} F_{\alpha\beta\gamma\delta} + 12 \gamma^{\alpha\beta} F_{\alpha\beta}{}^{\nu\rho} \right) \psi_\rho \,.$$

Add term $-\bar{\psi}_\nu \mathcal{J}^\nu$.

Leads to $\delta S \propto \epsilon F F \psi$.

Also from metric in $S = FF$, and

$$\begin{aligned} \delta\mathcal{L} &= \frac{1}{32 \cdot 144} \varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\rho\mu\nu} \bar{\epsilon} \gamma_{\nu\mu} \psi_\rho \, F_{\alpha'\beta'\gamma'\delta'} F_{\alpha\beta\gamma\delta} \,, \\ &= \frac{4}{3\sqrt{2} \cdot 32 \cdot 144} \varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\rho\mu\nu} (\delta A_{\mu\nu\rho}) \, F_{\alpha'\beta'\gamma'\delta'} F_{\alpha\beta\gamma\delta} \,. \end{aligned}$$

The Chern-Simons term

$$\delta\mathcal{L} = \frac{4}{3\sqrt{2}\cdot 32\cdot 144}\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\rho\mu\nu}(\delta A_{\mu\nu\rho})\, F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\delta}$$

This suggests to add a term in the Lagrangian of the following form to cancel this variation:

$$\begin{aligned} S_{\mathsf{C-S}} &= -\frac{\sqrt{2}}{(144\kappa)^2}\int \mathrm{d}^{11}x\,\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\mu\nu\rho}F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\delta}A_{\mu\nu\rho}\,, \\ &= -\frac{\sqrt{2}}{6\kappa^2}\int F^{(4)}\wedge F^{(4)}\wedge A^{(3)}\,, \end{aligned}$$

Chern-Simons term ! See:

$$\begin{aligned} \delta\int F^{(4)}\wedge F^{(4)}\wedge A^{(3)} &= \int\left[2\mathrm{d}\delta A^{(3)}\wedge F^{(4)}\wedge A^{(3)} + F^{(4)}\wedge F^{(4)}\wedge\delta A^{(3)}\right] \\ &= 3\int F^{(4)}\wedge F^{(4)}\wedge\delta A^{(3)}\,, \end{aligned}$$

No other variations ! Gauge invariant.

Final result

Further: transformation laws ‘covariant’.
Like covariant derivatives: transform without derivative on ϵ.

$$\widehat{F}_{\mu\nu\rho\sigma} = 4\,\partial_{[\mu}A_{\nu\rho\sigma]} + \tfrac{3}{2}\sqrt{2}\,\bar{\psi}_{[\mu}\gamma_{\nu\rho}\psi_{\sigma]}\,.$$

$\delta\widehat{F}$ has no $\partial_\mu\epsilon$. In action:

$$\begin{aligned} S \;=\; \frac{1}{2\kappa^2}\int \mathrm{d}^{11}x\, e\,\Big[& e^{a\mu}e^{b\nu}R_{\mu\nu ab}(\omega) - \bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu(\tfrac{1}{2}(\omega+\widehat{\omega}))\psi_\rho - \frac{1}{24}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma} \\ & - \frac{\sqrt{2}}{192}\bar{\psi}_\nu\left(\gamma^{\alpha\beta\gamma\delta\nu\rho} + 12\gamma^{\alpha\beta}g^{\gamma\nu}g^{\delta\rho}\right)\psi_\rho(F_{\alpha\beta\gamma\delta} + \widehat{F}_{\alpha\beta\gamma\delta}) \\ & - \frac{2\sqrt{2}}{(144)^2}\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\mu\nu\rho}F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\delta}A_{\mu\nu\rho}\Big]\,. \end{aligned}$$

Such that field equations are covariant. E.g.

$$\gamma^{\mu\nu\rho}D_\nu(\widehat{\omega})\psi_\rho - \frac{\sqrt{2}}{288}\gamma^{\mu\nu\rho}\left(\gamma^{\alpha\beta\gamma\delta}{}_\nu - 8\gamma^{\beta\gamma\delta}\delta^\alpha_\nu\right)\psi_\rho\widehat{F}_{\alpha\beta\gamma\delta} = 0\,.$$

10.4 The algebra of D=11 supergravity

$$\Big[\delta_Q(\epsilon_1),\, \delta_Q(\epsilon_2)\Big] \;=\; \delta_{\text{gct}}(\xi^\mu)+\delta_L(\lambda^{ab})+\delta_Q(\epsilon_3)+\delta_A(\theta_{\mu\nu})\,,$$

with

$$\begin{aligned}
\xi^\mu &= \tfrac{1}{2}\epsilon_2\gamma^\mu\epsilon_1\,,\\
\lambda^{ab} &= -\xi^\mu\widehat{\omega}_\mu{}^{ab} + \frac{1}{288}\sqrt{2}\bar{\epsilon}_1\left(\gamma^{ab\mu\nu\rho\sigma}\widehat{F}_{\mu\nu\rho\sigma} + 24\gamma_{\mu\nu}\widehat{F}^{ab\mu\nu}\right)\epsilon_2\,,\\
\epsilon_3 &= -\xi^\mu\psi_\mu\,,\\
\theta_{\mu\nu} &= -\xi^\rho A_{\rho\mu\nu} + \tfrac{1}{4}\sqrt{2}\bar{\epsilon}_1\gamma_{\mu\nu}\epsilon_2\,.
\end{aligned}$$

On the gravitino only if its field equations are satisfied.

See central charges.
Terms with $\bar{\epsilon}_1\Gamma^{(2)}\epsilon_2$ and $\bar{\epsilon}_1\Gamma^{(6)}\epsilon_2$.
Non-vanishing in BPS $M2$ and $M5$ brane solutions.

11. General gauge theory

- For matter-coupled theories we need more advanced methods.
- Sharpen knives and refine and extend concepts used before.
  Formalize manipulations, covariant derivatives.
- Will see how postulated supergravity transformations are determined by Poincaré supersymmetry algebra.

11.1 Symmetries acting on fields

- Rigid symmetries
- Local symmetries (gauge symmetries)
- Symmetries of the action
- Symmetries of field equations
- Symmetries of solutions

There are also: Symmetries acting on parameters:

sigma model symmetries, e.g. changing Kähler potential

Symmetries

- In this chapter: symmetries that leave action invariant.
- Continuous, infinitesimal: Lie algebra.
- Extend: structure functions.
- General treatment:
  spacetime, internal symmetries and susy.

11.1.1 Global symmetries

$$\delta(\epsilon) = \epsilon^A T_A \,,$$

T_A operator,
can in Hamiltonian be defined by Poisson brackets.
First linear:

$$T_A \phi^i = -(t_A)^i{}_j \phi^j \,, \qquad [t_A, t_B] = f_{AB}{}^C t_C \,.$$

Transformations act on fields !!

$$\begin{aligned} \delta(\epsilon_1)\delta(\epsilon_2)\phi^i &= \epsilon_1^A T_A \epsilon_2^B \left[-(t_B)^i{}_j \phi^j\right] \\ &= \epsilon_1^A \epsilon_2^B (-t_B)^i{}_j T_A \phi^j \\ &= \epsilon_1^A \epsilon_2^B (-t_B)^i{}_j (-t_A)^j{}_k \phi^k \,. \end{aligned}$$

Leads to

$$[T_A, T_B] = f_{AB}{}^C T_C \,,$$

T_A is more general notation.
Ψ^α in a complex representation of a compact symmetry group, their conjugates $\bar{\Psi}_\alpha$, and fields ϕ^B in the adjoint:

$$\begin{aligned} T_A \Psi^\alpha &= -(t_A)^\alpha{}_\beta \Psi^\beta \,, \\ T_A \bar{\Psi}_\alpha &= \bar{\Psi}_\beta (t_A)^\beta{}_\alpha \,, \\ T_A \phi^B &= -f_{AC}{}^B \phi^C \,. \end{aligned}$$

for the Poincaré group

Similar: (remember $\Sigma_{[\mu\nu]} = \frac{1}{2}\gamma_{\mu\nu}$)

$$M_{[\mu\nu]}\Psi(x) \;=\; -(L_{[\mu\nu]} + \tfrac{1}{2}\gamma_{\mu\nu})\Psi(x)\,.$$

While

$$P_\mu = \partial_\mu = \frac{\partial}{\partial x^\mu}\,,$$

for susy

$\epsilon^A \to \epsilon^\alpha$ and $T_A \to Q_\alpha$.

$$\delta(\epsilon) = \bar{\epsilon}^\alpha Q_\alpha = \bar{\epsilon} Q\,.$$

From

$$\delta Z = \frac{1}{\sqrt{2}} \bar{\epsilon} P_L \chi \qquad \to \qquad Q_\alpha Z = \frac{1}{\sqrt{2}} (P_L \chi)_\alpha\,.$$

Commutator:

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = (\bar{\epsilon}_2)^\beta (\bar{\epsilon}_1)^\alpha \Big(Q_\alpha Q_\beta + Q_\beta Q_\alpha \Big)$$

We write

$$\{Q_\alpha, Q_\beta\} = f_{\alpha\beta}{}^C T_C$$

E.g. we found

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = -\tfrac{1}{2} \bar{\epsilon}_1 \gamma^\mu \epsilon_2 P_\mu = \tfrac{1}{2} \bar{\epsilon}_2 \gamma^\mu \epsilon_1 P_\mu = -\tfrac{1}{2} \epsilon_2^\beta (\gamma^\mu)_{\beta\alpha} \epsilon_1^\alpha P_\mu$$

Leads to

$$\{Q_\alpha, Q_\beta\} = -\tfrac{1}{2} (\gamma^\mu)_{\alpha\beta} P_\mu\,, \qquad \to \qquad f_{\alpha\beta}{}^\mu = -\tfrac{1}{2} (\gamma^\mu)_{\alpha\beta} = f_{\beta\alpha}{}^\mu\,.$$

Thus: for bosonic and fermionic symmetries

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta(\epsilon_3^C = \epsilon_2^B \epsilon_1^A f_{AB}{}^C)$$

The nonlinear ¾-model and Killing symmetries.

Not linear

$$T_A \phi^i = k^i_A(\phi)\,.$$

with $k_A = k^i_A(\phi)\frac{\partial}{\partial \phi^i}$:

$$[k_A, k_B] = f_{AB}{}^C k_C\,,$$

11.1.2. Local symmetries and gauge fields

- Gauge theory $\delta(\epsilon)B_\mu{}^A \equiv \partial_\mu\epsilon^A + \epsilon^C B_\mu{}^B f_{BC}{}^A$

generic gauge symmetry T_A	parameter ϵ^A	gauge field B_μ^A
local translations P_a	ξ^a	e_μ^a
Lorentz transformations $M_{\{ab\}}$	λ^{ab}	$\omega_\mu{}^{ab}$
Supersymmetry Q_α	$\bar{\epsilon}^\alpha$	$\bar{\psi}_\mu^\alpha$
Internal symmetry T_A	θ^A	A_μ^A

Theories of gravity: Distinguish between coordinate indices $\mu,\nu,\rho,\ldots$ and local frame indices a,b,c….
We now use $M_{[ab]}$ and P_a to denote the generators of local Lorentz transformations and translations. There are some subtleties for local translations.

Good and bad. 1) Apply to susy on $e_\mu{}^a$. Gives ansatz from before.
2) Puzzling: $\delta_P \psi_{\mu\alpha} = 0$. (Also for other fields).
Need to modify the present setup before we can apply it to gravitational theories.

11.1.3. Modified symmetry algebras: soft algebra

Not mathematical Lie algebra

- When extra gauge symmetries, gauged by the vector multiplets, the derivatives become covariant

$$\left[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)\right]\phi = \xi^\mu D_\mu\phi = \xi^\mu\partial_\mu\phi - \xi^\mu A_\mu^A T_A\phi\,, \qquad \xi^\mu = \tfrac{1}{2}\bar\epsilon_2\gamma^\mu\epsilon_1$$

$$\{Q_\alpha, Q_\beta\} = -\tfrac{1}{2}(\gamma^\mu)_{\alpha\beta}(P_\mu - A_\mu^A T_A)$$

$$f_{\alpha\beta}{}^A = \tfrac{1}{2}A_\mu^A(\gamma^\mu)_{\alpha\beta}\,,$$

The algebra is ‘soft’:

structure constants become structure functions.

Modified Jacobi identities

For a solution: become again constants.

Leads to e.g. AdS or central charges.

Zilch symmetries and open algebras

$$\delta_{\text{triv}}\phi^i = \epsilon\,\eta^{ij}\frac{\delta S}{\delta\phi^j}$$

$$\delta_{\text{triv}}S = \frac{\delta S}{\delta\phi^i}\epsilon\,\eta^{ij}\frac{\delta S}{\delta\phi^j} = 0 \quad \text{if} \quad \eta^{ij} = -\eta^{ji}$$

Therefore: transformations not uniquely determined.

First principles: Symmetry: $S_{,i}$±(²)Á ^{i}=0
Thus: $S_{,ij}$±(²$_1$)Á ^{i}±(²$_2$)Á^{j}+$S_{,i}$± (²$_2$)±(²$_1$)Á ^{i}=0.
Taking the commutator, the first term vanishes by symmetry,
and the second term says that the commutator defines a symmetry.

But may include Zilch symmetries:

$$[\delta(\epsilon_1),\delta(\epsilon_2)]\,\phi^i = \text{susy algebra} + \eta^{ij}(\epsilon_1,\epsilon_2)\,\frac{\delta S}{\delta\phi^j}$$

‘Closed on-shell’ or ‘open algebra’
If basis without second term:
‘closed off-shell’, or ‘closed algebra'.

11.2 Covariant quantities

- Terminology: gauge fields ↔matter fields.
- For the latter $\delta(\epsilon)\phi^i(x) = \epsilon^A(T_A\phi^i)(x)$

  do not involve derivatives of the gauge parameters.

A **covariant quantity** is a local function that transforms under all local symmetries with no derivatives of a transformation parameter.

Note for below: special care needed for local translations. Will be discussed afterwards.

Covariant derivatives and curvatures

$$\begin{aligned}\mathcal{D}_\mu\phi^i &\equiv (\partial_\mu - \delta(B_\mu))\,\phi^i \\ &= \left(\partial_\mu - B_\mu^A T_A\right)\phi^i\,.\end{aligned}$$

is a covariant quantity.

Stronger: Gauge transformations and covariant derivatives commute on fields on which the algebra is off-shell closed.

$$\begin{aligned}[\mathcal{D}_\mu, \mathcal{D}_\nu] &= -\delta(R_{\mu\nu})\,, \\ R_{\mu\nu}{}^A &= 2\partial_{[\mu}B_{\nu]}{}^A + B_\nu{}^C B_\mu{}^B f_{BC}{}^A\,.\end{aligned}$$

is a covariant quantity.

11.3 Gauged spacetime translations

Remark as intro

$$R_{\mu\nu}(P^a) = 2\partial_{[\mu}e_{\nu]}{}^a + 2\omega_{[\mu}{}^{ab}e_{\nu]b} - \tfrac{1}{2}\bar{\psi}_\mu\gamma^a\psi_\nu = 0\,.$$

11.3.1 Gauge transformations for the Poincaré group

Poincaré on scalars

$$\delta(a,\lambda)\phi(x) = \left[a^\mu\partial_\mu - \tfrac{1}{2}\lambda^{\mu\nu}L_{[\mu\nu]}\right]\phi(x) = \left[a^\mu + \lambda^{\mu\nu}x_\nu\right]\partial_\mu\phi(x) = \xi^\mu(x)\partial_\mu\phi(x)$$

Orbital part can be included in $\xi^\mu(x)$.
Is change of basis from $a^\mu(x)$ and $\lambda^{ab}(x)$ to $\xi^\mu(x) = a^\mu(x) + \lambda^{\mu\nu}(x)x_\nu$ and $\lambda^{ab}(x)$.

spinors: global

$$\delta(a,\lambda)\Psi(x) = \left[a^\mu + \lambda^{\mu\nu}x_\nu\right]\partial_\mu\Psi(x) + -\tfrac{1}{4}\lambda^{ab}\gamma_{ab}\Psi(x)$$

Local

$$\delta(\xi,\lambda)\Psi(x) = \xi^\mu(x)\partial_\mu\Psi(x) + -\tfrac{1}{4}\lambda^{ab}(x)\gamma_{ab}\Psi(x)$$

Vectors

$$\begin{aligned}\delta(\xi,\lambda)V_\mu(x) &= \xi^\nu(x)\partial_\nu V_\mu(x) + V_\nu(x)\partial_\mu\xi^\nu(x)\\ \delta(\xi,\lambda)V_a &= \xi^\mu(x)\partial_\mu V_a(x) + V_b(x)\lambda^b{}_a(x)\,.\end{aligned}$$

11.3.2. Covariant derivatives and general coordinate transformations

There is a problem $\mathcal{D}_\mu\phi = \partial_\mu\phi - e_\mu^a(x)\partial_a\phi(x) = 0$

1. Remove gct from the sum over all symmetries: all the others are called 'standard gauge transformations'.

$$\begin{aligned} \mathcal{D}_\mu\phi^i &\equiv (\partial_\mu - \delta(B_\mu))\,\phi^i \\ &= \left(\partial_\mu - B_\mu^A T_A\right)\phi^i\,. \end{aligned}$$

2. We will always impose the constraint $R_{10}(P^a)=0$

3. We replace translations with 'covariant coordinate transformations'

$$\delta_{\mathrm{cgct}}(\xi) = \delta_{\mathrm{gct}}(\xi) - \delta(\xi^\mu B_\mu)$$

11.3.3 Covariant derivatives and curvatures in a gravity theory

- Some gauge fields have extra (non-gauge) terms

$$\delta(\epsilon)B_\mu^A = \partial_\mu\epsilon^A + \epsilon^C B_\mu{}^B f_{BC}{}^A + \epsilon^B \mathcal{M}_{\mu B}{}^A$$

- E.g. $$\delta A_\mu^A = \partial_\mu\theta^A + \theta^C A_\mu{}^B f_{BC}{}^A - \tfrac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^A$$

- Covariant curvature

$$\widehat{R}_{\mu\nu}^A = 2\partial_{[\mu}B_{\nu]}{}^A + B_\nu{}^C B_\mu{}^B f_{BC}{}^A - 2B_{[\mu}{}^B \mathcal{M}_{\nu]B}{}^A$$

- The covariant quantities in gravity have flat indices:

$$\mathcal{D}_a\phi = e_a{}^\mu \mathcal{D}_\mu\phi\,, \qquad \widehat{R}_{ab}^A = e_a{}^\mu e_b{}^\nu \widehat{R}_{\mu\nu}^A$$

- Gauge fields do not appear naked in covariant quantities: either in covariant derivative or in curvature.

12. Survey of supergravities

12.1 Minimal and extended superalgebras

- Minimal algebra $\left\{Q_\alpha, Q_\beta\right\} = -\frac{1}{2}\gamma^\mu_{\alpha\beta}P_\mu$

  $[P, Q] = 0$ $\quad [M_{\mu\nu}, Q] = -\frac{1}{4}\gamma_{\mu\nu}Q$

- Extension according to reality and Weyl properties, e.g. *D=4* (M) $\left\{Q^i_\alpha, Q_{\beta j}\right\} = (\gamma^\mu P_L)_{\alpha\beta}\delta^i_j P_\mu$

  D=5 (S) $\left\{Q^i_\alpha, Q^j_\beta\right\} = \gamma^\mu_{\alpha\beta}\Omega^{ij}P_\mu$

Haag-Lopuszanski-Sohnius, 1975

Central charges

Central charges: other symmetries in {Q,Q}:

e.g. D=4, N=2: $\left\{Q^i_\alpha, Q^j_\beta\right\} = \gamma^\mu_{\alpha\beta}\delta^i_j P_\mu + \varepsilon^{ij}\left[\mathcal{C}_{\alpha\beta}Z_1 + (\gamma_5)_{\alpha\beta}Z_2\right]$

D=11: $\left\{Q_\alpha, Q_\beta\right\} = \gamma^\mu_{\alpha\beta}P_\mu + \gamma^{\mu\nu}_{\alpha\beta}Z_{\mu\nu} + \gamma^{\mu_1\cdots\mu_5}_{\alpha\beta}Z_{\mu_1\cdots\mu_5}$

- The algebra gives limits on solutions.
- The left-hand side can be written as QQ^{y} ,hence positive !
- Without central charges:

  solution with preserved supersymmetry, i.e. $Q|soln> = 0$

  ! mass zero.
- With central charges: limit, typically $Z^2 \cdot M^2$.

  For preserved supersymmetry: bound saturated

BPS bound

BPS states

12.2 The R-symmetry group

- Supersymmetries may rotate under an automorphism group. E.g. for 4 dimensions:

$$[T_A, Q_{\alpha i}\ ] = (U_A)_i{}^j Q_{\alpha j} \qquad [T_A, Q_\alpha^i\ ] = (U_A)^i{}_j Q_\alpha^j$$

- related by charge conjugation: $(U_A)^j{}_i = \left((U_A)_j{}^i\right)^*$

- Jacobi identities [TTQ] : U forms a representation of T-algebra
- Jacobi identities [TQQ] :

$$(U_A)_i{}^j = -(U_A)^j{}_i \equiv -\left((U_A)_j{}^i\right)^* .$$

$\rightarrow$ forms U(N) group

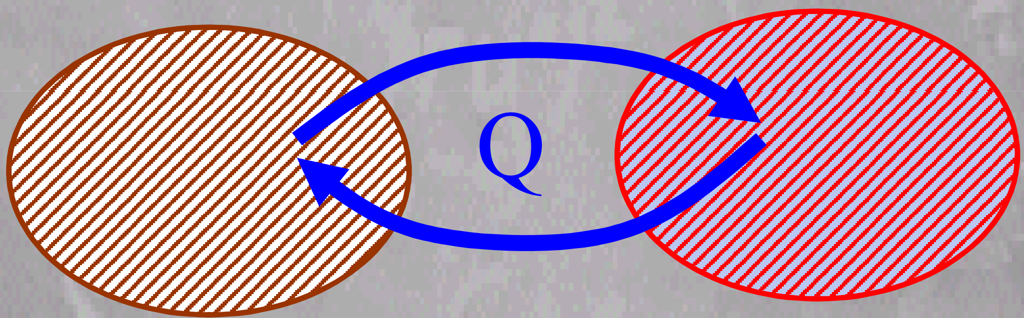
R-symmetry groups

group that rotates susys: $\left[T_A, Q^i_\alpha\right] = (U_A)^i{}_j Q^j_\alpha$

- Majorana spinors in odd dimensions:
  SO(N) (D=3,9)
- Majorana spinors in even dimensions:
  U(N) (D=4,8)
- Majorana-Weyl spinors:
  $SO(N_L)$ � $SO(N_R)$ (D=2,10)
- Symplectic spinors:
  USp(N) (D=5,7)
- Symplectic Majorana-Weyl spinors:
  $USp(N_L)$ � $USp(N_R)$ (D=6)

12.3 Multiplets

- There is an argument that
  \# bosonic d.o.f. = # fermionic d.o.f.,
  based on {Q,Q}=P (invertible)



- Should be valid for on-shell multiplets if eqs. of motion are satisfied: e.g. z : 2, χ : 2 **)** 2+2
- for off-shell multiplets counting all components:
  e.g. z : 2, χ : 4, h : 2 **)** 4+4

12.4 Supergravity theories: towards a catalogue

- basic theories and kinetic terms
- deformations and gauged supergravities

Strathdee,1987

The map: dimensions and # of supersymmetries

d	susy	32		24	20	16		12	8	4
11	M	M								
10	MW	IIA	IIB			I				
9	M	N=2				N=1				
8	M	N=2				N=1				
7	S	N=4				N=2				
6	SW	(2,2)		(2,1)		(1,1)	(2,0)		(1,0)	
5	S	N=8		N=6		N=4			N=2	
4	M	N=8		N=6	N=5	N=4		N=3	N=2	N=1
		SUGRA				SUGRA/SUSY		SUGRA	SUGRA/SUSY	

vector multiplets

tensor multiplet

vector multiplets + multiplets up to spin 1/2

Remarks

- 32 supersymmetries is maximal number for fermionic generators that square to translations. There may be others (e.g. special susy in the superconformal algebra).
- (4,0), (3,1) or (3,0) in 6 dimensions do not have a multiplet with a graviton
- Tensor is dual to vector in 5 dimensions, but non-Abelian theories can be different.
- *N=3* susy ´ *N=4* susy but not for sugra

Theories with 32 supersymmetries

- all in one multiplet, only on-shell known. Obtainable from *d=11* (except IIB in *d=10*): fields in SO(9) reps: (44+84)+128 $e_\mu^a,\ A_{\mu\nu\rho},\ \psi_\mu$

•E.g. :reduction to d=4:

spin	#
2	1
3/2	8
1	28
1/2	56
0	70

Theories with 16 supercharges

- Rigid supersymmetry is possible:
  - vector multiplets: Abelian or non-Abelian
  - tensor multiplets for (2,0) in D=6
- Supergravity theories
  - with or without matter (vector, tensor) multiplets
  - At the end the vectors in supergravity and in vector multiplets mix.
- Model fixed by giving:
  - the number of multiplets
  - the gauge group.

Theories with 8 or 4 supercharges

- Theories are not any more determined by a number of discrete choices,
  but by arbitrary functions.
- *N=1* supergravity consists of :
  - pure supergravity: spin 2 + spin 3/2 ("gravitino")
  - gauge multiplets: spin 1 + spin ½ ("gaugino")
    vectors gauge an arbitrary gauge group
  - chiral multiplets: complex spin 0 + spin ½
    in representation of gauge group
- E.g. chiral multiplets *N=1, D=4*
  - arbitrary function $K(z,z^*)$ for kinetic terms
  - potential determined by superpotential $W(z)$

Basic supergravities and deformations

- Basic supergravities:
  have only gauged supersymmetry and general coordinate transformations (and U(1)'s of vector fields).
  - No potential for the scalars.
  - Only Minkowski vacua.
- In any entry of the table there are 'deformations':
  without changing the kinetic terms of the fields, the couplings are changed.
  - Many deformations are 'gauged supergravities':
    gauging of a YM group, introducing a potential.
  - Produced by fluxes on branes
  - There are also other deformations
    (e.g. massive deformations, superpotential)

Deformations

- A basic supergravity theory can have different gaugings. E.g. N=8 :
  - Cremmer-Julia: pure N=8, the 28vectors are in $U(1)^{28}$. There is no potential. This is 'ungauged supergravity'.
  - de Wit-Nicolai: vectors gauge SO(8). This generates a potential: 'gauged supergravity'.
  - later other possibilities SO(p,q,r) gaugings, …
- What are the most general possibilities ?

Gauge symmetries: terminology

- Global ungauged susy:
  super-Poincaré group non gauged.
- Gauged supersymmetry:
  vector fields in matter multiplets gauge a Yang-Mills group.
  gauge group commutes with supersymmetry, but
  appears in commutator of 2 susys (soft algebra)
- Supergravity 'non-gauged':
  gauged super-Poincaré group.
  No other gauged symmetries.
- 'Gauged supergravity':
  vector fields gauge a Yang-Mills group.
  gauge group does **not** commute with supersymmetries
  (act partly as 'R-symmetries')
  appears in commutator of 2 susys (soft algebra)

Gauge group

- # generators = # vectors.
- Dropping trivial U(1) of Abelian vectors that do not act on other fields:

  # generators · # vectors
- This includes as well vectors in supergravity multiplet and those in vector multiplets (cannot be distinguished in general)
- The gauge group is arbitrary, but to have positive kinetic terms gives restrictions on possible non-compact gauge groups.
- Gauge symmetries are part of the isometries of the scalar manifold.

  One has to identify: which part of the isometry group is gauged. This is done by the embedding tensor

Embedding tensor formalism

- The gauge group is a subgroup of the isometry group G, defined by an embedding tensor. $\left(\partial_\mu - A_\mu{}^M \Theta_M{}^\alpha \delta_\alpha\right) \phi$

all the rigid symmetries

determines which symmetries are gauged, and how:
e.g. also the coupling constants.
There are several constraints on the tensor.

- More research is necessary to know all supergravities (even restricting to at most two spacetime derivatives in Lagrangian terms and Minkowski signature, …)

Cordaro, Frè, Gualtieri,Termonia and Trigiante, 9804056

Nicolai, Samtleben and Trigiante, 0010076

de Wit, Samtleben and Trigiante, 0507289

12.5 General characteristics of an action

A full action

$$e^{-1}\mathcal{L} = -\tfrac{1}{2}M_P^2\left[R + \bar\psi_\mu R^\mu + \mathcal{L}_{SG,torsion}\right]$$

$$-g_i{}^j\left[M_P^2(\hat\partial_\mu z^i)(\hat\partial^\mu z_j) + \bar\chi_j \not{\mathcal{D}}\chi^i + \bar\chi^i \not{\mathcal{D}}\chi_j\right]$$

$$+(\mathrm{Re}\ f_{\alpha\beta})\left[-\tfrac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu\,\beta} - \tfrac{1}{2}\bar\lambda^\alpha\,\hat{\not{\mathcal{D}}}\lambda^\beta\right]$$

$$+\tfrac{1}{4}i(\mathrm{Im}\ f_{\alpha\beta})\left[F^\alpha_{\mu\nu}\tilde F^{\mu\nu\,\beta} - \hat\partial_\mu\left(\bar\lambda^\alpha\gamma_5\gamma^\mu\lambda^\beta\right)\right]$$

$$-M_P^{-2}e^{\mathcal{K}}\left[-3WW^* + (\mathcal{D}^iW)g^{-1}{}_i{}^j(\mathcal{D}_jW^*)\right]$$

$$-\tfrac{1}{2}(\mathrm{Re}\ f)^{-1\,\alpha\beta}\mathcal{P}_\alpha\mathcal{P}_\beta$$

$$+\tfrac{1}{8}(\mathrm{Re}\ f_{\alpha\beta})\bar\psi_\mu\gamma^{\nu\rho}\left(F^\alpha_{\nu\rho} + \hat F^\alpha_{\nu\rho}\right)\gamma^\mu\lambda^\beta$$

$$+\left\{M_P g_j{}^i\bar\psi_{\mu L}(\hat{\not\partial} z^j)\gamma^\mu\chi_i + \bar\psi_R\cdot\gamma\left[\tfrac{1}{2}i\lambda^\alpha_L\mathcal{P}_\alpha + \chi_i Y^3 M_P^{-4}\mathcal{D}^iW\right]\right.$$

$$(\mathcal{D}^i\mathcal{D}^jW)\bar\chi_i\chi_j + \quad \bar\chi_i\gamma^{\mu\nu}\hat F^{-\alpha}_{\mu\nu}\lambda^\beta_L$$

$$-2M_P\xi_\alpha{}^i g_i{}^j\lambda^\alpha\chi_j$$

$$+\tfrac{1}{4}M_P^{-5}Y^3(\mathcal{D}^jW)g^{-1}{}_j{}^i f_{\alpha\beta i}\bar\lambda^\alpha_R\lambda^\beta_R$$

$$-\tfrac{1}{4}M_P^{-1}f^i_{\alpha\beta}\bar\psi_R\cdot\gamma\chi_i\bar\lambda \quad \bar\psi_\mu\chi^j\,\bar\psi^\mu\chi_i \quad \chi_j\bar\lambda^\alpha_L\lambda^\beta_L + \mathrm{h.c.}\left.\right\}$$

$$+g_j{}^i\left(\tfrac{1}{8}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_\mu\gamma_\nu\psi\right.$$

$$+M_P^{-2}\left(R^{k\ell}_{ij} - \tfrac{1}{2}g_i{}^k g_j{}^\ell\right)$$

$$+\tfrac{3}{64}M_P^{-2}\left((\mathrm{Re}\ f_{\alpha\beta})\bar\lambda^\alpha\gamma_\mu\gamma_5\lambda^\beta\right)^2$$

$$-\tfrac{1}{16}M_P^{-2}f^i_{\alpha\beta}\bar\lambda^\alpha_L\lambda^\beta_L g^{-1}{}_i{}^j f_{\gamma\delta j}\bar\lambda^\gamma_R\lambda^\delta_R$$

$$+\tfrac{1}{8}(\mathrm{Re}\ f)^{-1\,\alpha\beta}M_P^{-2}\left(f^i_{\alpha\gamma}\bar\chi_i\lambda^\gamma - f_{\alpha\gamma i}\bar\chi^i\lambda^\gamma\right)\left(f^j_{\beta\delta}\bar\chi_j\lambda^\delta - f_{\beta\delta j}\bar\chi^j\lambda^\delta\right)$$

Structure of the action (D=4)

- *(D=4)* with fields of spin 2, 1, 0, 3/2, 1/2

$$e_\mu^a\,,\, A_\mu^I\,,\, \varphi^u\,,\, \psi_\mu\,,\, \lambda^A$$

$$\begin{aligned}
e^{-1}\mathcal{L} = {} & \tfrac{1}{2}R + \\
& +\tfrac{1}{4}(\mathrm{Im}\ \mathcal{N}_{IJ})\mathcal{F}_{\mu\nu}^{I}\mathcal{F}^{\mu\nu J} \\
& -\tfrac{i}{8}(\mathrm{Re}\ \mathcal{N}_{IJ})\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}^{I}\mathcal{F}_{\rho\sigma}^{J} \\
& -\tfrac{1}{2}g_{uv}D_\mu\varphi^u\, D^\mu\varphi^v - V \\
& \Big\{-\bar{\psi}_{\mu i}\gamma^{\mu\nu\rho}D_\nu\psi_\rho{}^i \\
& -\tfrac{1}{2}g_A{}^B\bar{\lambda}^A \not{D}\lambda_B + \mathrm{h.c.}\Big\} + \ldots
\end{aligned}$$