Asymptotic analysis of 4d spinfoam models

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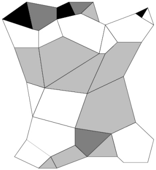

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Introduction

- A spinfoam model is a discretised functional integral usually constructed using:
  - A triangulation of the spacetime manifold (‘foam’)
  - Local variables (‘spins’)
  - Local amplitudes for the simplexes of the triangulation
- Main problem: how do spinfoam models relate to low energy physics?
- Key step: study the asymptotics (semi-classical limit) of the amplitude for the 4-simplexes
- Talk based on joined work with JW Barrett, RJ Dowdall, H Gomes, F Hellmann, and R Pereira

1. 4-simplex amplitudes for $\mathrm{SU}(2)$ BF and QG
2. Asymptotic formulae
3. Proof of the asymptotic results (Lorentzian QG)

Boundary state space for the Ooguri and EPRL models

- Let σ be a 4-simplex and consider the Lie group $\mathrm{SU}(2)$
- Consider the assignments

$$k : \quad \text{triangles} \rightarrow \mathrm{Irrep}(\mathrm{SU}(2)) \cong \{k, k \in \mathbb{N}/2\}$$

- One can then associate to each tetrahedron a state

$$\Psi \quad \in \quad \mathcal{H}_{\mathrm{tet}} = \mathrm{Inv}_{\mathrm{SU}(2)}(k_1 \otimes \ldots \otimes k_4)$$

- The state space for the boundary of σ yields

$$\mathcal{H}_{\partial\sigma} = \bigotimes_{\mathrm{tet}} \mathcal{H}_{\mathrm{tet}},$$

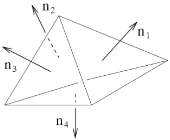
- The amplitude for a 4-simplex σ is given by a map

$$A_\sigma : \mathcal{H}_{\partial\sigma} \rightarrow \mathbb{C}$$

Geometry of the boundary: coherent states

- Space $\bigoplus_k \mathcal{H}_{\text{tet}}$: quantum tetrahedron [Barbieri - 98; Baez, Barrett - 99]
- Parametrisation of $\mathcal{H}_{\text{tet}}$: coherent states
- A coherent state for the direction **n** and spin k is a unit vector ξ in k defined up to a phase and satisfying $(\mathbf{J} \cdot \mathbf{n})\,\xi = ik\,\xi$
- Coherent tetrahedron [Livine, Speziale - 07]

$$\Psi = \int_{\mathrm{SU}(2)} dX\, X\,\xi_1 \otimes \ldots \otimes X\,\xi_4 \quad \in \mathcal{H}_{\text{tet}}$$



- Coherent triangulated 3-manifold described by a state:

$$\Psi(k, \mathbf{n}) = \bigotimes_{\text{tet}} \Psi,$$

together with a canonical choice of phase (Regge-like)

Amplitude for the Ooguri model [Ooguri - 92; Baez - 99; Livine, Speziale - 07]

- Ingredients:
  - $\mathrm{SU}(2)$ anti-linear structure $J : \mathbb{C}^2 \to \mathbb{C}^2$; $(z_0, z_1) \mapsto (-\bar{z}_1, \bar{z}_0)$
  - Hermitian inner product $\langle , \rangle$ on $\mathbb{C}^2$
- If $a = 1, ..., 5$ labels the five tetrahedra of $\partial\sigma$, the couple ab labels the triangle shared by tetrahedra a and b
- Let $\Psi(k_{ab}, \mathbf{n}_{ab})$ be a coherent state for $\partial\sigma$
- The amplitude $A_\sigma(\Psi) \in \mathbb{C}$ is given by

$$15\mathrm{j}(k_{ab}, \mathbf{n}_{ab}) = \int_{\mathrm{SU}(2)^5} \prod_a dX_a \prod_{a<b} \langle J\xi_{ab}, X_a^{-1} X_b \, \xi_{ba} \rangle^{2k_{ab}},$$

where $\xi \in \mathbb{C}^2$ is a coherent state in the fundamental representation

Amplitude for the Euclidean EPRL model [Engle, Livine, Pereira, Rovelli - 08]

- QG with Immirzi $\gamma \neq 0$
- Model based on $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$, $\mathrm{Irrep}(G) = \{(j_+, j_-)\}$
- Idea: the boundary representation k is mapped to the highest or lowest diagonal $\mathrm{SU}(2)$ subgroup factor of (j_+, j_-)

$$\phi : k \to \left(\frac{1}{2}(1+\gamma)k, \frac{1}{2}|1-\gamma|k\right) \subset \mathrm{Irrep}(G)$$

- In the $\gamma < 1$ case, for a boundary state $\Psi(k_{ab}, \mathbf{n}_{ab})$:

$$A_\sigma(k_{ab}, \mathbf{n}_{ab}) = \int_{G^5} \prod_a dX_a^+ dX_a^- \prod_{a<b} \langle J\xi_{ab}, (X_a^+)^{-1} X_b^+ \, \xi_{ba}\rangle^{2j_{ab}^+} \times \langle J\xi_{ab}, (X_a^-)^{-1} X_b^- \, \xi_{ba}\rangle^{2j_{ab}^-}$$

- Rem: A_σ is an 'unbalanced' square of the 15j ($\gamma < 1$)

Amplitude for the Lorentzian EPRL model [Engle, Livine, Pereira, Rovelli - 08]

- Model based on $G = \mathrm{SL}(2, \mathbb{C})$,
  $\mathrm{Irrep}(G) = \{(n, p), \;\; n \in \mathbb{Z}/2, \; p \in \mathbb{R}\}$
- Idea: the boundary representation k is identified with the lowest $\mathrm{SU}(2)$ subgroup factor of the (n, p) representation

$$\phi : k \to (k, \gamma k) \subset \mathrm{Irrep}(G)$$

- For a boundary state $\Psi(k_{ab}, \mathbf{n}_{ab})$:

$$A_\sigma(k_{ab}, \mathbf{n}_{ab}) = \int_{G^5} \prod_a dX_a \, \delta(X_5) \prod_{a<b} P_{ab},$$

where

$$P_{ab} = c_{ab} \int_{\mathbb{CP}^1} \Omega_z \, \langle X_a^\dagger z, X_a^\dagger z \rangle^{-1-ip_{ab}-k_{ab}} \langle X_a^\dagger z, \xi_{ab} \rangle^{2k_{ab}} \times \langle X_b^\dagger z, X_b^\dagger z \rangle^{-1+ip_{ab}-k_{ab}} \langle J\xi_{ba}, X_b^\dagger z \rangle^{2k_{ab}},$$

with z in $\mathbb{C}^2$ and Ω_z the standard two-form on $\mathbb{C}^2 - \{0\}$

Asymptotic results

- Assumption: the boundary data is Regge-like and the phase of the boundary state is the canonical phase
- For large spins k:
  - If boundary data is that of an Euclidean 4-simplex σ_E:
    - Ooguri : $A_\sigma \sim ae^{iS_E} + be^{-iS_E}$
    - Euclidean EPRL : $A_\sigma \sim c\cos\gamma S_E + ae^{iS_E} + be^{-iS_E}$
    - Lorentzian EPRL : $A_\sigma \sim ae^{iS_E} + be^{-iS_E}$
  - If boundary data is that of a Lorentzian 4-simplex σ_L:
    - Ooguri : $A_\sigma \sim 0$
    - Euclidean EPRL : $A_\sigma \sim 0$
    - Lorentzian EPRL : $A_\sigma \sim ce^{i\gamma S_L} + c'e^{-i\gamma S_L}$
- The Regge action

$$S = \sum_{a<b} k_{ab}\Theta_{ab}, \quad \Theta_{ab} \text{ dihedral angle,}$$

is noted S_E (resp. S_L) for a simplex σ_E (resp. σ_L)

Stationary phase framework

- The 4-simplex amplitude can be re-expressed as

$$A_\sigma = \int_{(\mathrm{SL}(2,\mathbb{C}))^5} \delta(X_5) \prod_a dX_a \int_{(\mathbb{CP}^1)^{10}} \prod_{a<b} \Omega_{ab}\, e^S$$

- The action S for the asymptotic problem is given by

$$S[X,z] = \sum_{a<b} k_{ab} \ln \frac{\langle Z_{ab}, \xi_{ab}\rangle^2 \langle J\xi_{ba}, Z_{ba}\rangle^2}{\langle Z_{ab}, Z_{ab}\rangle \langle Z_{ba}, Z_{ba}\rangle} + i p_{ab} \ln \frac{\langle Z_{ba}, Z_{ba}\rangle}{\langle Z_{ab}, Z_{ab}\rangle},$$

where the notations Z_{ab} and Z_{ba} are used as a shorthand for

$$Z_{ab} = X_a^\dagger z_{ab} \quad \text{and} \quad Z_{ba} = X_b^\dagger z_{ab}, \quad \forall a < b$$

- The asymptotics of A_σ can therefore be studied using (extended) stationary phase methods

Critical points: I.

- The asymptotic formula is dominated by the critical points of S, i.e., stationary points for which $\mathrm{Re}\, S$ is a maximum
- The real part of the action is negative

$$\mathrm{Re}\, S = \sum_{a<b} k_{ab} \ln \frac{|\langle Z_{ab}, \xi_{ab}\rangle|^2 |\langle J\xi_{ba}, Z_{ba}\rangle|^2}{\langle Z_{ab}, Z_{ab}\rangle\langle Z_{ba}, Z_{ba}\rangle} \leq 0$$

- The maximality condition $\mathrm{Re}\, S = 0$ leads to one spinor equation for each triangle ab, $a < b$,

$$(X_a^\dagger)^{-1}\, \xi_{ab} = \frac{\| Z_{ba} \|}{\| Z_{ab} \|} e^{i\theta_{ab}} (X_b^\dagger)^{-1} J\, \xi_{ba}, \qquad (1)$$

where θ_{ab} is a phase, and $\| Z \|^2 = \langle Z, Z\rangle$

Critical points: II.

- The other critical point equations are obtained by evaluating the first variation of the action S w.r.t the variables (X, z) on the motion (1)
- For the spinor variables $(\bar{z}_{ab}, z_{ab})$, this leads to the equation

$$X_a \, \xi_{ab} = \frac{\| Z_{ab} \|}{\| Z_{ba} \|} e^{i\theta_{ab}} X_b \, J \, \xi_{ba}, \qquad (2)$$

  and its complex conjugate for each triangle ab
- For the $\mathrm{SL}(2, \mathbb{C})$ variables X_a, we obtain one equation for each tetrahedron a

$$\sum_{b: b \neq a} k_{ab} \mathbf{n}_{ab} = 0, \qquad (3)$$

  where $\mathbf{n} \in \mathbb{R}^3$ is the unit vector corresponding to the coherent state ξ (i.e. $\langle \xi, \mathbf{J} \xi \rangle = \frac{i}{2} \mathbf{n}$)

Geometry of the critical points: null vectors

- Idea : use the identification between spinors and null vectors
  - Let $\gamma : \mathbb{R}^{3,1} \rightarrow \mathbb{H}$ be the isomorphism between $\mathbb{R}^{3,1}$ and the space of 2×2 hermitian matrices $\mathbb{H}$ ($\det \gamma(x) = -\eta(x,x)$)
  - Call $\mathbb{H}_0^+$ the subset defined by

$$\mathbb{H}_0^+ = \{h \in \mathbb{H} \mid \det h = 0, \text{ and } \operatorname{Tr} h > 0\}$$

    The isomorphism γ identifies the future null cone C^+ with $\mathbb{H}_0^+$
  - Therefore, using

$$\zeta : \mathbb{C}^2 \rightarrow \mathbb{H}_0^+, \quad z \mapsto \zeta(z) = z \otimes z^\dagger,$$

    one can construct a map $\iota : \mathbb{C}^2 \rightarrow C^+ \subset \mathbb{R}^{3,1}$
- The map ι associates the two null vectors

$$\iota(\xi) = \frac{1}{2}(1, \mathbf{n}) \quad \text{and} \quad \iota(J\xi) = \frac{1}{2}(1, -\mathbf{n})$$

  to the coherent state ξ

Geometry of the critical points: bivectors

- To each ξ_{ab}, one can associate the space-like bivector

$$b_{ab} = 2 * \iota(J\xi_{ab}) \wedge \iota(\xi_{ab}) = * \begin{bmatrix} 0 & \mathbf{n}_{ab} \\ -\mathbf{n}_{ab} & 0 \end{bmatrix},$$

where the star $*$ is the Hodge operator on $\Lambda^2(\mathbb{R}^{3,1})$

- Construct ten space-like bivectors by rotating the b_{ab}'s :

$$B_{ab} = k_{ab}\, \hat{X}_a \otimes \hat{X}_a\, b_{ab}$$

- The critical point equations (1), (2), (3) reduce to

$$B_{ab} = -B_{ba} \quad \text{and} \quad \sum_{b:b\neq a} B_{ab} = 0$$

Bivectors constructed as such and satisfying these equations (almost) determine a geometric 4-simplex [Barrett, Crane - 98, 00]

The Regge action

- On all critical points, the action S yields

$$S = i \sum_{a<b} p_{ab} \ln \frac{\| Z_{ba} \|^2}{\| Z_{ab} \|^2} + 2k_{ab}\,\theta_{ab}$$

- The dihedral angle associated to the triangle ab is defined as

$$\cosh \Theta_{ab} := |N_a \cdot N_b| = \cosh\, r_{ab}, \quad e^{r_{ab}} = \frac{\| Z_{ba} \|^2}{\| Z_{ab} \|^2}$$

  where $N_a = X_a X_a^\dagger$
- The angle θ_{ab} vanishes with the canonical phase choice
- Thus, the action yields

$$S = i\gamma \sum_{a<b} k_{ab}\,\Theta_{ab} = i\gamma S_{\text{Regge}}$$

- Second term in asymptotic formula: parity related solution

Conclusion

- We have analysed the asymptotic properties of the Ooguri and EPRL models, in both Euclidean and Lorentzian signatures
- All 4-simplex amplitudes are asymptotic to functions of the Regge action
- Unexpected result: this is also the case for BF theory
- We need to go beyond one simplex and look at the asymptotics of the whole state sums