Sigma Models and Complex Geometry

Ulf Lindström

Theoretical Physics
Department of Physics and Astronomy
University of Uppsala
Sweden

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Outline

- Sigma models
- Supersymmetry
- Superspace
- SUSY sigma models
- Complex geometry
- Quotients
- Hyperkähler quotient

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- Generalized complex geometry
- Generalized Kähler geometry
- (2,2) superspace
- Sigma model description of GKG
- The role of K
- New vector multiplets
- T-duality
- (4,4)

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Collaborators

Tom Buscher,
Anders Karlhede,
Nigel Hitchin,
Chris Hull,
Martin Roček,
Rikard von Unge,
Maxim Zabzine,
Malin Göteman,
Itai Ryb

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Origin

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The Axial Vector Current in Beta Decay (*).

M. Gell-Mann (**)

Collège de France and Ecole Normale Supérieure - Paris (***)

M. Lévy

Faculte des Sciences, Orsay, and Ecole Normale Supérieure - Paris (***)

(ricevuto il 19 Febbraio 1960)

Summary. — In order to derive in a convincing manner the formula of Goldberger and Treiman for the rate of charged pion decay, we consider the possibility that the divergence of the axial vector current in β-decay may be proportional to the pion field. Three models of the pion-nucleon

Yet we have evidence that the weak interactions are symmetrical between V and A, particularly their apparent equality of strength and the fact that for the leptons, which have no strong couplings, the weak coupling is just $\gamma_\alpha(1+\gamma_5)$.

5. – The σ model.

We have another example of a theory in which eq. (5) holds, if we take a Lagrangian for the strong interactions that is essentially one proposed by SCHWINGER [16] and then for the axial vector current the form suggested by POLKINGHORNE [17].

Again, for simplicity, we restrict ourselves to nucleons and pions only, except that we introduce (following SCHWINGER) a new scalar meson σ, with isotopic spin zero. It has strong interactions, and thus might easily have escaped observation if it is much heavier than π, so that it would disintegrate immediately into two pions. It would appear experimentally as a resonant state of two pions with $J=0$, $I=0$.

We take for our Lagrangian the following one, which leads to a renormalizable theory of the strong interactions:

$$(36) \qquad \mathscr{L}_2 = -\bar{N}[\gamma\partial + m_0 - g_0(\sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi}\gamma_5)]N - \frac{(\partial\boldsymbol{\pi})^2}{2} - \frac{(\partial\sigma)^2}{2} - \frac{\mu_0^2\pi^2}{2} -$$

Sigma models

$$\phi^i : \Sigma \to \mathcal{T}$$

$$S = \int_\Sigma d\phi^i G_{ij}(\phi) \star d\phi^j$$

$$\nabla^2\phi^i := \partial^2\phi^i + \partial\phi^j \Gamma_{jk}{}^i \partial\phi^k = 0$$

$$S = \mu^{d-2} \int_{\Sigma_B} d\xi \left\{ \eta^{\mu\nu}\partial_\mu X^i G_{ij}(X) \partial_\nu X^j + \dots \right\} + \int_{\partial\Sigma} \dots .$$

i) The mass-scale μ shows that the model typically will be non-renormalizable for $d \geqslant 3$ but renormalizable and classically conformally invariant in $d = 2$.

ii) We have not included a potential for X and thus excluded Landau-Ginsburg models.

iii) There is also the possibility to include a Wess-Zumino term. We shall return to this when discussing $d = 2$

iv) From a quantum mechanical point of view it is useful to think of $G_{ij}(X)$ as an infinite number of coupling constants:

$$G_{ij}(X) = G^0_{ij} + G^1_{ij,k}X^k + \dots$$

v) Classically, it is more rewarding to emphasize the geometry and think of $G_{ij}(X)$ as a metric on the target space $\mathcal{T}$. This is the aspect we shall be mainly concerned with.

vi) The invariance of the action S under $Diff(\mathcal{T})$:

$$X^i \to X^{i'}(X)\ , \quad G_{ij}(X) \to G_{i'j'}(X')$$

(field-redefinitions from the point of view of the field theory on Σ), implies that the sigma model is defined by an equivalence class of metrics. N.B. This is not a symmetry of the model since the "coupling constants" also transform. It is an important property, however. Classically it means that the model is extendable beyond a single patch in $\mathcal{T}$, and quantum mechanically it is needed for the effective action to be well-defined.

Supersymmetry

The algebra depends on the dimension d and the number N of supersymmetries. In $d = 4$ we have

$$\{Q^a_\alpha, Q^b_\beta\} = 2\delta^{ab}(\gamma^\mu C)_{\alpha\beta}P_\mu + C_{\alpha\beta}Z^{ab} + (\gamma_5 C)_{\alpha\beta}Y^{ab}$$

Q^a are translation-invariant spinors that satisfy a Majorana reality condition and transform under some internal symmetry group $\mathcal{G} \subset U(N)$ (corresponding to the index a).

Weyl-spinors and $N = 1$:

$$\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = 2i\partial_{\alpha\dot\alpha}$$

Superspace

Just like, e.g., translations are represented by differential operators acting on functions on Minkowski space

$$\delta_P \Phi(x) = i[\xi^\mu P_\mu, \Phi(x)] \ ,$$

$$P_\mu = i\partial_\mu \ ,$$

supersymmetry transformations may be represented by differential operators acting on functions on superspace:

$$\delta_Q \Phi(x,\theta) = i[\epsilon^\alpha Q_\alpha + \bar{\epsilon}^{\dot\alpha} \overline{Q}_{\dot\alpha}, \Phi(x,\theta)]$$

where

$$Q_\alpha = i\partial_\alpha + \frac{1}{2}\bar{\theta}^{\dot\alpha}\partial_{\alpha\dot\alpha} \ , \quad \overline{Q}_{\dot\alpha} = i\partial_{\dot\alpha} + \frac{1}{2}\theta^\alpha \partial_{\alpha\dot\alpha} \ .$$

and $\partial_\alpha := \frac{\partial}{\partial\theta^\alpha}$.

Covariant derivatives

$$\{D_\alpha, \overline{D}_{\dot\alpha}\} = 2i\partial_{\alpha\dot\alpha}$$

$$\{D, Q\} = \{\bar{D}, Q\} = \{D, \bar{Q}\} = \{\bar{D}, \bar{Q}\} = 0$$

$$D_\alpha = \partial_\alpha + i\frac{1}{2}\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}\ , \quad \bar{D}_{\dot\alpha} = \partial_{\dot\alpha} + i\frac{1}{2}\theta^{\alpha}\partial_{\alpha\dot\alpha}\ ,$$

Using these we may impose covariant constraints. E.g., chirality:

$$\bar{D}_{\dot\alpha}\phi = 0 = D_\alpha\bar\phi\ .$$

SUSY sigma models

Ex. ($d = 2, N = (2,2)$ chiral fields)

$$\{D_\alpha, \bar{D}_\beta\} = 2i\partial_{\alpha\beta}$$

$$\phi(z) \to \phi(z,\theta):$$

$$X = \phi| \ , \quad \Psi_\alpha = D_\alpha\phi| \ , \quad F = D^2\phi|$$

$$S \to \int dz d\bar{z} D^2 \bar{D}^2 \ K(\phi, \bar{\phi})$$

$$= \int dz d\bar{z} (\partial X \ G_{X\bar{X}}(X, \bar{X}) \bar{\partial} \bar{X} + ...)$$

where

$$G_{X\bar{X}}(X,\bar{X}) = \partial_X \partial_{\bar{X}} K(X,\bar{X})$$

$\iff$ $\mathcal{T}$ carries Kähler Geometry

Susy σ models $\iff$ Geometry of $\mathcal{T}$

d=	6	4	2	Geometry
N=	1	2	4	Hyperkähler
N=		1	2	Kähler
N=			1	Riemannian

(Odd dimensions have the same structure as the even dimension lower.)

Complex Geometry

Complex structure: $J : TM \circlearrowleft \qquad J^2 = -1$

Projectors: $\pi_\pm := \frac{1}{2}\left(\mathbf{1} \pm iJ\right)$

Nijenhuis: $\mathcal{N}(J) = 0 \iff \pi_\mp[\pi_\pm u, \pi_\pm v] = 0$

Hermitean Metric: $J^t G J = G$

Kähler: $\nabla J = 0, \qquad G_{z\bar{z}} = \partial_z \partial_{\bar{z}} K(z, \bar{z})$

Hyperkähler: $J^A, A = 1, 2, 3 \quad J^A J^B = -\delta^{AB} + \epsilon^{ABC} J^C$

Hyperkähler

$$S = \int d^4 x D^2 \bar{D}^2 K(\phi, \bar{\phi})$$

Extra, non-manifest SUSY:

$$\delta\phi^i = \bar{D}^2(\bar{\varepsilon}\bar{\Omega}^i) \ , \quad \delta\bar{\phi}^{\bar{i}} = D^2(\varepsilon\Omega^{\bar{i}}) \ .$$

Invariance of the action and closure of the algebra (on-shell) iff the following $J^{(A)}$ represent a Hyperkähler geometry:

$$J^{(1)} = \begin{pmatrix} 0 & \Omega^{\bar{i}}{}_j \\ \bar{\Omega}^i{}_{\bar{j}} & 0 \end{pmatrix} \qquad J^{(2)} = \begin{pmatrix} 0 & i\Omega^{\bar{i}}{}_j \\ -i\bar{\Omega}^i{}_{\bar{j}} & 0 \end{pmatrix}$$

$$J^{(3)} = \begin{pmatrix} i\delta^i_j & 0 \\ 0 & -i\delta^{\bar{i}}_{\bar{j}} \end{pmatrix}$$

Quotient

Back to bosonic:

$$S = \int dx \partial_\mu \phi^i G_{ij}(\phi) \partial^\mu \phi^j \ .$$

$$\delta \phi^i = \lambda^A k_A(\phi) = [\lambda k, \phi]^i \equiv \mathcal{L}_{\lambda k} \phi^i \ .$$

Under such a transformation the action varies as

$$\delta S = \int dx \partial_\mu \phi^i \mathcal{L}_{\lambda k} G_{ij}(\phi) \partial^\mu \phi^j \ .$$

It is thus an invariance of the action if it is an isometry

$$\mathcal{L}_{\lambda k} G_{ij} = 0 \ .$$

Algebra

$$[k_A, k_B] = c_{AB}{}^C k_C \ , \quad k_A \equiv k_A^i \frac{\partial}{\partial \phi^i} \ .$$

We may gauge the isometry using minimal coupling

$$\partial_\mu \phi^i \to \partial_\mu \phi^i - A_\mu^A k_A^i = (\partial_\mu - A_\mu^A k_A)\phi^i \equiv \nabla_\mu \phi^i \ ,$$

$$S \to \int dx \nabla_\mu \phi^i G_{ij}(\phi) \nabla^\mu \phi^j \ .$$

Extremizing this action w.r.t. A yields a new sigma model on the space of orbits of the gauge group:

$$S = \int dx \partial_\mu \phi^i \left(G_{ij} - \mathbb{H}^{-1AB} k_{i_A} k_{j_B} \right) \partial^\mu \phi^j \ ,$$

where

$$\mathbb{H}_{AB} \equiv k_A^i G_{ij} k_B^j \ .$$

SUSY quotient

Such a quotient yields a new sigma model with a new target space. But we need to preserve additional structure such as SUSY.

Ex. Flat space, (i=1,2)

$$S = \int d^4x D^2 \bar{D}^2 K(\phi\bar{\phi}) = \int d^4x D^2 \bar{D}^2 \phi^i \bar{\phi}^i$$

Isometry:

$$\delta\phi^i = i\lambda\phi^i \, , \delta\bar{\phi} = -i\lambda\bar{\phi}^i$$

Gauged action:

$$S = \int d^4x D^2 \bar{D}^2 (\phi^i \bar{\phi}^i e^V - cV)$$

Extremizing:

$$V = -ln\left(\frac{\phi^i\bar{\phi}^i}{c}\right)$$

Quotient potential:

$$\tilde{K} = c\left(ln\left(\frac{\phi^i\bar{\phi}^i}{c}\right) + 1\right) = c\ ln(1 + \zeta\bar{\zeta}) + ... ,$$

where

$$\zeta = \phi^1/\phi^2 .$$

$\tilde{K}$ is the potential for the Fubini-Study metric on $\mathbb{CP}^1$.

In the example the Kähler geometry of the original model was inherited by the quotient geometry. It is much more difficult to preserve hyperkähler geometry. This requires gauging a tri-holomorphic isometry and performing a quotient which respect to the complexified action of this isometry. The latter point arises already in the Kähler quotient just illustrated: The general formula is

$$K(\phi,\bar{\phi}) \to \hat{K}(\phi,\bar{\phi},V) = K(\phi,\bar{\phi}) + \int_0^1 dt e^{(-\frac{1}{2}tJV)}\mu^V$$

(D) Hyperkähler Quotients

Suppose finally that M^{4n} is a *hyperkähler* manifold having a metric g and covariantly constant complex structures **I**, **J**, **K** which behave algebraically like quaternions:

$$\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = -1\,, \qquad \mathbf{IJ} = -\mathbf{JI} = \mathbf{K} \quad \text{etc.} \tag{3.25}$$

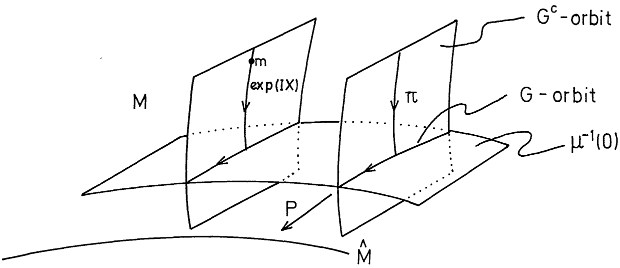

Fig. 3. The orbits of the group G and of its complexification G^C. G acts on $\mu^{-1}(0)$ and $\hat{M}$ is the quotient space corresponding to this action. The same space is obtained if one considers the extension of $\mu^{-1}(0)$ by $\exp(IX)$ and takes the quotient by G^C

Summary and conclusions part I

- Supersymmetric sigma models provide a powerful tool to probe complex geometry.
- The more supersymmetries, the more specialized geometry
- N=2 in d=4 has a hyperkähler target space.
- Gauging isometries and taking a quotient leads to new models.
- The hyperkähler reduction is suggested to us by superspace.
- Additional supersymmetries, when examined at the $(2,2)$ level, lead to interesting new structures on the target space.

Sigma models in d=2

The (1,1)-D-algebra:

$$\mathbb{D}_{\pm}^2 = i\partial_{\stackrel{+}{=}}$$

$$\mathcal{S} \quad = \quad \int d^2x \mathbb{D}_+\mathbb{D}_- \left(\mathbb{D}_+\varphi^i(G_{ij} + B_{ij})\mathbb{D}_-\varphi^j\right).$$

The (1,1) analysis by Gates Hull and Roček gives:

Susy	(0,0) (1,1)	(2,2)	(2,2)	(4,4)	(4,4)
Bgd	G, B	G	G, B	G	G, B
Geom	Riem.	Kähler	biherm.	hyperk.	bihyperc.

The (1,1)-D-algebra:

$$\mathbb{D}_{\pm}^2 = i\partial_{\pm\!\!\!\!\pm}$$

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Bgd	G, B	G	G, B	G	G, B
Geom	Riem.	Kähler	biherm.	hyperk.	bihyperc.

Ansatz for the extra supersymmetries:

$$\delta\varphi^i = \epsilon^+ J^i_{(+)j}\mathbb{D}_+\varphi^j + \epsilon^- J^i_{(-)j}\mathbb{D}_-\varphi^j$$

Invariance of the action and closure of the algebra requires the geometry to be bi-hermitean:

$$J^2_{(\pm)} = -\mathbf{1}$$

$$J^t_{(\pm)} G J_{(\pm)} = G$$

$$N(J_{(\pm)}) = 0$$

$$\nabla^{(\pm)} J^2_{(\pm)} = 0\ , \quad \Gamma^{(\pm)} = \Gamma^0 \pm G^{-1}H\ , \quad H := dB$$

$$H \simeq H J_{(\pm)} J_{(\pm)}$$

Generalized Complex Geometry

Complex structure:

$$\mathcal{J} : TM \oplus T^*M \circlearrowleft \qquad \mathcal{J}^2 = -1$$

$$\Pi_\pm := \frac{1}{2}(\mathbf{1} \pm \mathcal{J})$$

"Nijenhuis":

$$\mathcal{N}_C(\mathcal{J}) = 0 \quad \Longleftrightarrow \quad \Pi_\mp[\Pi_\pm u, \Pi_\pm v]_C = 0$$

where

$$u = (U, \xi) \ , \quad v = (V, \rho)$$

$$[u, v]_C = [U, V] + \mathcal{L}_U \rho - \mathcal{L}_V \xi - \frac{1}{2} d(\imath_U \rho - \imath_V \xi)$$

The automorphisms of this courant bracket are diffeomorphisms and *b*-transforms:

$$e^b(U, \xi) = (U, \xi + ib) \ , \quad db = 0 \ .$$

In a coordinate basis (∂_x, dx) a b-transform acts on $\mathcal{J}$ as follows:

$$\left(\begin{array}{cc} 1 & 0 \\ b & 1 \end{array}\right) \mathcal{J} \left(\begin{array}{cc} 1 & 0 \\ -b & 1 \end{array}\right) ,$$

In such a basis, the natural pairing

$$< (U, \xi), (V, \rho) >= \imath_U \rho + \imath_V \xi$$

is represented by the matrix

$$\mathcal{I} = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)$$

A final requirement ofn GCG is that

$$\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}$$

Generalized Kähler:

$$\exists \ (\mathcal{J}_1, \mathcal{J}_2) \ ; \ [\mathcal{J}_1, \mathcal{J}_2] = 0$$

$$\mathcal{G} = -\mathcal{J}_1 \mathcal{J}_2 \ , \quad \mathcal{G}^2 = 1$$

Ex. Kähler $(\omega = GJ)$:

$$\mathcal{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix} \quad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \quad \mathcal{G} = \begin{pmatrix} 0 & G^{-1} \\ G & 0 \end{pmatrix}$$

GKG ↔ Bi-Hermitean (the G-map):

$$\mathcal{J}^{(1,2)} =$$

$$\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} J^{(+)} \pm J^{(-)} & -(\omega^{-1}_{(+)} \mp \omega^{-1}_{(-)}) \\ \omega_{(+)} \mp \omega_{(-)} & -(J^{t(+)} \pm J^{t(-)}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}$$

General description of GKG

So Bi-Hermitean and GKG data are equivalent. But what is the most general description? Again, superspace has the answer.

The description should be $(2,2)$ symmetric, as we know from GHR. They found the complete description of $ker[J_{(+)}, J_{(-)}]$ but its complement was not described.

The kernel corresponds to the target space geometry of a sigma model with chiral and twisted chiral $(2,2)$-superfields. The complement is coordinatized by semi-chiral fields.

(2,2) superspace

The (2,2)-D-algebra:

$$\{D_\pm, \bar{D}_\pm\} = 2i\partial_{\pm\!\!\!\!=}$$

Reduction to (1,1):

$$\mathbb{D}_\pm := \frac{1}{\sqrt{2}}\left(D_\pm + \bar{D}_\pm\right)$$
$$\mathbb{Q}_\pm := \frac{i}{\sqrt{2}}\left(D_\pm - \bar{D}_\pm\right)$$

The (1,1)-D-algebra:

$$\mathbb{D}^2_\pm = i\partial_{\pm\!\!\!\!=}$$

(2,2) superfields

Chiral fields ϕ:

$$\bar{D}_\pm \phi = 0 \Rightarrow D_\pm \bar{\phi} = 0$$

Twisted chiral fields χ:

$$\bar{D}_+ \chi = D_- \chi = 0 \Rightarrow D_+ \bar{\chi} = \bar{D}_- \bar{\chi} = 0$$

Left/Right semi-chiral fields $\mathbb{X}_{L/R}$:

$$\bar{D}_+ \mathbb{X}_L = 0 \Rightarrow D_+ \bar{\mathbb{X}}_L = 0$$

$$\bar{D}_- \mathbb{X}_R = 0 \Rightarrow D_- \bar{\mathbb{X}}_R = 0$$

These are all the fields needed.

Complex linear fields Σ_ϕ:

$$\bar{D}_+\bar{D}_-\Sigma_\phi = 0 \Rightarrow D_+D_-\bar{\Sigma}_\phi = 0$$

Dual to chiral fields

Complex twisted linear fields Σ_χ:

$$\bar{D}_+D_-\Sigma_\chi = 0 \Rightarrow D_+\bar{D}_-\bar{\Sigma}_\chi = 0$$

Dual to twisted chiral fields

N=(1,1) content

Define:

$$J := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Chiral fields:

$$\Phi := \begin{pmatrix} \phi \\ \bar{\phi} \end{pmatrix} \Rightarrow \mathbb{Q}_\pm \Phi = J \mathbb{D}_\pm \Phi$$

Twisted chiral fields:

$$\chi := \begin{pmatrix} \chi \\ \bar{\chi} \end{pmatrix} \Rightarrow \mathbb{Q}_\pm \chi = \pm J \mathbb{D}_\pm \chi$$

Read off the non-manifest second susy by projecting to the θ_2 independent part.

Semi-chiral fields:

$$X_{L/R} := \mathbb{X}_{L/R}|, \quad \psi_{L-/R+} := \mathbb{Q}_{\mp}\mathbb{X}_{L/R}|$$

$$\mathbf{X}_{L/R} := \begin{pmatrix} X_{L/R} \\ \bar{X}_{L/R} \end{pmatrix}, \quad \Psi_{L-/R+} := \begin{pmatrix} \psi_{L-/R+} \\ \bar{\psi}_{L-/R+} \end{pmatrix}$$

$$\mathbb{Q}_{+}\mathbf{X}_{L} = J\mathbb{D}_{+}\mathbf{X}_{L}, \quad \mathbb{Q}_{-}\mathbf{X}_{R} = J\mathbb{D}_{-}\mathbf{X}_{R}$$

and

$$\mathbb{Q}_{+}\Psi_{L-} = J\mathbb{D}_{+}\Psi_{L-}, \quad \mathbb{Q}_{-}\Psi_{L-} = -i\partial_{=}\mathbf{X}_{L}$$

$$\mathbb{Q}_{-}\Psi_{R+} = J\mathbb{D}_{-}\Psi_{R+}, \quad \mathbb{Q}_{+}\Psi_{R+} = -i\partial_{+\!\!+}\mathbf{X}_{R}$$

The Ψ's are auxiliary fermions.

Relation to GKG

★

$$\begin{aligned}\mathcal{S} &= \int d^2x D^2 \bar{D}^2 K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_{L/R}, \bar{\mathbb{X}}_{L/R}) \\ &\to \int d^2x \left(\partial_{+\!\!+} \varphi^i (G_{ij} + B_{ij}) \partial_{=} \varphi^j + ...\right) .\end{aligned}$$

In $(1,1)$:

$$\delta_\Psi \mathcal{S} = 0 : \quad \Rightarrow (J_{(\pm)},\ G, H = dB) \ ,$$

$$J^2_{(\pm)} = -1, \qquad N(J) = 0, \quad [J_{(+)}, J_{(-)}] \neq 0 \ ,$$

$$J^t_{(\pm)} G J_{(\pm)} = G, \quad H = d^c_{(+)} \omega_{(+)} = -d^c_{(-)} \omega_{(-)}$$

A complete description of GKG.

The dependence on the generalized Kähler potential is non-linear (for simplicity consider semi-schiral fields only): E.g.,

$$J_{(+)} = \begin{pmatrix} j & 0 \\ K^{RL}C_{LL} & K^{RL}jK_{LR} \end{pmatrix}, J_{(-)} = \begin{pmatrix} K^{LR}jK_{RL} & K^{RL}C_{RR} \\ 0 & j \end{pmatrix}$$

where

$$j := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad C := [j, K], \quad K^{RL}K_{LR} = \mathbf{1} .$$

Only a symplectic form Ω depends linearly on the Hessian of K:

$$\Omega = \begin{pmatrix} 0 & K_{LR} \\ -K_{RL} & 0 \end{pmatrix} .$$

The metric and B-field depend non-linearly:

$$G = \Omega[J_{(+)}, J_{(-)}] , \quad B = \Omega\{J_{(+)}, J_{(-)}\}$$

Generating function

There are two special sets of Darboux coordinates for the symplectic form Ω. One set, $(\mathbb{X}^L, \mathbb{Y}_L)$, is also canonical coordinates for $J_{(+)}$ and the other set, $(\mathbb{X}^R, \mathbb{Y}_R)$ is canonical coordinates for $J_{(-)}$. The symplectomorphism that relates the two sets of coordinates has thus a generating function. This generating function is in fact the generalized Kähler-potential $K(\mathbb{X}^L, \mathbb{X}^R)$.

$(\mathbb{X}^L, \mathbb{Y}_L)$	$\leftarrow K(\mathbb{X}^L, \mathbb{X}^R) \rightarrow$	$(\mathbb{X}^R, \mathbb{Y}_R)$
$J_{(+)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$		$J_{(-)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

This fact is a key ingredient in the proof that we have a complete description or GKG.

The roles of K

- $K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_{L/R}, \bar{\mathbb{X}}_{L/R})$ is the superspace Lagrangian for a $(2,2)$ sigma model with Generalized Kähler target space geometry.
- $K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_{L/R}, \bar{\mathbb{X}}_{L/R})$ is generalized Kähler potential for the metric G and B-field
- For fixed chiral and twisted chiral fields, $K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_{L/R}, \bar{\mathbb{X}}_{L/R})$ generates symplectomorphisms.

New vector multiplets

$$K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_{L/R}, \bar{\mathbb{X}}_{L/R})$$

(Abelian) Isometries:

$$k_\phi = i(\partial_\phi - \partial_{\bar{\phi}})$$

$$k_{\phi\chi} = i(\partial_\phi - \partial_{\bar{\phi}} - \partial_\chi + \partial_{\bar{\chi}})$$

$$k_{LR} = i(\partial_L - \partial_{\bar{L}} - \partial_R + \partial_{\bar{R}})$$

The corresponding gauged Lagrangians:

$$K_\phi(\phi + \bar\phi + V^\phi, x)$$

$$K_{\phi\chi}(\phi + \bar\phi + V^\phi, \chi + \bar\chi + V^\chi, i(\phi - \bar\phi + \chi - \bar\chi) + V', x)$$

$$K_{\mathbb{X}}(\mathbb{X}_L + \bar{\mathbb{X}}_L + \mathbb{V}^L, \mathbb{X}_R + \bar{\mathbb{X}}_R + \mathbb{V}^R, i(\mathbb{X}_L - \bar{\mathbb{X}}_L + \mathbb{X}_R - \bar{\mathbb{X}}_R) + \mathbb{V}', x)$$

with gauge transformations for the vectors;

$$\delta V^\phi = i(\bar\Lambda - \Lambda)$$

$$\delta V^\chi = i(\bar{\tilde\Lambda} - \tilde\Lambda)$$

$$\delta V' = \bar\Lambda + \Lambda + \bar{\tilde\Lambda} + \tilde\Lambda$$

$$\delta \mathbb{V}^{L/R} = i(\bar\Lambda_{L/R} - \Lambda_{L/R})$$

$$\delta \mathbb{V}' = \bar\Lambda_L + \Lambda_L + \bar\Lambda_R + \Lambda_R$$

The invariant field strengths are the usual ones

$$W = iD_-\bar{D}_+V^\phi, \quad \bar{W} = i\bar{D}_-D_+V^\phi$$

$$\tilde{W} = i\bar{D}_-\bar{D}_+V^\chi, \quad \bar{\tilde{W}} = iD_-D_+V^\chi$$

and the new

$$\mathbb{F} = \frac{1}{2}\bar{D}_+\bar{D}_-(\mathbb{V}' + i(\mathbb{V}^L + \mathbb{V}^R))$$

$$\tilde{\mathbb{F}} = \frac{1}{2}\bar{D}_+D_-(\mathbb{V}' + i(\mathbb{V}^L - \mathbb{V}^R))$$

$$\mathbb{G}_+ = \frac{1}{2}\bar{D}_+(V' + i(V^\phi + V\chi)) := \frac{1}{2}\bar{D}_+V$$

$$\mathbb{G}_- = \frac{1}{2}\bar{D}_-(V' + i(V^\phi - V^\chi)) := \frac{1}{2}\bar{D}_-\tilde{V}$$

Reduction to (1,1). Non-abelian extensions. Applied to T-duality.

T-duality

$$K_{\phi\chi}(\phi + \bar{\phi} + V^{\phi}, \chi + \bar{\chi} + V^{\chi}, i(\phi - \bar{\phi} + \chi - \bar{\chi}) + V')$$

$$-\frac{1}{2}\mathbb{X}_L V - \frac{1}{2}\bar{\mathbb{X}}_L \bar{V} - \frac{1}{2}\mathbb{X}_R \tilde{V} - \frac{1}{2}\bar{\mathbb{X}}_R \bar{\tilde{V}}$$

$\delta\mathbb{X}_{L,R}$:

$\Rightarrow$ V and $\tilde{V}$ pure gauge. Plug back to find $K_{\phi\chi}(\phi, \bar{\phi}, \chi, \bar{\chi})$

$\delta V, \delta\tilde{V}$:

$\Rightarrow \partial_V K_{\phi\chi} = \mathbb{X}_L$ etc. Solve to give $V(\mathbb{X}_{L,R}, \bar{\mathbb{X}}_{L,R})$,......

Plug back to find $\hat{K}(\mathbb{X}_{L,R}, \bar{\mathbb{X}}_{L,R})$

A similar relation starting from the gauged semi-chiral action also displays the duality between (twisted) chiral and semi-chiral models.

Additional supersymmetry

The chiral sector, same as described for $d = 4$ above:

$$K \to K(\phi, \bar{\phi})$$

$$\delta\phi^a = \bar{\epsilon}^\alpha \bar{D}_\alpha \Omega^a(\phi, \bar{\phi}), \quad \delta\bar{\phi}^{\bar{a}} = \epsilon^\alpha D_\alpha \bar{\Omega}^{\bar{a}}(\phi, \bar{\phi})$$

On-shell algebra.

$$J^{(3)i}{}_j = \begin{pmatrix} i\delta^a_b & 0 \\ 0 & -i\delta^{\bar{a}}_{\bar{b}} \end{pmatrix},$$

$$J^{(1)i}{}_j = \begin{pmatrix} 0 & \Omega^a_{\bar{b}} \\ \bar{\Omega}^{\bar{a}}_b & 0 \end{pmatrix}, \quad J^{(2)i}{}_j = \begin{pmatrix} 0 & -i\Omega^a_{\bar{b}} \\ i\bar{\Omega}^{\bar{a}}_b & 0 \end{pmatrix}$$

The semi-sector:

$$K \to K(\mathbb{X}_L, \mathbb{X}_R, \bar{\mathbb{X}}_L, \bar{\mathbb{X}}_R)$$

The general situation not known at the $(2,2)$ level.

Linear tf:

$$\delta\mathbb{X}_L = i\bar{\epsilon}^+\bar{\mathbb{D}}_+(\bar{\mathbb{X}}_L + \mathbb{X}_R + \tfrac{1}{\kappa}\bar{\mathbb{X}}_R) + i\kappa\bar{\epsilon}^-\bar{\mathbb{D}}_-\mathbb{X}_L + i\kappa^{-1}\epsilon^-\mathbb{D}_-\mathbb{X}_L,$$

$$\begin{aligned}\delta\mathbb{X}_R = {} & i\bar{\epsilon}^-\bar{\mathbb{D}}_-(\bar{\mathbb{X}}_R - (|\kappa|^2 - 1)\mathbb{X}_L + \tfrac{|\kappa|^2-1}{\bar{\kappa}}\bar{\mathbb{X}}_L) - i\bar{\kappa}\bar{\epsilon}^+\bar{\mathbb{D}}_+\mathbb{X}_R \\ & - i\bar{\kappa}^{-1}\epsilon^+\mathbb{D}_+\mathbb{X}_R,\end{aligned}$$

Invariance:

$$\begin{aligned} K_{1\bar{1}} - K_{12} - \kappa K_{\bar{1}2} &= 0, \\ (|\kappa|^2 - 1)K_{2\bar{2}} + K_{12} - \bar{\kappa}K_{1\bar{2}} &= 0. \end{aligned}$$

$$
\begin{aligned}
&\{J_+, J_-\} = 2c\ ,\\
&\Longleftrightarrow\\
&(1+c)|K_{12}|^2 + (1-c)|K_{1\bar{2}}|^2 = 2K_{1\bar{1}}K_{2\bar{2}}.
\end{aligned}
$$

Using the invariance condition we find:

$$
\boxed{c = -\frac{|\kappa|^2+1}{|\kappa|^2-1}}
$$

Since $c^2 > 1$ is a constant we can form the following two local product structures:

$$
\begin{aligned}
S &:= \frac{1}{\sqrt{c^2-1}}(J_- + cJ_+), \qquad && S^2 = 1\ ,\\
T &:= \frac{1}{2\sqrt{c^2-1}}[J_+, J_-], \qquad && T^2 = 1\ ,
\end{aligned}
$$

such that the commutator algebra of (J_+, S, T) is $SL(2, \mathbb{R})$.

The structures (J_+, S, T) preserve a metric of signature $(2,2)$ and this geometry of the target space is called neutral hypercomplex.

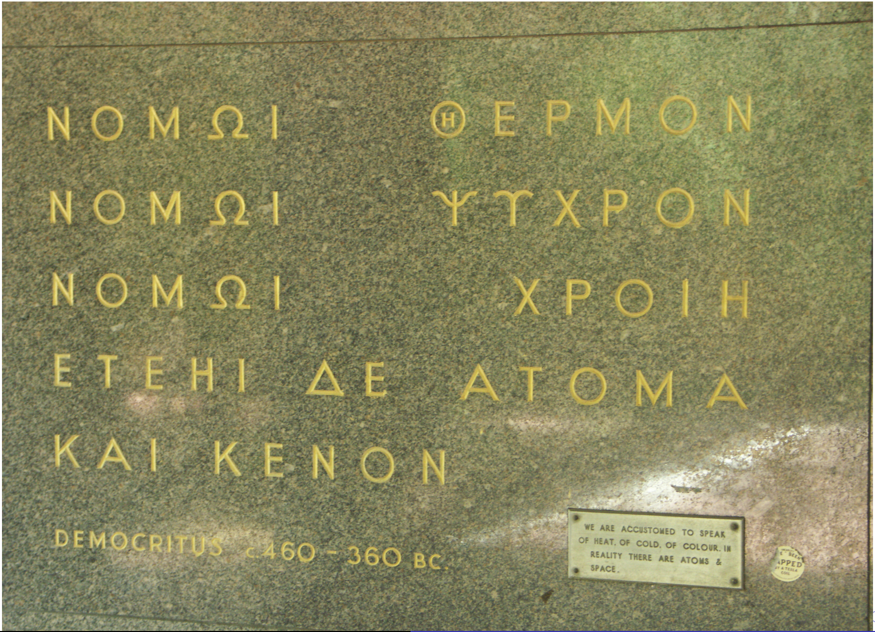
When $c^2 < 1$, the corresponding construction yields a triple of complex structures, the metric is positive definite and the geometry hyperkähler.

The general case is presently under investigation, i.e., 2d-dimensions and non-linear transformations. We do not expect that it will give a constant c, but it seems to have other interesting geometric properties related to Yano f-structures $f^3 + f = 0$.

Summary and conclusions part II

• A complete description of GKG uses chiral, twisted chiral and semi-chiral superfields.
• The generalized Kähler potential doubles as a (non-linear) potential for the metric and B-field and as a generating function of symplectomorphisms.
• New vector-multiplets are available for gauging an important class of isometries.
• T-duality and quotients may be discussed in terms of these multiplets.
• Global issues (bi-holomorphic gerbes...) can be addressed.
• Additional supersymmetries, when examined at the $(2,2)$ level, lead to interesting new structures on the target space.

Telos


ΝΟΜΩΙ ΘΕΡΜΟΝ
ΝΟΜΩΙ ΨΥΧΡΟΝ
ΝΟΜΩΙ ΧΡΟΙΗ
ΕΤΕΗΙ ΔΕ ΑΤΟΜΑ
ΚΑΙ ΚΕΝΟΝ
DEMOCRITUS c.460 - 360 BC.
WE ARE ACCUSTOMED TO SPEAK OF HEAT, OF COLD, OF COLOUR. IN REALITY THERE ARE ATOMS & SPACE.

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Back to the chiral complex linear duality:

$$
\begin{aligned}
\mathcal{S} &= \int d^2x D^2 \bar{D}^2 K(\Sigma, \bar{\Sigma}) \\
\to \tilde{\mathcal{S}} &= \int d^2x D^2 \bar{D}^2 \left(K(S, \bar{S}) - \phi S - \bar{\phi} \bar{S} \right)
\end{aligned}
$$

$$
\begin{aligned}
&\delta_\phi \tilde{\mathcal{S}} = \delta_{\bar{\phi}} \tilde{\mathcal{S}} = 0 \Rightarrow \bar{D}_+ \bar{D}_- S = 0, D_+ D_- S = 0 \\
&\Rightarrow S = \Sigma, \bar{S} = \bar{\Sigma}, \quad \tilde{\mathcal{S}} \to \mathcal{S}
\end{aligned}
$$

$$
\begin{aligned}
&\delta_S \tilde{\mathcal{S}} = \delta_{\bar{S}} \tilde{\mathcal{S}} = 0 \iff K_S = \phi, K_{\bar{S}} = \bar{\phi} \\
&\Rightarrow K - \phi S - \bar{\phi} \bar{S} = \hat{K}(\phi, \bar{\phi})
\end{aligned}
$$