Noncommutative Black Holes

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Introduction/Motivation

Quantum Black Holes

- ideal theoretical laboratory for physics beyond QFT/GR
- challenges: information paradox, entropy, holography, singularities, artificial cutoffs, ...
- need to reconsider cherished physical principles: locality, unitarity, Lorentz invariance, ...

$\rightarrow$ parallels to discovery of quantum mechanics

Quantum/Noncommutative Spactime

- model of quantum geometry, spacetime uncertainty
- controlled LI violation and non-locality, UV/IR mixing
- mixing of internal & spacetime symmetries
- holographic properties natural

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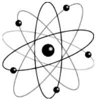
Introduction/Motivation

Spacetime noncommutativity



Heuristic argument: quantum + gravity

$\rightarrow$ fundamental lengthscale, spacetime uncertainty

$$\Delta x \geq \sqrt{\frac{\hbar G}{c^3}}$$

$\leftrightarrow$ noncommutative spacetime structure

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}(x)$$

("first quantized" geometry)

But: obvious problems with spacetime symmetries

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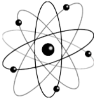

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Star product

General x-dependent NC structure:

$$f \star g = f \cdot g + \frac{i}{2} \sum \theta^{ij}\, \partial_i f \cdot \partial_j g - \frac{\hbar^2}{4} \sum \theta^{ij} \theta^{kl}\, \partial_i \partial_k f \cdot \partial_j \partial_l g$$
$$- \frac{\hbar^2}{6} \Big(\sum \theta^{ij} \partial_j \theta^{kl}\, \partial_i \partial_k f \cdot \partial_l g - \partial_k f \cdot \partial_i \partial_l g \Big) + \dots$$

ditto on coordinates:

$$[x^i \overset{\star}{,} x^j] = i\theta^{ij}$$

need to generalize star product, deform symmetry

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Noncommutative Gravity

- simple model of quantum spacetime, captures features of quantum geometry/gravity
- has deformed analog of diffeomorphism symmetry
- fuzzy black hole solutions and cosmological models; toy models to study quantum gravitational effects

Other Approaches

Spectral action, specific models, phenomenology, matrix models; strings, branes, spins, foams, loops. . .

Noncommutative Gravity and Quantum Geometry

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Twisted tensor calculus

- Tensors must be star-multiplied (generalized star product!)
- The transformation of individual tensors is undeformed
- The Leibniz rule is deformed (e.g. via Drinfel'd twist)

Covariant derivative

$$D_\mu \star V_\nu = \partial_\mu V_\nu - \Gamma^\alpha_{\mu\nu} \star V_\alpha$$

Curvature and torsion

$$[D_\mu \stackrel{\star}{,} D_\nu] \star V_\rho = V_\sigma \star R^\sigma{}_{\rho\mu\nu} + T_{\mu\nu}{}^\alpha \star D_\alpha \star V_\rho$$

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Connection and metric

Metric compatibility and

$$(G_{\mu\nu})^* = G_{\nu\mu}\,, \qquad (\Gamma^{\alpha}_{\mu\nu})^* = \Gamma^{\alpha}_{\nu\mu}$$

fixes the connection in terms of the metric:

$$\Gamma^{\sigma}_{\alpha\beta} \star G_{\sigma\gamma} + G_{\gamma\sigma} \star \Gamma^{\sigma}_{\alpha\beta} = \partial_\alpha G_{\beta\gamma} + \partial_\beta G_{\gamma\alpha} - \partial_\gamma G_{\alpha\beta}$$

Riemann and Ricci tensors

$$R^{\sigma}{}_{\rho\nu\mu} = \partial_\nu \Gamma^{\sigma}_{\mu\rho} - \partial_\mu \Gamma^{\sigma}_{\nu\rho} + \Gamma^{\beta}_{\nu\rho} \star \Gamma^{\sigma}_{\mu\beta} - \Gamma^{\beta}_{\mu\rho} \star \Gamma^{\sigma}_{\nu\beta}\,, \qquad R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$$

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Kinematics

Geodesic equation

For hermitean (or real symmetric) connection coefficients

$$\frac{du^\gamma}{d\lambda} = u^\alpha \star \Gamma^\gamma_{\alpha\beta} \star u^\beta \qquad u^\alpha = \frac{dx^\alpha}{d\lambda}$$

Interpretation: Heisenberg-type equations for operators u^α, x^α

trajectories $\rightarrow$ transition amplitudes

Alternative: Path integral (generalizes variational approach)

More generaly: Fields on NC spacetime

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Dynamics

Deformed Einstein Equations

For "noncommutative sources" $T_{\mu\nu}$

$$R_{\mu\nu}(G, \star) = T_{\mu\nu} - \frac{1}{2} G_{\mu\nu} T$$

Note: Ordering ambiguities disappear in known solutions.

Solutions?

Solution = mutually compatible pair of

- an algebra (with twist) $\star$
- a metric $G_{\mu\nu}$

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Noncommutative Schwarzschild Black Hole

Vacuum solution (origin excluded), i.e.

$$R_{\mu\nu} = 0$$

Spherical symmetry via Killing vectors

$$[\xi_i, \xi_j] = i\epsilon_{ijk}\xi_k \qquad \mathcal{L}_{\xi_i} g^{\mu\nu} = 0$$

NC Schwarzschild Solution

Compatible metric (in isotropic coordinates)

$$ds^2 = -A(\rho)dt^2 + B(\rho)(dx^2 + dy^2 + dz^2) + C(\rho)d\rho^2$$

$$\rho^2 = g_{ij}x^i x^j = x^2 + y^2 + z^2$$

Compatible algebra

$$[x_i \stackrel{\star}{,} x_j] = 2i\lambda\epsilon_{ijk}x_k$$

Note:

- λ can be function of ρ (central), $a = 2M$
- Star product acts nontrivially on tensors

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NC Schwarzschild Solution

Star product

on functions:

$$f \star g = fg + \sum_{n=1}^{\infty} C_n(\frac{\lambda}{\rho}) {\xi_+}^n f \, {\xi_-}^n g$$

with left-invariant (Killing) vector fields $\xi_\pm = \xi_1 \pm i\xi_2$

$$\begin{aligned} C_n(\frac{\lambda}{\rho}) &= B(n, \frac{\rho}{\lambda}) \\ &= \frac{\lambda^n}{n! \, \rho(\rho-\lambda)(\rho-2\lambda)\cdots(\rho-(n-1)\lambda)} \end{aligned}$$

Grosse, Presnajder; Alekseev, Lachowska; Kürkcüoglu, Sämann

NC Schwarzschild Solution

Star product

on tensors:

$$V \star W = VW + \sum_{n=1}^{\infty} C_n(\frac{\lambda}{\rho}) \mathcal{L}^n_{\xi_+} V \, \mathcal{L}^n_{\xi_-} W$$

Metric and algebra are compatible

$$T^{\alpha\ldots\omega} \star g^{\mu\nu} = T^{\alpha\ldots\omega} g^{\mu\nu} = g^{\mu\nu} \star T^{\alpha\ldots\omega}$$

furthermore

$$T^{\alpha\ldots\omega} \star \partial_{\sigma_1} \ldots \partial_{\sigma_k} g^{\mu\nu} = T^{\alpha\ldots\omega} \partial_{\sigma_1} \ldots \partial_{\sigma_k} g^{\mu\nu} = \partial_{\sigma_1} \ldots \partial_{\sigma_k} g^{\mu\nu} \star T^{\alpha\ldots\omega}$$

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NC Schwarzschild Solution

Metric

The metric is now determined using standard methods.
In isotropic coordinates:

$$ds^2 = -\left(1 - \frac{a}{\rho}\right) dt^2 + \frac{r^2}{\rho^2} \left(dx^2 + dy^2 + dz^2\right)$$

where

$$r = (\rho + a/4)^2/\rho, \quad a = 2M$$

and

$$\rho^2 = g_{ij} x^i x^j = x^2 + y^2 + z^2$$

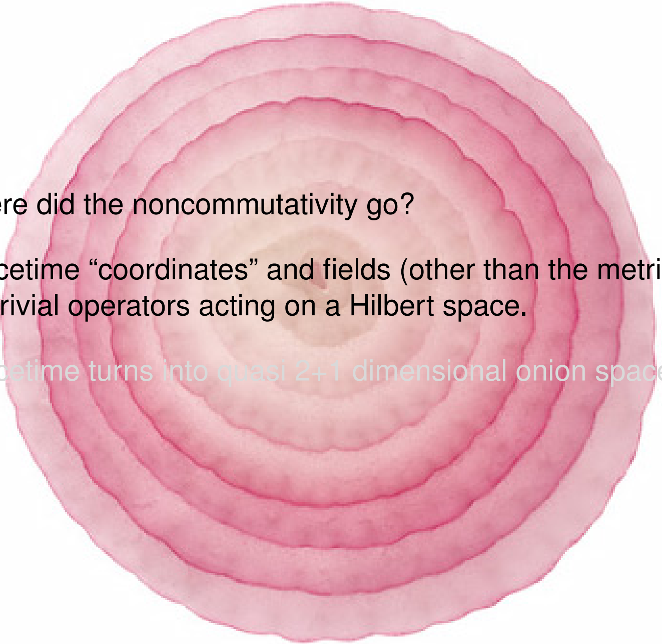
Onion Spacetime

Where did the noncommutativity go?

Spacetime “coordinates” and fields (other than the metric) are nontrivial operators acting on a Hilbert space.

Spacetime turns into quasi 2+1 dimensional onion spacetime

Onion Spacetime



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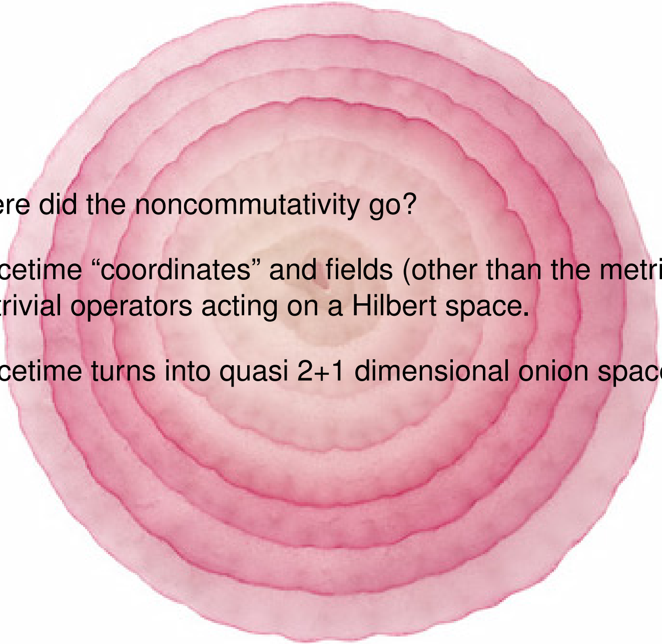

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Unitary representations

Coordinate measurements should give “real” results
$\Rightarrow$ Consider unitary representations of

$$[x_i \overset{\star}{,} x_j] = 2i\lambda\epsilon_{ijk}x_k$$

(i.e. choose an appropriate class of functions)

$$(\vec{x})^{\star 2}|j, m\rangle = (2\lambda)^2 j(j+1)|j, m\rangle, \quad 2j = 0, 1, 2, \ldots$$

Onion Spacetime

Ditto, in terms of

$$\rho^2 = g_{ij} x^i x^j = x^2 + y^2 + z^2 \ :$$

using

$$(\vec{x})^{\star 2} \equiv \sum x_i \star x_i = \rho(\rho + 2\lambda)$$

we get:

$$\boxed{\rho = 2j\lambda = n\lambda; \quad n = 0, 1, 2, \dots}$$

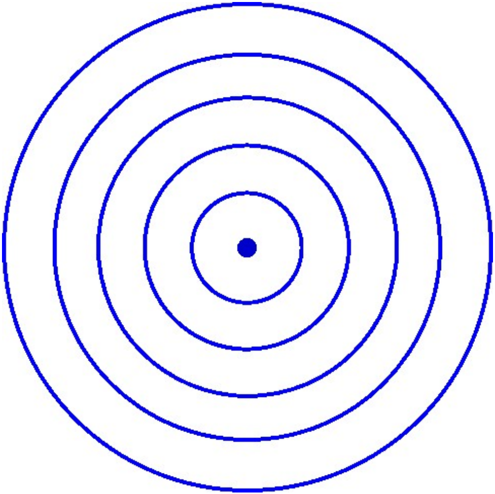
Recall: $\lambda = \lambda(\rho, a)$ in general, but restricted by physics.
e.g. $\lambda(a, a) \sim l_p^2/a$.

Outside horizon condition:

$$\rho > a/4 \ \Rightarrow n > a/4\lambda$$

NC Schwarzschild Spacetime

(schematically, in isotropic coordinates)



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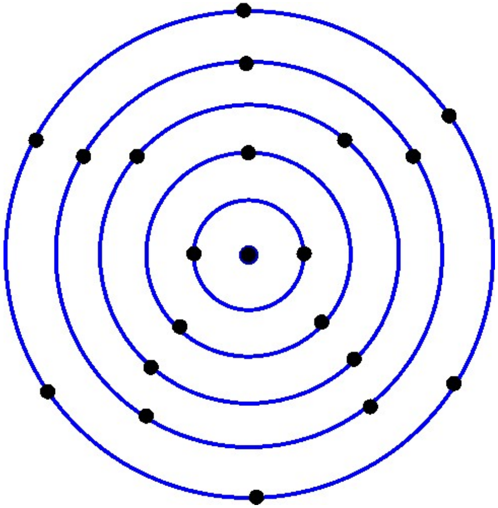

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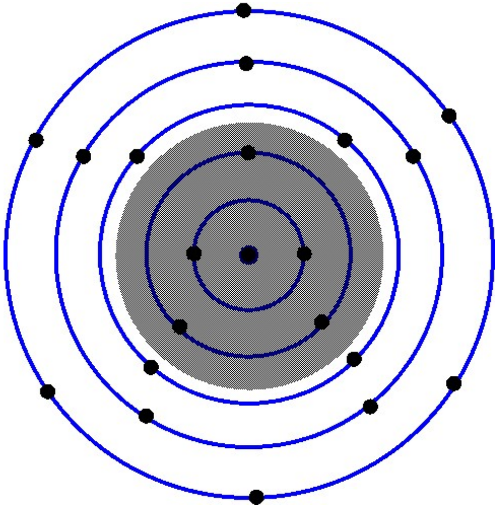

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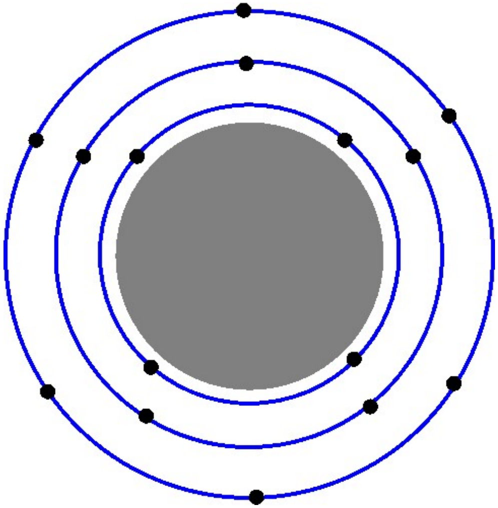
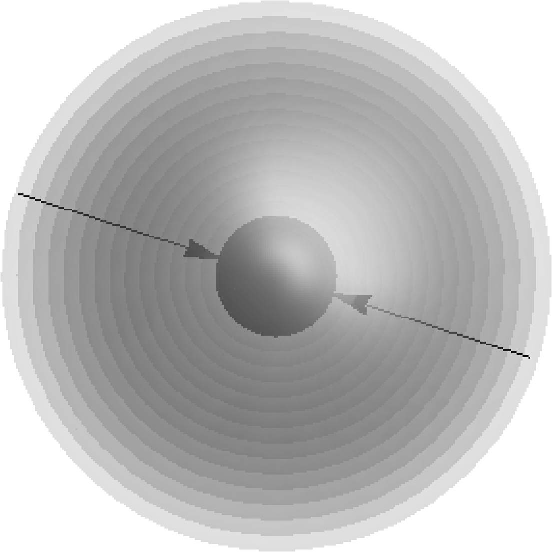


Hilbert Space



Hilbert Space

Hilbert Space

Outside event horizon

Sum over representations on “onion shells”:

$$\mathcal{H} = \bigoplus_{n>N} \mathbb{C}^{n+1} \qquad N \sim a/4\lambda_0$$

Here: $n = 2j$ and $\mathbb{C}^{2j+1} \equiv [j]$

“Inside” event horizon

Hidden by the event horizon are all states with $n \leq N$:

$$\mathcal{H}_{\text{hidden}} = [0] \oplus [\frac{1}{2}] \oplus \ldots \oplus [j_{\text{max}}]$$

That is equal to $([J] \otimes [J]) \oplus ([J] \otimes [J \mp \frac{1}{2}])$,
i.e. a scalar plus a spinorial function (field) on a fuzzy sphere.

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Hilbert Space

$L_2(\mathbb{R}^3) \to L_2(S_2)$

We find the following surprising result:
States describing the location of an event in the 3 dim NC bulk are equivalent to "ordinary" wave functions on a sphere (minus a fuzzy sphere)

$$\mathcal{H} = \bigoplus_{n>N} \mathbb{C}^{n+1} = \bigoplus \mathbb{C}^{n+1} - H_{\text{hidden}}$$

$\Rightarrow$ NC Schwarzschild = "Fuzzy Black Hole" ($\to$ Brian Dolan)
Holographic behavior appears quite naturally

NC: bulk (3D) $\to$ surface (2D)

Heuristically:
(1) coordinates are no longer independent: $z \sim [x, y]$
(2) number of commuting operators = two

Fuzzy Event Horizon

Area quantization

The fuzzy sphere that represents the states hidden by the event horizon has N "cells" ($N \equiv n_{\text{min}}$). This result can be obtained by either by counting states or by the uncertainty principle.

An equidistant spectrum of the area operator was originally conjectured by Bekenstein and Mukhanov.

Mass quantization?

The parameter $a = 2M$ is not necessarily discrete in our model. But $TdS = dM$?!

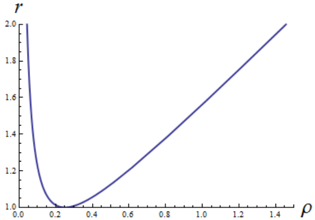
There are indications of a modification of the Planck black body radiation spectrum for small quantum numbers.

NC Schwarzschild Solution

NC Schwarzschild BH in Schwarzschild coordinates

Change of coordinates

$$r = (\rho + a/4)^2/\rho, \quad a = 2M$$



(the coordinates are still quantized!)

horizon: $r = a \;\Leftrightarrow\; \rho = a/4$
$\rho < a/4$: second copy of Schwarzschild spacetime

Inside NC black hole I

Inside the fuzzy black hole

Constant time slices are now de Sitter.
$\rightarrow$ introduce auxiliary coordinate

$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$$

and constraint

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 = \alpha^2 > 0$$

Now quantize x^1, x^2, x^3 as before.

$$x_0^2 = \rho^2 - \alpha^2 \qquad \rho^2 = \frac{a}{4r(1 - r/a)}$$

$\rho = \infty$: either $r = 0$ (singularity) or $r = a$ (horizon)
$\rightarrow$ two sequences of fuzzy spheres

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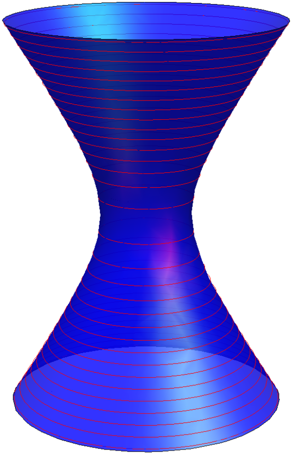

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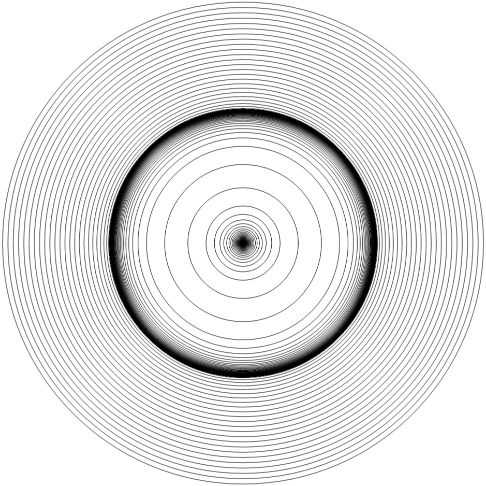
Fuzzy black hole inside and outside, in one figure:



Solutions apparently do not match . . .

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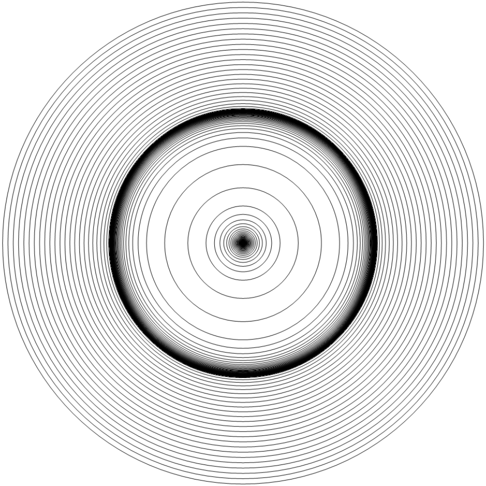

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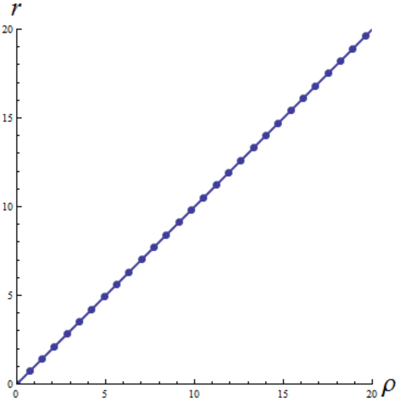
Inside NC black hole II

So far: NC Solutions inside event horizon exist; important proof of principle. Matching solutions: choose algebras carefully.

NC “wormhole”

Reconsider solution in isotropic coordinates



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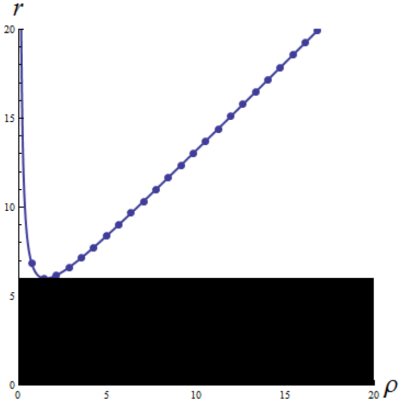

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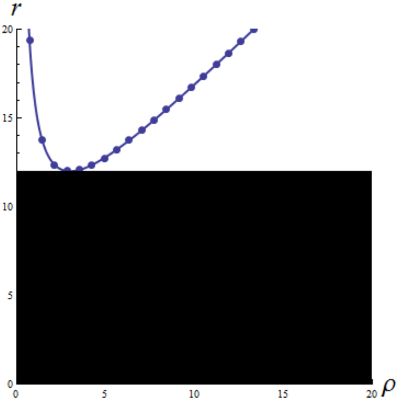

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NC "wormhole"

Reconsider solution in isotropic coordinates

NC "wormhole" =
two Schwarzschild spacetimes glued together at the horizon

inside $\leftrightarrow$ outside: $a/4\rho \leftrightarrow 4\rho/a$

Entropy

Outside observer (far away, stationary): Not relevant to ask what is "really" going on inside the black hole. Only relevant to consider degrees of freedom of part of original spacetime now hidden by event horizon. That space is discreet.

Naive state counting "inside" NC black hole

1st quantized description: wave function on fuzzy sphere
2nd quantized description: wave function $\rightarrow$ quantum field

For field operators with finite spectrum: dim $\mathcal{H} = f^N$

Entropy = missing information $\propto N$ (just like black hole area)

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Entanglement Entropy

Pure state on “whole” NC spacetime:

$$|\Psi\rangle = \sum c_i |\psi_i^A\rangle \otimes |\psi_i^B\rangle$$

with appropriately chosen states $|\psi_i^A\rangle \in \mathcal{H}_{\text{out}}$ and $|\psi_i^B\rangle \in \mathcal{H}_{\text{in}}$.

Note: $\dim \mathcal{H}_{\text{out}} \gg \dim \mathcal{H}_{\text{in}}$
# of nonzero c_i = Schmidt rank of $|\Psi\rangle$

Partial traces $\rightarrow$ density matrices ρ_A, ρ_B

Entanglement entropy $S = -\text{tr}\rho_A \ln \rho_A = -\text{tr}\rho_B \ln \rho_B$

For an initial pure state with maximal Schmidt number we recover the naive state counting result.

Entanglement Entropy

Pure state on "whole" NC spacetime:

$$|\Psi\rangle = \sum c_i |\psi_i^A\rangle \otimes |\psi_i^B\rangle$$

with appropriately chosen states $|\psi_i^A\rangle \in \mathcal{H}_{\mathrm{out}}$ and $|\psi_i^B\rangle \in \mathcal{H}_{\mathrm{in}}$.

Note: $\dim \mathcal{H}_{\mathrm{out}} \gg \dim \mathcal{H}_{\mathrm{in}}$
# of nonzero c_i = Schmidt rank of $|\Psi\rangle$

Partial traces $\rightarrow$ density matrices ρ_A, ρ_B

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Entanglement Entropy

A more sophisticated computation starts with a scalar field in the ground state of the Rindler Hamiltonian.

For large quantum number N:
NC computation $\sim$ regularized classical computation.

- Partial trace gives thermal Unruh density matrix with temperature $T = 1/2\pi$. Redshifted this yields the Hawking temperature. Subtleties occur for $2M = a \neq N\lambda$.
- Entanglement entropy is proportional to area of horizon. It is finite (!). (Automatic UV cutoff at $l \sim a/\lambda_0$.)
- Entropy of thermal gas near horizon is also finite.

Entropy

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Generalized coherent state ($SU(2)$, spin j representation)

$$|\Omega\rangle = \mathcal{R}_\Omega |j, j\rangle, \quad \mathcal{R}_\Omega \in SU(2)/U(1); \qquad (2j+1)\int \frac{d\Omega}{4\pi} |\Omega\rangle\langle\Omega| = 1_j$$

Star product

For $A(\Omega) := \langle\Omega|A|\Omega\rangle$ and $B(\Omega) := \langle\Omega|B|\Omega\rangle$ define:

$$A(\Omega) \star B(\Omega) = \langle\Omega|AB|\Omega\rangle = (2j+1)\int \frac{d\tilde{\Omega}}{4\pi} \langle\Omega|A|\tilde{\Omega}\rangle\langle\tilde{\Omega}|B|\Omega\rangle$$

Coherent States, Star Products, Entropy II

Quantum mechanical entropy for a density operator ρ

Entropy

$$S_Q(\rho) = -\mathrm{tr}\rho \ln \rho = -(2j+1) \int \frac{d\Omega}{4\pi} \rho(\Omega) \star \ln_\star \rho(\Omega)$$

Now “switch off” (or ignore) noncommutativity $\Rightarrow$

Lieb-Wehrl entropy

$$\begin{aligned} S_W(\rho) &= -(2j+1) \int \frac{d\Omega}{4\pi} \rho(\Omega) \ln \rho(\Omega) \\ &\geq -(2j+1) \int \frac{d\Omega}{4\pi} |\langle \Omega | \Psi \rangle|^2 \ln |\langle \Omega | \Psi \rangle|^2 \\ &> 0 \quad \text{even for pure states} \end{aligned}$$

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Coherent States, Star Products, Entropy II

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$\sim$ entropy of classical coarse-graining: "quantum ignorance"

scales like area for a field on the fuzzy event horizon

Exact NC Gravity Solutions

Other exact solutions?

Given enough isometries, we can find compatible algebra-metric pairs with the same method as for the Schwarzschild case. Some examples:

- Rotating solution (Kerr) with time-space noncommutativity
- Charged solution (Reissner-Nordström), time commutative
- Robertson-Walker (hard to make isotropic)
- BTZ black hole solution in 2+1 dimensions

Higher dimensions

The simple Lie-type coordinate is an artefact of 3+1 dimensions. In higher dimensions we find higher algebras

$$[x_i, x_j, \ldots, x_k] = 2i\lambda\epsilon_{ij\ldots kl}x_l$$

and have to deal with nonassociativity or projectors.

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Exact NC Gravity Solutions

Noncommutative BTZ black hole

2+1 dimensions: ϕ, ρ, t

Quantization of metric-compatible Poisson structure gives

$$[\phi, t] = i\tau \qquad \text{(angle-time noncommutativity)}$$

where $\tau > 0$ is a fundamental unit of time.

Better: $[e^{i\phi}, t] = \tau e^{i\phi}$ (since $\phi = \phi + 2\pi$ etc.)

Irreducible representations (labeled by $\alpha \in [0, \tau)$)

$$t|n, \alpha\rangle = (n\tau + \alpha)|n, \alpha\rangle$$

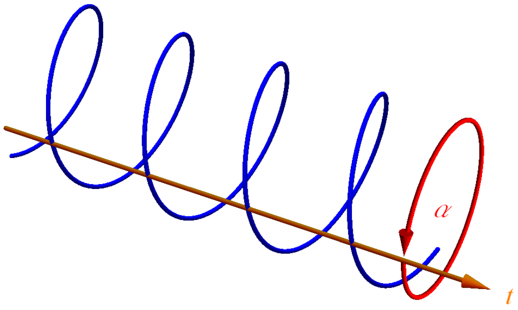
Dolan, Gupta, Stern

Turns out to be an exact solution to the NC gravity equations.

Noncommutative BTZ black hole

2+1 dimensions: ϕ, ρ, t

$$t|n, \alpha\rangle = (n\tau + \alpha)|n, \alpha\rangle\,, \qquad \alpha \in [0, \tau)$$



Summary

noncommutative gravity

- simple construction via twisted tensor calculus
- fully covariant

solutions

- use isometries, metric central
- provide simple models of quantum geometry

fuzzy black hole

- discrete, quasi-2D onion-type spacetime
- natural holographic properties